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Research Article

Some Results on Super Quasi-Einstein Manifolds

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This paper deals with the study of super quasi-Einstein manifolds admitting W_2-curvature tensor. The totally umbilical hypersurfaces of $S(QE)_n$ are also studied. Among others, the existence of such a manifold is ensured by a nontrivial example.

1. Introduction

It is well known that a Riemannian manifold (M^n, g) $(n > 2)$ is Einstein if its Ricci tensor S of type $(0, 2)$ is of the form $S = ag$, where a is a constant, which turns into $S = (r/n)g$, r being the scalar curvature (constant) of the manifold. Let (M^n, g) $(n > 2)$ be a Riemannian manifold. Let $U_S = \{x \in M : S \neq (r/n)g \text{ at } x\}$, then the manifold (M^n, g) is said to be quasi-Einstein manifold [1–12] if on $U_S \subset M$, we have

$$S - ag = bA \otimes A, \tag{1.1}$$

where A is a 1-form on U_S and a, b are some functions on U_S. It is clear that the 1-form A as well as the function b are nonzero at every point on U_S. From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g., Schwarzschild spacetime) is quasi-Einstein. The scalars a, b are known as the associated scalars of the manifold. Also, the 1-form A is called the associated 1-form of the manifold defined by $g(X, \rho) = A(X)$ for any vector field X, ρ being a unit vector field, called the generator of the manifold. Such an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$. The quasi-Einstein manifolds have also been studied by De and Ghosh [13], Shaikh et al. [14], and Shaikh and Patra [15].

As a generalization of quasi-Einstein manifold, Chaki [16] introduced the notion of generalized quasi-Einstein manifolds. A Riemannian manifold (M^n, g) $(n > 2)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the following:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)], \tag{1.2}$$

where a, b, and c are scalars of which $b \neq 0$, $c \neq 0$, A, B are nonzero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all X and ρ, μ are two unit vector fields mutually orthogonal to each other. In such a case, a, b, and c are called the associated scalars, A, B are called the associated 1-forms, and ρ, μ are the generators of the manifold. Such an n-dimensional manifold is denoted by $G(QE)_n$.

In [17], Chaki also introduced the notion of super quasi-Einstein manifold. A Riemannian manifold (M^n, g) $(n > 2)$ is called super quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the following:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \tag{1.3}$$

where a, b, c, and d are nonzero scalars, A, B are two nonzero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X, and ρ, μ are unit vectors such that ρ is perpendicular to μ and D is a symmetric $(0,2)$ tensor with zero trace, which satisfies the condition $D(X,\rho) = 0$ for all vector fields X. Here, a, b, c, and d are called the associated scalars, A, B are the associated 1-forms of the manifold, and D is called the structure tensor. Such an n-dimensional manifold is denoted by $S(QE)_n$. The super quasi-Einstein manifolds have also been studied by Debnath and Konar [18], Özgür [19], and many others.

In 1970, Pokhariyal and Mishra [20] introduced new tensor fields, called W_2 and E tensor fields, in a Riemannian manifold and studied their properties. According to them, a W_2-curvature tensor on a manifold (M^n, g) $(n > 2)$ is defined by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[g(X,Z)QY - g(Y,Z)QX\right], \tag{1.4}$$

where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ for all X, Y. In this connection, it may be mentioned that Pokhariyal and Mishra [20, 21] and Pokhariyal [22] introduced some new curvature tensors defined on the line of Weyl projective curvature tensor.

The W_2-curvature tensor was introduced on the line of Weyl projective curvature tensor, and by breaking W_2 into skew-symmetric parts, the tensor E has been defined. Rainich conditions for the existence of the nonnull electrovariance can be obtained by W_2 and E if we replace the matter tensor by the contracted part of these tensors. The tensor E enables to extend Pirani formulation of gravitational waves to Einstein space [23, 24]. It is shown that [20] except the vanishing of complexion vector and property of being identical in two spaces which are in geodesic correspondence, the W_2-curvature tensor possesses the properties almost similar to the Weyl projective curvature tensor. Thus, we can very well use W_2-curvature tensor in various physical and geometrical spheres in place of the Weyl projective curvature tensor.

The W_2-curvature tensor has also been studied by various authors in different structures such as De and Sarkar [25], Matsumoto et al. [26], Pokhariyal [23, 24, 27], Shaikh et al. [28], Shaikh et al. [29], Taleshian and Hosseinzadeh [30], Tripathi and Gupta [31], Venkatesha et al. [32], and Yíldíz and De [33].

