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Research Article

A Note on the Growth of Periodic Points for Commuting Toral Automorphisms

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In this note we study the growth of the number of periodic points for non-degenerate actions of commuting hyperbolic toral automorophisms.

1. Introduction

There are well-known formulae for the number of fixed points for powers of a given orientation preserving hyperbolic linear toral automorphism $T : \mathbb{T}^d \to \mathbb{T}^d$. In particular, if $A \in \mathrm{SL}(d, \mathbb{R})$ is the associated hyperbolic matrix, then the number of fixed points for T^n is given by

$$\mathrm{Card}\{T^n x = x\} = |\det(A^n - 1)|. \tag{1.1}$$

Since the hyperbolicity of T is equivalent to the fact that the eigenvalues for A do not lie on the unit circle, it is easy to see that the number of fixed points for T^n grows exponentially fast in n. We then recall that the growth rate of the number of periodic points for T is given by

$$h(T) = \lim_{k \to +\infty} \frac{1}{k} \log\ \mathrm{Card}\left\{x\ : T^k x = x\right\} > 0, \tag{1.2}$$

where $h(T)$ is the topological entropy of $T : \mathbb{T}^d \to \mathbb{T}^d$ (or, equivalently, the sum of the logarithms of the eigenvalue of the matrix A with absolute value at least 1).

In this note, we want to consider fixed points for commuting hyperbolic toral automorphisms. This necessarily requires the torus to have dimension $d \geq 3$, and for simplicity of exposition we shall initially assume that $d = 3$. Let us, therefore, consider a pair of commuting hyperbolic matrices $A_1, A_2 \in \mathrm{SL}(3,\mathbb{Z})$ (i.e., $A_1A_2 = A_2A_1$ and neither matrix has an eigenvalue of modulus one) and associate the natural $\mathbb{Z}^2$-action on the three dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ defined by

$$\mathcal{A} : \mathbb{Z}^2 \times \mathbb{T}^3 \longrightarrow \mathbb{T}^3 \quad \text{given by } \mathcal{A}(n_1, n_2, x) = A_1^{n_1} A_2^{n_2} x + \mathbb{Z}^3. \tag{1.3}$$

We will also ask for this action to be *nondegenerate*, that is, if $n_1, n_2 \in \mathbb{Z}$ satisfy $A_1^{n_1} A_2^{n_2} = I$, then this necessarily implies that $n_1 = n_2 = 0$. We say that A_1 and A_2 are independent.

We can now consider the growth of the number of fixed points for the action associated to any element $(n_1, n_2) \in \mathbb{Z}^2$.

Definition 1.1. We denote the number of fixed points of by $A_1^{n_1} A_2^{n_2}$ on $\mathbb{T}^3$ by

$$N(n_1, n_2) = \mathrm{Card}\left\{x \in \mathbb{T}^3 : \mathcal{A}(n_1, n_2, x) = x\right\}. \tag{1.4}$$

We want to give uniform estimates on the rate of growth of the number of fixed points for the actions $\mathcal{A}(n_1, n_2, \cdot) : \mathbb{T}^3 \to \mathbb{T}^3$ in terms of $(n_1, n_2) \in \mathbb{Z}^2$. In particular, we want to give a lower bound on the growth of the fixed points in terms as $\|(n_1, n_2)\|_2 = \sqrt{n_1^2 + n_2^2} \to +\infty$. In the present context, we can assume without loss of generality that the eigenvalues $\alpha_1, \alpha_2, \alpha_3$ of A_1 and the eigenvalues $\beta_1, \beta_2, \beta_3$ of A_2 are real.

Definition 1.2. We denote

$$\begin{aligned} \overline{\lambda} &:= \sup_{0\le\theta\le 2\pi} \left\{ \max_{i=1,2,3} \left\{ \cos\theta \log|\alpha_i| + \sin\theta \log\left|\beta_i\right| \right\} \right\}, \\ \underline{\lambda} &:= \inf_{0\le\theta\le 2\pi} \left\{ \max_{i=1,2,3} \left\{ \cos\theta \log|\alpha_i| + \sin\theta \log\left|\beta_i\right| \right\} \right\}. \end{aligned} \tag{1.5}$$

Our main result, in the particular case $d = 3$, is the following.

Theorem 1.3. *Let $A_1, A_2 \in SL(3,\mathbb{Z})$ be commuting independent hyperbolic matrices. The growth rates of the fixed points*

$$\begin{aligned} \overline{\lambda} &= \limsup_{\|(n_1,n_2)\|_2 \to +\infty} \frac{1}{\|(n_1, n_2)\|_2} \log N(n_1, n_2), \\ \underline{\lambda} &= \liminf_{\|(n_1,n_2)\|_2 \to +\infty} \frac{1}{\|(n_1, n_2)\|_2} \log N(n_1, n_2) > 0 \end{aligned} \tag{1.6}$$

satisfy $0 < \underline{\lambda} < \overline{\lambda} < +\infty$.

Related problems have been studied for $\mathbb{Z}^d$-actions in algebraic and symbolic examples by Miles and Ward [1]. Interestingly, whereas their analysis relies on deep results in diophantine approximation, in the present context the required analysis is completely elementary.

Table 1: The number of fixed points $N(n_1, n_2)$ for $|n_1|, |n_2| \le 4$. The columns correspond to n_1 and the rows correspond to n_2.

	−4	−3	−2	−1	0	1	2	3	4
4	533	27	203	377	533	448	1261	11857	68411
3	377	91	13	64	91	27	559	3913	21463
2	203	64	13	7	13	13	203	1247	6656
1	27	13	7	1	1	7	64	377	2009
0	533	91	13	1	∞	1	13	91	533
−1	2009	377	64	7	1	1	7	13	27
−2	6656	1247	203	13	13	7	13	64	203
−3	21463	3913	559	27	91	64	13	91	377
−4	68411	11857	1261	448	533	377	203	27	533

The quantity $\underline{\lambda}$ is related to the supremum of the sum of the Lyapunov exponents for the action. In particular, the bound $\underline{\lambda} > 0$ can then be deduced from ([2], Lemma 4.3 (a)).

Remark 1.4. The values $\overline{\theta}$ and $\underline{\theta}$ realizing the supremum and infimum, respectively, in (1.6) can be understood as giving the "approximate directions" of largest and smallest growth in the number of fixed points points.

Remark 1.5. There is no analogous result for rates of mixing. The reason for this is simply because any hyperbolic toral automorphism mixes superexponentially with respect to the Haar measure and C^∞ test functions. In particular, the rate of mixing is infinite and there is no useful way to distinguish between the actions. By the same token, there is no analogous result for rates of equidistribution for closed orbits [3, Theorem 1.6].

The calculations in this paper were inspired by a lecture by Tom Ward, who presented tables similar to those in this note in the context of $\mathbb{Z}^2$-subshifts of finite type.

2. Examples

Let us consider some examples that illustrate Theorem 1.3.

Example 2.1. Consider the commuting matrices $A_1, A_2 \in \mathrm{SL}(3, \mathbb{Z})$ given by

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}. \tag{2.1}$$

The number of fixed points $N(n_1, n_2)$ for $|n_1|, |n_2| \le 4$ is presented in Table 1.

The eigenvalues of A_1 are $\alpha_1 = 3.24698\ldots$, $\alpha_2 = 1.55496\ldots$, and $\alpha_3 = 0.198062\ldots$, and the eigenvalues of A_2 are $\beta_1 = 0.198062\ldots$, $\beta_2 = 3.24698\ldots$, and $\beta_3 = 1.55496\ldots$ (which

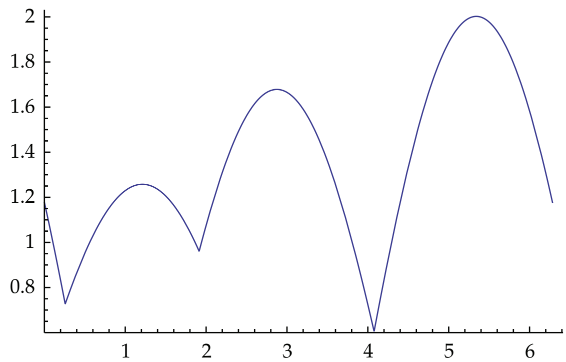

Figure 1: A plot of $\{\max_{i=1,2,3}\{\cos\theta\log|\alpha_i| + \sin\theta\log|\beta_i|\}$ as a function of $0 \le \theta < 2\pi$.

happen to be a permutation of those for A_1). Corresponding to these eigenvalues are the common eigenvectors

$$e_1 = \begin{pmatrix} -0.327985\ldots \\ 0.736976\ldots \\ -0.591009\ldots \end{pmatrix}, \qquad e_2 = \begin{pmatrix} -0.591009\ldots \\ 0.327985\ldots \\ 0.736976\ldots \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0.736976\ldots \\ 0.591009\ldots \\ 0.327985\ldots \end{pmatrix}. \tag{2.2}$$

Using these eigenvalues we can now plot the function $\theta \mapsto \{\max_{i=1,2,3}\{\cos\theta\log|\alpha_i| + \sin\theta\log|\beta_i|\}$ (cf. Figure 1) and then read off the values of $\overline{\lambda}$ and $\underline{\lambda}$ as the maximum and minimum values, respectively.

In this example, we see that $\underline{\lambda} = 0.60501\ldots$ (occurring at $\underline{\theta} = 4.07742\ldots$) and $\overline{\lambda} = 2.00219\ldots$ (occurring at $\overline{\theta} = 5.34124\ldots$).

Example 2.2 (cf. [4]). We can let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -11 & 5 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 11 & -5 \end{pmatrix}. \tag{2.3}$$

The number of fixed points $N(n_1, n_2)$ for $|n_1|, |n_2| \le 4$ is presented in Table 2.

The eigenvalues for A are $\alpha_1 = 4.70928$, $\alpha_2 = 0.0967881$, and $\alpha_3 = 2.19394$, and the eigenvalues for A_2 are $\beta_1 = -2.70928$, $\beta_2 = 1.90321$, and $\beta_3 = -0.193937$. These correspond to the eigenvectors

$$\begin{pmatrix} -0.0440649 \\ -0.207514 \\ -0.977239 \end{pmatrix}, \qquad \begin{pmatrix} -0.995305 \\ -0.0963337 \\ -0.00932395 \end{pmatrix}, \qquad \begin{pmatrix} 0.185754 \\ 0.407532 \\ 0.894099 \end{pmatrix}. \tag{2.4}$$

In this case, we can compute $\underline{\lambda} = 0.689643\ldots$ (occurring at $\underline{\theta} = 4.17448\ldots$) and $\overline{\lambda} = 2.2481\ldots$ (occurring at $\overline{\theta} = 5.43348\ldots$) (cf. Figure 2).

Table 2: The number of fixed points $N(n_1, n_2)$ for $|n_1|, |n_2| \leq 4$. The columns correspond to n_1 and the rows correspond to n_2.