Motivated by the above studies, in Section 3, we study W_2-curvature tensor of a super quasi-Einstein manifold. It is proved that if in an $S(QE)_n$ $(n > 2)$ the associated scalars are constants, the structure tensor is of Codazzi type and the generators ρ and μ are vector fields with the associated 1-forms A and B not being the 1-forms of recurrences, then the manifold is W_2-conservative.

Recently, Özen and Altay [34] studied the totally umbilical hypersurfaces of weakly and pseudosymmetric spaces. Again, Özen and Altay [35] also studied the totally umbilical hypersurfaces of weakly concircular and pseudoconcircular symmetric spaces. In this connection, it may be mentioned that Shaikh et al. [36] studied the totally umbilical hypersurfaces of weakly conharmonically symmetric spaces. Section 4 deals with the study of totally umbilical hypersurfaces of $S(QE)_n$. It is proved that the totally umbilical hypersurfaces of $S(QE)_{n+1}$ are $S(QE)_n$ if and only if the hypersurface is a totally geodesic hypersurface.

Finally, in the last section, the existence of super quasi-Einstein manifold is ensured by a nontrivial example.

2. Preliminaries

In this section, we will obtain some formulas of $S(QE)_n$, which will be required in the sequel. Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal frame field at any point of the manifold, then setting $X = Y = e_i$ in (1.3) and taking summation over i, $1 \le i \le n$, we obtain

$$r = na + b, \tag{2.1}$$

where r is the scalar curvature of the manifold.

Also from (1.3), we have

$$S(X, \rho) = (a + b)A(X) + cB(X), \tag{2.2}$$

$$S(\rho, \rho) = (a + b), \tag{2.3}$$

$$S(X, \mu) = aB(X) + cA(X) + dD(X, \mu), \tag{2.4}$$

$$S(\mu, \mu) = a + dD(\mu, \mu), \tag{2.5}$$

$$S(\rho, \mu) = c. \tag{2.6}$$

3. W_2-Curvature Tensor of $S(QE)_n$

Let a manifold M be an $S(QE)_n$ $(n > 2)$, which is W_2-flat, then from (1.4), we get

$$R(Y, Z, U, V) = \frac{1}{n-1}\left[g(Z, U)S(Y, V) - g(Y, U)S(Z, V)\right]. \tag{3.1}$$

Setting $U = \rho$ and $V = \mu$ in (3.1) and using (2.2) and (2.4), we obtain

$$R(Y, Z, \rho, \mu) = \frac{a}{n-1}[A(Z)B(Y) - A(Y)B(Z)] + \frac{d}{n-1}\left[A(Z)D(Y, \mu) - A(Y)D(Z, \mu)\right]. \tag{3.2}$$

Again, plugging $U = \mu$ and $V = \rho$ in (3.1) and using (2.2) and (2.4), we get

$$R(Y, Z, \mu, \rho) = \frac{a+b}{n-1}[A(Y)B(Z) - A(Z)B(Y)]. \tag{3.3}$$

From (3.2) and (3.3), we have

$$A(Z)[bB(Y) - dD(Y, \mu)] = A(Y)[bB(Z) - dD(Z, \mu)], \tag{3.4}$$

that is,

$$A(Z)E(Y) = A(Y)E(Z), \tag{3.5}$$

where $E(Y) = g(Y, \sigma) = bB(Y) - dD(Y, \mu)$ for all Y. From (3.5), we may conclude that the two vector fields ρ and σ are codirectional, provided $E \neq 0$.

If $E(Y) = 0$, then we have

$$D(Y, \mu) = \frac{b}{d}B(Y) = \frac{b}{d}g(Y, \mu) \quad \text{since } \delta \neq 0, \tag{3.6}$$

which implies that b/d is an eigenvalue of the tensor D corresponding to the eigenvector σ. Thus, we have the following result.

Theorem 3.1. *Let a manifold M be a W_2-flat $S(QE)_n$ $(n > 2)$ such that b/d is not an eigenvalue of the tensor D corresponding to the eigenvector σ defined by $E(Y) = g(Y, \sigma) = bB(Y) - dD(Y, \mu)$, then the vector fields ρ and σ corresponding to the 1-forms A and E, respectively, are codirectional.*

From (1.4), we get that

$$(\operatorname{div} W_2)(Y, Z)U = (\operatorname{div} R)(Y, Z)U + \frac{1}{2(n-1)}[dr(Z)g(Y, U) - dr(Y)g(Z, U)], \tag{3.7}$$

where "div" denotes the divergence.

Again, it is known that in a Riemannian manifold, we have

$$(\operatorname{div} R)(Y, Z)U = (\nabla_Y S)(Z, U) - (\nabla_Z S)(Y, U). \tag{3.8}$$

Consequently, by virtue of the above relation, (3.7) takes the form

$$(\operatorname{div} W_2)(Y, Z)U = (\nabla_Y S)(Z, U) - (\nabla_Z S)(Y, U) + \frac{1}{2(n-1)}\{dr(Z)g(Y, U) - dr(Y)g(Z, U)\}. \tag{3.9}$$

We now consider the associated scalars a, b, c, and d as constants, then (2.1) yields that the scalar curvature r is constant, and hence $dr(\mathrm{X}) = 0$ for all X. Consequently, (3.9) yields

$$(\operatorname{div} W_2)(Y, Z)U = (\nabla_Y S)(Z, U) - (\nabla_Z S)(Y, U). \tag{3.10}$$

Since a, b, c, and d are constants, we have from (1.3) that

$$\begin{aligned}(\nabla_Y S)(Z,U) = b&[(\nabla_Y A)(Z)A(U) + A(Z)(\nabla_Y A)(U)]\\ &+ c[(\nabla_Y A)(Z)B(U) + A(Z)(\nabla_Y B)(U)\\ &\quad +(\nabla_Y A)(U)B(Z) + A(U)(\nabla_Y B)(Z)] + d(\nabla_Y D)(Z,U).\end{aligned} \tag{3.11}$$

We now assume that the structure tensor D of such as $S(QE)_n$ is of Codazzi type [37], then for all vector fields Y, Z, and U, we have

$$(\nabla_Y D)(Z,U) = (\nabla_Z D)(Y,U). \tag{3.12}$$

By virtue of (3.11) and (3.12), (3.10) yields

$$\begin{aligned}(\operatorname{div} W_2)(Y,Z)U = b&[(\nabla_Y A)(Z)A(U) + A(Z)(\nabla_Y A)(U)\\ &-(\nabla_Z A)(Y)A(U) - A(Y)(\nabla_Z A)(U)]\\ &+ c[(\nabla_Y A)(Z)B(U) + A(Z)(\nabla_Y B)(U)\\ &\quad + (\nabla_Y A)(U)B(Z) + A(U)(\nabla_Y B)(Z)\\ &\quad - (\nabla_Z A)(Y)B(U) - A(Y)(\nabla_Z B)(U)\\ &\quad -(\nabla_Z A)(U)B(Y) - A(U)(\nabla_Z B)(Y)].\end{aligned} \tag{3.13}$$

Now, if the generators ρ and μ of the manifold are recurrent vector fields [38], then we have $\nabla_Y\rho = \pi_1(Y)\rho$ and $\nabla_Y\mu = \pi_2(Y)\mu$, where π_1 and π_2 are called the 1-forms of recurrence such that π_1 and π_2 are different from A and B. Consequently, we get

$$\begin{aligned}(\nabla_Y A)(Z) &= g(\nabla_Y\rho, Z) = g(\pi_1(Y)\rho, Z) = \pi_1(Y)A(Z),\\ (\nabla_Y B)(Z) &= g(\nabla_Y\mu, Z) = g(\pi_2(Y)\rho, Z) = \pi_2(Y)B(Z).\end{aligned} \tag{3.14}$$

In view of (3.14), (3.13) turns into

$$\begin{aligned}(\operatorname{div} W_2)(Y,Z)U = 2b&\pi_1(Y)A(Z)A(U)\\ &+ c[\{\pi_1(Y) + \pi_2(Y)\}\{A(Z)B(U) + A(U)B(Z)\}\\ &\quad -\{\pi_1(Z) + \pi_2(Z)\}\{A(Y)B(U) + A(U)B(Y)\}].\end{aligned} \tag{3.15}$$

Since $g(\rho,\rho) = g(\mu,\mu) = 1$, it follows that $(\nabla_Y A)(\rho) = g(\nabla_Y\rho,\rho) = 0$, and hence (3.14) reduces to $\pi_1(Y) = 0$ for all Y. Similarly, we have $\pi_2(Y) = 0$. Hence, from (3.15), we have $(\operatorname{div} W_2)(Y,Z)U = 0$, that is, the manifold under consideration is W_2-conservative [39]. Hence, we can state the following.