	−4	−3	−2	−1	0	1	2	3	4
4	25600	8132	464	4652	10880	21388	40592	75356	133120
3	5476	1928	460	296	988	2008	3700	6152	7004
2	1040	452	128	4	80	172	256	76	2000
1	68	152	20	8	4	8	20	232	1404
0	640	148	16	4	∞	4	16	148	640
−1	1404	232	20	8	4	8	20	152	68
−2	2000	76	256	172	80	4	128	452	1040
−3	7004	6152	3700	2008	988	296	460	1928	5476
−4	133120	75356	40592	21388	10880	4652	464	8132	25600

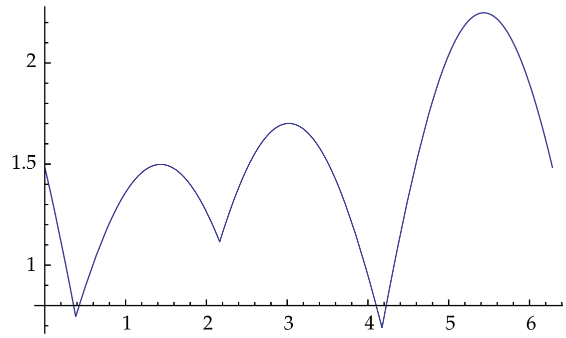

Figure 2: A plot of $\{\max_{i=1,2,3}\{\cos\theta \log|\alpha_i| + \sin\theta \log|\beta_i|\}$ as a function of $0 \leq \theta < 2\pi$.

3. Proof of Theorem 1.3

We begin by fixing our notation. Let $A_1, A_2 \in \mathrm{SL}(3, \mathbb{Z})$ be commuting hyperbolic matrices (i.e., none of the eigenvalues has modulus unity. In this particular case, it is not possible to have ergodic nonhyperbolic toral automorphisms.) We shall assume the associated action is nondegenrate (i.e., $A_1^{n_1} A_2^{n_2} = I$ implies $(n_1, n_2) = (0, 0)$).

We next recall the following standard results.

Lemma 3.1. *Under the above hypotheses,*

(1) *the eigenvalues $\alpha_1, \alpha_2, \alpha_3$ of A_1 are real, and the eigenvalues $\beta_1, \beta_2, \beta_3$ of A_2 are real;*

(2) *each of the common eigenvectors e_1, e_2, e_3 for A_1 and A_2 has irrational slope (i.e., each $\mathbb{R}v_i + \mathbb{Z}^3$ is dense in $\mathbb{T}^3$);*

(3) *each of the real numbers* $\log|\alpha_i| / \log|\beta_i|$, $i = 1, 2, 3$, *is irrational.*

Proof. The first result is a consequence of a standard general result for more general Cartan actions, applied in the particular case of $\mathbb{Z}^2$-actions [4].

For the second part, we can restrict to the case e_1, with the other cases being similar. It is easy to see that we can make an appropriate choice of $n, m \in \mathbb{Z}^3$ such that matrix $A_1^n A_2^m$ either has $|\alpha_1^n \beta_1^m| > 1 > |\alpha_2^n \beta_2^m|, |\alpha_3^n \beta_3^m|$ or $|\alpha_1^n \beta_1^m| < 1 < |\alpha_2^n \beta_2^m|, |\alpha_3^n \beta_3^m|$. In particular, $T = \mathcal{A}(n_1, n_2, \cdot)$ corresponds to a linear hyperbolic toral automorphism T for which $\mathbb{R}v_i + \mathbb{Z}^3$ is a leaf of either the one-dimensional stable or one-dimensional unstable manifold foliation. In particular, this is dense by the well known minimality of the stable and unstable manifolds.

Finally, for the last part, the irrationality of the ratio of the logarithm of the eigenvalues is a consequence of the nontriviality assumption and part 2. More precisely, if we assume for a contradiction that $\log\alpha_i/\log\beta_i$ is a rational p/q, say, then by comparing the actions of A_1 and A_2 on the dense $\mathbb{R}e_1 + \mathbb{Z}^3$ set, we then see from the second part that $A_1^pA_2^q = I$. This contradicts the nondegeneracy condition, completing the proof. □

In particular, we see from parts 2 and 3 of Lemma 3.1 that for all $(n_1, n_2) \in \mathbb{Z}^2 - (0,0)$ we have that $A_1^{n_1}A_2^{n_2}$ has no eigenvalues of modulus 1.

We recall the following standard result for the fixed points of the single transformation $\mathcal{A}(n_1, n_2, \cdot) : \mathbb{T}^3 \to \mathbb{T}^3$.

Lemma 3.2. *For each $(n_1, n_2) \in \mathbb{Z}^2 - \{(0,0)\}$, we can write*

$$N(n_1, n_2) = \left|\det\left(I - A_1^{n_1}A_2^{n_2}\right)\right|. \tag{3.1}$$

Proof. This is a standard result, which can also be easily deduced from the Lefschetz fixed point theorem. □

Lemma 3.2 is particularly useful in computing the numerical values of fixed points in the tables we have for the examples. We also have the following simple, but useful, corollary.

Lemma 3.3. *For each $(n_1, n_2) \in \mathbb{Z}^2 - \{(0,0)\}$, we can write*

$$N(n_1, n_2) = \left|1 - \left(\alpha_1^{n_1}\beta_1^{n_2} + \alpha_2^{n_1}\beta_2^{n_2} + \alpha_2^{n_1}\beta_2^{n_2}\right) + \left(\alpha_1^{-n_1}\beta_1^{-n_1} + \alpha_2^{-n_2}\beta_2^{-n_2} + \alpha_3^{-n_3}\beta_3^{-n_3}\right) - 1\right|. \tag{3.2}$$