Theorem 3.2. *Suppose that a manifold M is an $S(QE)_n$ ($n > 2$) such that associated scalars are constants and the structure tensor is of Codazzi type. If the generators ρ and μ corresponding to the associated 1-forms A and B are not being the 1-forms of recurrences, then the manifold is W_2-conservative.*

4. Totally Umbilical Hypersurfaces of $S(QE)_n$

Let $(\overline{V}, \overline{g})$ be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let (V, g) be a hypersurface of $(\overline{V}, \overline{g})$ defined in a locally coordinate system by means of a system of parametric equation $y^\alpha = y^\alpha(x^i)$, where Greek indices take values $1, 2, \ldots, n+1$ and Latin indices take values $1, 2, \ldots, n$. Let N^α be the components of a local unit normal to (V, g), then we have

$$\begin{gathered} g_{ij} = \overline{g}_{\alpha\beta} y_i^\alpha y_j^\beta, \\ \overline{g}_{\alpha\beta} N^\alpha y_j^\beta = 0, \qquad \overline{g}_{\alpha\beta} N^\alpha N^\beta = e = 1, \\ y_i^\alpha y_j^\beta g^{ij} = \overline{g}^{\alpha\beta} - N^\alpha N^\beta, \quad y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}. \end{gathered} \tag{4.1}$$

The hypersurface (V, g) is called a totally umbilical hypersurface [40, 41] of $(\overline{V}, \overline{g})$ if its second fundamental form Ω_{ij} satisfies

$$\Omega_{ij} = H g_{ij}, \qquad y_{i,j}^\alpha = g_{ij} H N^\alpha, \tag{4.2}$$

where the scalar function H is called the mean curvature of (V, g) given by $H = (1/n) \sum g^{ij} \Omega_{ij}$. If, in particular, $H = 0$, that is,

$$\Omega_{ij} = 0, \tag{4.3}$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\overline{V}, \overline{g})$.

The equation of Weingarten for (V, g) can be written as $N_{,j}^\alpha = -(H/n) y_j^\alpha$. The structure equations of Gauss and Codazzi [40, 41] for (V, g) and $(\overline{V}, \overline{g})$ are, respectively, given by

$$\begin{gathered} R_{ijkl} = \overline{R}_{\alpha\beta\gamma\delta} F_{ijkl}^{\alpha\beta\gamma\delta} + H^2 G_{ijkl}, \\ \overline{R}_{\alpha\beta\gamma\delta} F_{ijk}^{\alpha\beta\gamma} N^\delta = H_{,i} g_{jk} - H_{,j} g_{ik}, \end{gathered} \tag{4.4}$$

where R_{ijkl} and $\overline{R}_{\alpha\beta\gamma\delta}$ are curvature tensors of (V, g) and $(\overline{V}, \overline{g})$, respectively, and

$$F_{ijkl}^{\alpha\beta\gamma\delta} = F_i^\alpha F_j^\beta F_k^\gamma F_l^\delta, \quad F_i^\alpha = y_i^\alpha, \qquad G_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl}. \tag{4.5}$$

Also we have [40, 41]

$$\overline{S}_{\alpha\delta} F_i^\alpha F_j^\delta = S_{ij} - (n-1) H^2 g_{ij}, \tag{4.6}$$

$$\overline{S}_{\alpha\delta} N^\alpha F_i^\delta = (n-1) H_{,i}, \tag{4.7}$$

$$\overline{r} = r - n(n-1) H^2, \tag{4.8}$$

where S_{ij} and $\overline{S}_{\alpha\delta}$ are the Ricci tensors of (V, g) and $(\overline{V}, \overline{g})$, respectively, and r and $\overline{r}$ are the scalar curvatures of (V, g) and $(\overline{V}, \overline{g})$, respectively.