Proof. The matrix $A_1^{n_1}A_2^{n_2}$ has eigenvalues $\alpha_1^{n_1}\beta_1^{n_2}$, $\alpha_2^{n_1}\beta_2^{n_2}$, and $\alpha_3^{n_1}\beta_3^{n_2}$. Multiplying out this expression for $N(n_1, n_2)$ gives

$$\begin{aligned}
N(n_1, n_2) &= \left|\det\left(I - A_1^{n_1}A_2^{n_2}\right)\right| \\
&= \left|\left(1 - \alpha_1^{n_1}\beta_1^{n_2}\right)\left(1 - \alpha_2^{n_1}\beta_2^{n_2}\right)\left(1 - \alpha_3^{n_1}\beta_3^{n_2}\right)\right| \\
&= \Big|1 - \left(\alpha_1^{n_1}\beta_1^{n_2} + \alpha_2^{n_1}\beta_2^{n_2} + \alpha_2^{n_1}\beta_2^{n_2}\right) \\
&\quad + \left((\alpha_1\alpha_2)^{n_1}(\beta_1\beta_2)^{n_2} + (\alpha_1\alpha_3)^{n_1}(\beta_1\beta_3)^{n_2} + (\alpha_2\alpha_3)^{n_1}(\beta_2\beta_3)^{n_2}\right) - 1\Big| \\
&= \left|1 - \left(\alpha_1^{n_1}\beta_1^{n_2} + \alpha_2^{n_1}\beta_2^{n_2} + \alpha_2^{n_1}\beta_2^{n_2}\right) + \left(\alpha_1^{-n_1}\beta_1^{-n_2} + \alpha_2^{-n_1}\beta_2^{-n_2} + \alpha_3^{-n_1}\beta_3^{-n_2}\right) - 1\right|,
\end{aligned} \tag{3.3}$$

where we have used the identities $\alpha_1\alpha_2\alpha_3 = \det A_1 = 1$ and $\beta_1\beta_2\beta_3 = \det A_2 = 1$ for the last line. □

We want to use this lemma to estimate the growth of $N(n_1, n_2)$. In particular, we want to get bounds based on the largest of the terms (in modulus) contributing to the right hand side of (3.2). In order to formulate these estimates, it is convenient to introduce the vectors in $\mathbb{R}^2$ defined by

$$v_1 = \begin{pmatrix} \log|\alpha_1| \\ \log|\beta_1| \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \log|\alpha_2| \\ \log|\beta_2| \end{pmatrix}, \qquad v_3 = \begin{pmatrix} \log|\alpha_3| \\ \log|\beta_3| \end{pmatrix}. \tag{3.4}$$

Each of these has irrational slope, by the final part of Lemma 3.1.

Lemma 3.4. *All of the vectors v_1, v_2, and v_3 are nonzero and satisfy $v_1 + v_2 + v_2 = 0$.*

Proof. For the first part, we need only observe that if $v_i = 0$, say, then this would require $|\alpha_i| = |\beta_i| = 1$, that is, at least one of the eigenvalues for the matrices is of modulus one which would contradict the hyperbolicity assumption.

For the second part, we observe that since $\alpha_1\alpha_2\alpha_3 = \det A_1 = 1$ and $\beta_1\beta_2\beta_3 = \det A_2 = 1$ we immediately see that $v_1 + v_2 + v_2 = 0$. □

We now parameterize the unit vectors in $\mathbb{R}^2$ by

$$w_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \text{for } 0 \le \theta < 2\pi. \tag{3.5}$$

We can then write that

$$\langle v_i, w_\theta \rangle = \cos\theta \log\alpha_i + \sin\theta \log\beta_i, \quad \text{for } i = 1, 2, 3. \tag{3.6}$$

In particular, if we write $(n_1, n_2) = (R\cos\theta, R\sin\theta)$, say, where $R = \|(n_1, n_2)\|_2$, then we can write

$$\left|\alpha_i^{n_1}\beta_i^{n_2}\right| = \exp\left(R\left(\cos\theta \log|\alpha_i| + \sin\theta \log\left|\beta_i\right|\right)\right). \tag{3.7}$$

To prove Theorem 1.3, it suffices to show that the vectors v_1, v_2, v_3 are not collinear. Since $v_1 + v_2 + v_3 = 0$ and the vectors v_1, v_2, v_3 are nonzero, and additionally we know that the vectors are noncollinear, it is then easy to see that this is enough to know that for any $0 \le \theta < 2\pi$ there is some i such that $\langle v_i, v_\theta \rangle > 0$ (se Figure 3). For typical θ, there will be a single dominant term of the form (3.7) contributing to the right hand side of (3.2).

Assume for a contradiction that the vectors v_1, v_2, and v_3 are collinear. Then we can choose $\delta \neq 0$ such that

$$\delta = \frac{\log|\alpha_1|}{\log\left|\beta_1\right|} = \frac{\log|\alpha_2|}{\log\left|\beta_2\right|} = \frac{\log|\alpha_3|}{\log\left|\beta_3\right|}. \tag{3.8}$$

First we observe that δ cannot be irrational since otherwise $\{n\log|\alpha_1| + m\log|\beta_1| \ : \ n, m \in \mathbb{Z}\}$ will be dense on the real line $\mathbb{R}$. However, since $\mathbb{R}^+ v_1 + \mathbb{Z}^3$ is dense in $\mathbb{T}^3$ this means that we can choose $n_k, m_k \in \mathbb{Z}$ such that $A^{n_k}, B^{m_k} \to I$ as $k \to +\infty$, but with $A^{n_k}, B^{m_k} \neq I$. However, this is clearly false in the lattice $\mathrm{SL}(3, \mathbb{Z})$. On the other hand, if $\delta = p/q$ were a rational then by again considering the action on the dense set $\mathbb{R}^+ w_1 + \mathbb{Z}^3$ we see that $A^p B^q = I$, which contradicts the nondegeneracy hypothesis.

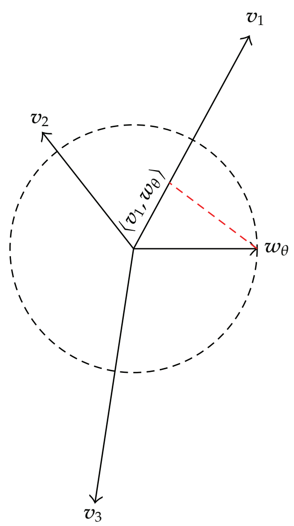

Figure 3: The projection of w_θ onto one of the vectors v_1, v_2, v_3 must have a strictly positive component.

4. Generalizations to $\mathbb{Z}^k$-Actions

We will consider the more general setting of higher-dimensional actions. The basic results are similar to the case of Theorem 1.3.

Hypothesis 1. Let $2 \le k \le d-1$.

(1) We shall assume that $A_1, \ldots, A_k \in \mathrm{SL}(d, \mathbb{Z})$ are commuting matrices, that is, $A_i A_j = A_j A_i$ for $1 \le i, j \le k$.

(2) We shall assume that each matrix $A_1^{n_1} \cdots A_k^{n_k}$, $(n_1, \ldots, n_k) \in \mathbb{Z}^k - (0, \ldots, 0)$ is ergodic (i.e., they do not have eigenvalues, which are roots of unity).

(3) We shall assume that the action is nondegenerate, that is, if there exist $n_1, \ldots, n_k \in \mathbb{Z}$ such that

$$A_1^{n_1} A_2^{n_2} \cdots A_k^{n_k} = I \quad \text{then } n_1 = \cdots = n_k = 0. \tag{4.1}$$

(4) We shall assume that the action is irreducible, that is, no $\mathcal{A}(n_1, \ldots, n_k) : \mathbb{T}^d \to \mathbb{T}^d$ preserves a proper invariant toral subgroup of $\mathbb{T}^d$.

(5) We shall additionally assume, mainly for convenience, that the matrices are semisimple (i.e., they diagonalize over the complex numbers) and A_i has complex eigenvalues $\alpha_1^{(i)}, \ldots, \alpha_d^{(i)}$ for $i = 1, \ldots, k$.

The special case that bears closest comparison with the special case of $k = 2$ and $d = 3$ is when $k = d - 1$. In particular, in this case A_i has real eigenvalues $\alpha_1^{(i)}, \ldots, \alpha_d^{(i)}$ for $i = 1, \ldots, k$.

We now generalize two definitions from the first section.

Definition 4.1. Let $\mathcal{A} : \mathbb{Z}^k \times \mathbb{T}^d \to \mathbb{T}^d$ be the action given by $\mathcal{A}(n_1, \ldots, n_d, x) = A_1^{n_1} \cdots A_k^{n_k} x + \mathbb{Z}^d$, then we denote

$$N(n_1, \ldots, n_k) = \mathrm{Card}\left\{ x \in \mathbb{T}^d : \mathcal{A}(n_1, \ldots n_k, x) = x \right\}. \tag{4.2}$$

Definition 4.2. We can define

$$\underline{\lambda} = \inf_{\|v\|_2=1} \left\{ \sup_{w} \langle v, w \rangle \right\}, \qquad \overline{\lambda} = \sup_{\|v\|_2=1} \left\{ \sup_{w} \langle v, w \rangle \right\}, \tag{4.3}$$

where the supreme ranges over all unit vectors v in $\mathbb{R}^k$.

The natural generalization of Theorem 1.3 is the following.

Theorem 4.3. *The growth rates of the number of fixed points*

$$\begin{aligned} \overline{\lambda} &:= \limsup_{\|(n_1,\ldots,n_k)\|_2 \to +\infty} \frac{1}{\|(n_1, \ldots, n_k)\|_2} \log N(n_1, \ldots, n_k) \}, \\ \underline{\lambda} &:= \liminf_{\|(n_1,\ldots,n_k)\|_2 \to +\infty} \frac{1}{\|(n_1, \ldots, n_k)\|_2} \log N(n_1, \ldots, n_k) \} > 0 \end{aligned} \tag{4.4}$$

satisfy $0 < \underline{\lambda} < \overline{\lambda} < +\infty$.

To begin the proof, we need the following standard generalization of Lemma 3.2.

Lemma 4.4. *For each* $(n_1, \ldots, n_k) \in \mathbb{Z}^2 - \{(0, \ldots, 0)\}$, *we can write*

$$N(n_1, \ldots, n_k) = \left| \det\left(I - A_1^{n_1} \cdots A_k^{n_k}\right) \right|. \tag{4.5}$$

Proof. This is again a standard application of the Lefschetz formula. □

In particular, we can use Lemma 4.4 to write

$$N(n_1, \ldots, n_k) = \left| \det\left(I - A_1^{n_1} \cdots A_k^{n_k}\right) \right| = \prod_{j=1}^{d} \left| 1 - \prod_{i=1}^{d-1} \left(\alpha_j^{(i)} \right)^{n_i} \right|. \tag{4.6}$$

It is convenient to use the parameterization $(n_1, \ldots, n_k) = (p_1 R, \ldots, p_k R)$, where

(1) $0 \le p_1, \ldots, p_k \le 1$ with $p_1^2 + \cdots + p_{\mathrm{k}}^2 = 1$;

(2) $R = \|(n_1, \ldots, n_k)\|_2$.

We can now introduce the notation $v_p = (p_1, \ldots, p_k)$ and $v_j = (\log|\alpha_j^{(1)}|, \ldots, \log|\alpha_j^{(k)}|)$, for $j = 1, \ldots, k$. We can now easily see from (4.6) that for any $\delta > 0$, there exists $R_0 = R_0(\delta)$ such that

$$\begin{aligned} N(n_1, \ldots, n_k) &\geq \prod_{j:\langle v_p, v_j\rangle > 0} (\exp(R\langle v_p, v_j\rangle) - 1) \prod_{j:\langle v_p, v_j\rangle < 0} (1 - \exp\ (R\langle v_p, v_j\rangle)) \\ &\geq (1-\delta) \prod_{j:\langle v_p, v_j\rangle \geq 0} \exp(R\langle v_p, v_j\rangle) \\ &\geq (1-\delta) \exp\left(R\left\langle v_p, \left(\sum_{j:\langle v_p, v_j\rangle \geq 0} v_j \right) \right\rangle \right), \end{aligned} \tag{4.7}$$

for $R \geq R_0$. In particular, we see that

$$N(n_1, \ldots, n_k) \geq (1-\delta) \exp\left(\underline{\lambda} \|(n_1, \ldots, n_k)\|_2\right), \tag{4.8}$$

where

$$\underline{\lambda} = \inf_p \left\{ \left\langle v_p, \left(\sum_{j:\langle v_p, v_j\rangle \geq 0} v_j \right) \right\rangle \right\}. \tag{4.9}$$

Similarly, we see that for $R \geq R_0$,

$$N(n_1, \ldots, n_k) \leq (1+\delta) \exp\left(\overline{\lambda} \|(n_1, \ldots, n_k)\|_2\right), \tag{4.10}$$

where

$$\overline{\lambda} = \sup_p \left\{ \left\langle v_p, \left(\sum_{j:\langle v_p, v_j\rangle \geq 0} v_j \right) \right\rangle \right\}. \tag{4.11}$$

To see that $\underline{\lambda} > 0$, we need to know that $v_1, \ldots, v_k$ are not confined to a codimension one hyperplane in $\mathbb{R}^k$ orthogonal to some v_p. Assume for a contradiction that there is a unit vector v_p such that $\langle v_p, v_i\rangle = 0$ for $i = 1, \ldots, k$. Let $v_p = (v_p^{(1)}, \ldots, v_p^{(k)})$, then by Dirichlet's theorem of simultaneous diophantine approximation, for any $\epsilon > 0$, we choose $1 \leq q \leq ([1/\epsilon] + 1)^k$ and $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ with

$$\left\|(n_1, \ldots, n_k) - q v_p\right\|_\infty \leq \epsilon. \tag{4.12}$$

In particular, the eigenvalues $(\alpha_j^1)^{n_1}\cdots(\alpha_j^k)^{n_k}$, $j=1,\ldots,k$, for the matrix $A_1^{n_1}\cdots A_k^{n_k}\in \mathrm{SL}(d,\mathbb{R})$ satisfy

$$\left|\log\left|\left(\alpha_j^{(1)}\right)^{n_1}\cdots\left(\alpha_j^{(k)}\right)^{n_k}\right|\right| = \left|\sum_{l=1}^{k} n_l \log\left|\alpha_j^{(l)}\right|\right| \le k\epsilon + q\underbrace{\left|\sum_{l=1}^{k} v_l \log\left|\alpha_j^{(l)}\right|\right|}_{=\left|\langle v_p, v_i\rangle\right|=0}. \tag{4.13}$$

In particular, the algebraic integers, and its conjugates, occurring as zeros of the characteristic polynomial $\det(zI - A_1^{n_1}\cdots A_k^{n_k}) = 0$ can be arbitrarily close to one. It only remains to show this cannot happen, which we deduce from the following two results.

Lemma 4.5 (Krönecker, [5]). *Any algebraic integer α whose conjugate roots $\alpha=\alpha_1,\ldots,\alpha_d$ all lie on the unit circle must necessarily be a root of unity.*

Proof. We include the simple proof for completeness. Let us define a sequence of monomials

$$P_n(x) := \prod_{i=1}^{d}\left(x-\alpha_i^n\right) = x^d + a_{d-1}^{(n)}x^{d-1} + \cdots + a_k^{(n)}x^k + \cdots + a_1^{(n)}x + a_0^{(n)}. \tag{4.14}$$

In particular, since

$$\left|a_k^{(n)}\right| = \left|\sum_{i_1<\cdots<i_{d-k}} \alpha_{i_1}^{(n)}\cdots\alpha_{i_{d-k}}^{(n)}\right| \le K := d!\ , \tag{4.15}$$

we see that $\{P_n(x) \ :\ n\ge 1\}$ is a finite set as is the set of roots α of these polynomials. Thus for any such root, the pigeonhole principle applied to $\{\alpha^n \ :\ n\ge 0\}$ shows that there exists $0\le p<q\le K+1$ such that $\alpha^p=\alpha^q$, and thus $\alpha^{q-p}=1$. □

We can also prove the following variant.

Lemma 4.6. *Given $d\ge 2$, there exists $\epsilon>0$ such that if α is an algebraic number of degree, which is not an algebraic integer, then the conjugate values $\alpha=\alpha_1,\ldots,\alpha_d$ cannot all be contained in the annulus*

$$A(\epsilon) = \{z\in\mathbb{C} : 1-\epsilon \le |z| \le 1+\epsilon\}. \tag{4.16}$$

Proof. Since the proof is elementary, we include it for convenience. Assume for a contradiction that for some $d\ge 2$ we can find an infinite sequence of monomials

$$P_n(x) = x^d + a_{d-1}^{(n)}x^{d-1} + \cdots + a_k^{(n)}x^k + \cdots + a_1^{(n)}x + a_0^{(n)} \in \mathbb{Z}[x]\,], \quad \text{for } n\ge 2, \tag{4.17}$$

Table 3: The number of fixed points $N(n_1, n_2)$ for $|n_1|, |n_2| \leq 4$. The columns correspond to n_1 and the rows correspond to n_2.

	−4	−3	−2	−1	0	1	2	3	4
4	13395375	1295405	61440	2645	15	125	48735	6452480	560164815
3	1613760	722000	37500	3920	540	1280	37500	524880	269340
2	141135	182405	3375	1280	15	1805	59535	2415125	79626240
1	48735	3920	15	125	15	405	19440	1245005	56745375
0	10786560	188180	960	20	∞	20	960	188180	10786560
−1	56745375	1245005	19440	405	15	125	15	3920	48735
−2	79626240	2415125	59535	1805	15	1280	3375	182405	141135
−3	269340	524880	37500	1280	540	3920	37500	722000	1613760
−4	560164815	6452480	48735	125	15	2645	61440	1295405	13395375

whose roots $\alpha_1^{(n)}, \ldots, \alpha_d^{(n)} \in A(1/n)$ do not lie on the unit circle. In particular, since $P_n(x) = \prod_{i=1}^d (x - \alpha_i^{(n)})$, we see that

$$\left|a_k^{(n)}\right| = \left|\sum_{i_1 < \cdots < i_{d-k}} \alpha_{i_1}^{(n)} \cdots \alpha_{i_{d-k}}^{(n)}\right| \leq K := \left(1 + \frac{1}{n}\right)^{d!}. \tag{4.18}$$

Since for each k, we have $a_k^{(n)} \in \mathbb{Z} \cap [-K, K]$, for all $n \geq 1$, we can use the pigeonhole principle to choose an infinite subsequence with $P(x) := P_{n_1}(x) = P_{n_2}(x) = P_{n_3}(x) = \cdots$ for which the coefficients all agree. But this contradicts the zeros of each polynomial not lying on the unit circle. □

Remark 4.7. In fact, Schinzel and Zassenhaus showed that if α is not a root of unity, then $|\alpha| \geq 1 + 1/4^{2+d/2}$. (cf. [6]).

Remark 4.8. The formula (4.1) and the description of the growth of periodic points for a single hyperbolic matrix were a core ingredient in Manning's famous work on the classification of Anosov toral automorphisms [7].

Example 4.9 (cf. [2]). We can consider the action on $\mathbb{T}^6$ defined by the matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}. \tag{4.19}$$

The number of fixed points $N(n_1, n_2)$ for $|n_1|, |n_2| \leq 4$ is presented in Table 3.

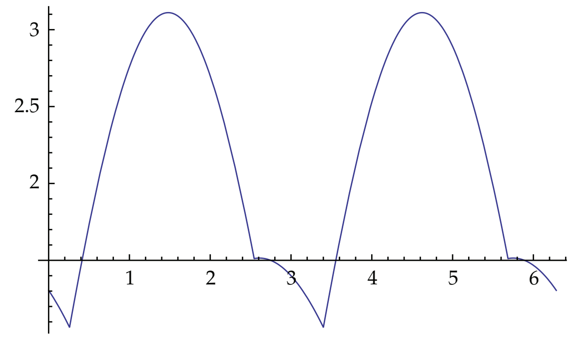

Figure 4: A plot of $\{\max_{i=1,2,3}\{\cos\theta \log|\alpha_i| + \sin\theta \log|\beta_i|\}$ as a function of $0 \le \theta < 2\pi$.

The matrix A_1 has eigenvalues

$$\begin{gathered}\alpha_1 = 3.68631\ldots, \qquad \alpha_2 = -1.32361\ldots, \qquad \alpha_3 = 0.0607659\ldots + 0.998152i\ldots, \\ \alpha_4 = 0.0607659\ldots - 0.998152i\ldots, \qquad \alpha_5 = -0.75551\ldots, \qquad \alpha_6 = 0.271274\ldots,\end{gathered} \tag{4.20}$$

and the matrix A_2 has corresponding eigenvalues

$$\begin{gathered}\beta_1 = -0.463258\ldots, \qquad \beta_2 = -22.1542\ldots, \qquad \beta_3 = 0.910592\ldots - 0.413307i\ldots, \\ \beta_4 = 0.910592\ldots + 0.413307i\ldots, \qquad \beta_5 = -0.0451382\ldots, \qquad \beta_6 = -2.15862\ldots\end{gathered} \tag{4.21}$$

In this example, we see that $\underline{\lambda} = 1.06415\ldots$ (occurring at $\underline{\theta} = 0.258896\ldots$) and $\overline{\lambda} = 3.11069\ldots$ (occurring at $\overline{\theta} = 4.62214\ldots$) (cf. Figure 4).

5. A Sector Theorem and Directional Growth

A natural refinement is to estimate the number of fixed points for (n_1, n_2) lying in a sector of the form $\mathcal{S}(\theta_1, \theta_2) := \{(n_1, n_2) \in \mathbb{Z}^2 \ : \ n_2 \tan(\theta_1) \le n_1 \le n_2 \tan(\theta_2)\}$, for $0 \le \theta_1 < \theta_2 \le 2\pi$.

Definition 5.1. We can denote

$$\begin{aligned}\overline{\lambda}(\theta_1, \theta_2) &= \sup_{\theta_1 \le \theta \le \theta_2} \left\{ \max_{i=1,2,3} \{\cos\theta \log|\alpha_i| + \sin\theta \log|\beta_i|\} \right\}, \\ \underline{\lambda}(\theta_1, \theta_2) &= \inf_{\theta_1 \le \theta \le \theta_2} \left\{ \max_{i=1,2,3} \{\cos\theta \log|\alpha_i| + \sin\theta \log|\beta_i|\} \right\}.\end{aligned} \tag{5.1}$$

We then have the following natural refinement of Theorem 1.3.

Theorem 5.2 (Sector Theorem). *Let $A_1, A_2 \in SL(3,\mathbb{Z})$ be commuting independent hyperbolic matrices. Let $0 \le \theta_1 < \theta_2 \le 2\pi$. The growth rates of the fixed points in the sector $\mathcal{S}(\theta_1,\theta_2)$*

$$\begin{aligned}\overline{\lambda}(\theta_1,\theta_2) &:= \limsup_{\|(n_1,n_2)\|_2\to+\infty,(n_1,n_2)\in\mathcal{S}(\theta_1,\theta_2)} \frac{1}{\|(n_1,n_2)\|_2}\log N(n_1,n_2),\\ \underline{\lambda}(\theta_1,\theta_2) &:= \liminf_{\|(n_1,n_2)\|_2\to+\infty,(n_1,n_2)\in\mathcal{S}(\theta_1,\theta_2)} \frac{1}{\|(n_1,n_2)\|_2}\log N(n_1,n_2)\end{aligned} \tag{5.2}$$

satisfy $0 < \underline{\lambda}(\theta_1,\theta_2) < \overline{\lambda}(\theta_1,\theta_2) < +\infty$.

Proof. The proof follows easily by modifying the proof of Theorem 1.3. Recall that the number of fixed points of the single transformation $\mathcal{A}(n_1,n_2,\cdot) : \mathbb{T}^3 \to \mathbb{T}^3$, this time restricting to $(n_1,n_2)\in\mathcal{S}$, can be written as

$$\begin{aligned}N(n_1,n_2) &= \left|\det\left(I - A_1^{n_1}A_2^{n_2}\right)\right|\\ &= \left|1 - \left(\alpha_1^{n_1}\beta_1^{n_2} + \alpha_2^{n_1}\beta_2^{n_2} + \alpha_2^{n_1}\beta_2^{n_2}\right) + \left(\alpha_1^{-n_1}\beta_1^{-n_1} + \alpha_2^{-n_2}\beta_2^{-n_2} + \alpha_3^{-n_3}\beta_3^{-n_3}\right) - 1\right|.\end{aligned} \tag{5.3}$$

We can again consider the vectors v_1, v_2, v_3, but this time we only need to consider unit vectors v_θ with $\theta_1 \le \theta \le \theta_2$. We can again write that $\langle v_i, w_\theta\rangle = \cos\theta\log\alpha_i + \sin\theta\log\beta_i$ for $i = 1,2,3$. In particular, if we write $(n_1,n_2) = (R\cos\theta, R\sin\theta) \in \mathcal{S}(\theta_1,\theta_2)$, say, where $R = \|(n_1,n_2)\|_2$, then we have that

$$\left|\alpha_i^{n_1}\beta_i^{n_2}\right| = \exp\left(R\left(\cos\theta\log|\alpha_i| + \sin\theta\log\left|\beta_i\right|\right)\right). \tag{5.4}$$

We now want to estimate $N(n_1,n_2)$ in terms of the largest expression of the form (5.4) where $(n_1,n_2)\in\mathcal{S}(\theta_1,\theta_2)$. In particular, modifying the proof of Theorem 1.3, we observe that

$$\overline{\lambda}(\theta_1,\theta_2) = \sup_{\theta_1\le\theta\le\theta_2}\max_{i=1,2,3}\{\langle v_i, w_\theta\rangle\} \ge \inf_{\theta_1\le\theta\le\theta_2}\max_{i=1,2,3}\{\langle v_i, w_\theta\rangle\} = \underline{\lambda}(\theta_1,\theta_2), \tag{5.5}$$

as required. □

Definition 5.3. Let us denote

$$\lambda(\theta) := \max_{i=1,2,3}\left\{\cos\theta\log|\alpha_i| + \sin\theta\log\left|\beta_i\right|\right\}. \tag{5.6}$$

We then have the following corollary.

Corollary 5.4 (Directional growth). *Let $A_1, A_2 \in SL(3,\mathbb{Z})$ be commuting independent hyperbolic matrices. Let $0 \leq \theta < 2\pi$. The following limits exist and agree:*

$$\overline{\lambda}(\theta) := \lim_{\epsilon \to 0} \limsup_{\|(n_1,n_2)\|_2 \to +\infty, (n_1,n_2) \in \mathcal{S}(\theta-\epsilon, \theta_2+\epsilon)} \frac{1}{\|(n_1,n_2)\|_2} \log N(n_1,n_2),$$
$$\underline{\lambda}(\theta) := \lim_{\epsilon \to 0} \liminf_{\|(n_1,n_2)\|_2 \to +\infty, (n_1,n_2) \in \mathcal{S}(\theta-\epsilon, \theta+\epsilon)} \frac{1}{\|(n_1,n_2)\|_2} \log N(n_1,n_2), \tag{5.7}$$

and $\overline{\lambda}(\theta) = \underline{\lambda}(\theta) = \lambda(\theta)$.

Proof. This follows immediately from Theorem 5.2 and continuity of $\lambda(\theta)$. □

Remark 5.5. We have that for each fixed choice $(n_1, n_2) \in \mathcal{S}(\theta_1, \theta_2)$ that

$$h(\mathcal{A}(n_1, n_2, \cdot)) = \lim_{k \to +\infty} \frac{1}{k} \log \operatorname{Card}\{x \ : A(kn_1, kn_2)x = x\}. \tag{5.8}$$

We see that for any $\epsilon > 0$ we have that

$$\underline{\lambda}(\theta_1, \theta_2) - \epsilon \leq \frac{h(\mathcal{A}(n_1, n_2))}{\|(n_1,n_2)\|_2} \leq \overline{\lambda}(\theta_1, \theta_2) + \epsilon \tag{5.9}$$

providing $\|(n_1,n_2)\|_2$ is sufficiently large. In particular, by continuity we see that we have the limit

$$\lim_{R \to +\infty} \frac{h(\mathcal{A}([R\cos\theta, R\sin\theta]))}{R} = \lambda(\theta). \tag{5.10}$$

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