In terms of local coordinates, the relation (1.3) can be written as

$$S_{ij} = ag_{ij} + bA_iA_j + c\left[A_iB_j + A_jB_i\right] + dD_{ij}. \tag{4.9}$$

Let $(\overline{V}, \overline{g})$ be an $S(QE)_{n+1}$, then we get

$$S_{\alpha\beta} = ag_{\alpha\beta} + bA_\alpha A_\beta + c\left[A_\alpha B_\beta + A_\beta B_\alpha\right] + dD_{\alpha\beta}. \tag{4.10}$$

Multiplying both sides of (4.10) by $F_{ij}^{\alpha\beta}$ and then using (4.6) and (4.9), we obtain $H = 0$, which implies that the hypersurface is a totally geodesic hypersurface.

Conversely, we now consider that the hypersurface (V, g) is totally geodesic hypersurface, that is,

$$H = 0. \tag{4.11}$$

In view of (4.11), (4.6) yields

$$\overline{S}_{\alpha\delta}F_i^\alpha F_j^\delta = S_{ij}. \tag{4.12}$$

Using (4.12) in (4.10), we have the relation (4.9). Thus, we can state the following.

Theorem 4.1. *The totally umbilical hypersurface of an $S(QE)_{n+1}$ is an $S(QE)_n$ if and only if the hypersurface is a totally geodesic hypersurface.*

Note that the theorem is a statement on the hypersurface based on the restrictions of the associated scalars and 1-forms coming from the manifold.

5. Example of a Super Quasi-Einstein Manifold

This section deals with a nontrivial example of $S(QE)_4$.

Example 5.1. We define a Riemannian metric g on $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij}dx^i dx^j = \left(dx^1\right)^2 + e^{x^1}\left[e^{x^2}\left(dx^2\right)^2 + e^{x^3}\left(dx^3\right)^2 + e^{x^4}\left(dx^4\right)^2\right], \quad (i, j = 1, 2, 3, 4). \tag{5.1}$$

Then, the only nonvanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, and the scalar curvature are given by

$$\begin{gathered}
\Gamma^1_{22} = -\frac{1}{2}e^{x^1+x^2}, \qquad \Gamma^1_{33} = -\frac{1}{2}e^{x^1+x^3}, \qquad \Gamma^1_{44} = -\frac{1}{2}e^{x^1+x^4}, \\
\Gamma^2_{22} = \frac{1}{2} = \Gamma^3_{33} = \Gamma^4_{44} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14}, \\
R_{1221} = \frac{1}{4}e^{x^1+x^2}, \qquad R_{1331} = \frac{1}{4}e^{x^1+x^3}, \qquad R_{1441} = \frac{1}{2}e^{x^1+x^4}, \\
R_{2332} = \frac{1}{4}e^{2x^1+x^2+x^3}, \qquad R_{2442} = \frac{1}{4}e^{2x^1+x^2+x^4}, \qquad R_{3443} = \frac{1}{4}e^{2x^1+x^3+x^4}, \\
S_{11} = \frac{3}{4}, \qquad S_{22} = \frac{3}{4}e^{x^1+x^2}, \qquad S_{33} = \frac{3}{4}e^{x^1+x^3}, \qquad S_{44} = \frac{3}{4}e^{x^1+x^4}, \qquad r = 3
\end{gathered} \tag{5.2}$$

and the components which can be obtained from these by the symmetry properties. Therefore, $\mathbb{R}^4$ is a Riemannian manifold (M^4, g) of nonvanishing scalar curvature. We will now show that M^4 is an $S(QE)_4$, that is, it satisfies (1.3). Let us now consider the associated scalars as follows:

$$a = \frac{3}{4}, \qquad b = 2e^{x^1}, \qquad c = e^{x^1+x^3}, \qquad d = \frac{1}{2e^{x^4}}. \tag{5.3}$$

In terms of local coordinate system, let us consider the 1-forms A and B and the structure tensor D as follows:

$$\begin{gathered}
A_i(x) = \begin{cases} \dfrac{1}{2} & \text{for } i = 1, \\ \dfrac{1}{\sqrt{e^{x^1}}} & \text{for } i = 2, \\ 0 & \text{otherwise}, \end{cases} \\
B_i(x) = \begin{cases} -\dfrac{1}{e^{x^3}} & \text{for } i = 1, \\ \dfrac{1}{e^{x^1}} & \text{for } i = 3, \\ 0 & \text{otherwise}, \end{cases} \\
D_{ij}(x) = \begin{cases} e^{x^1+x^4} & \text{for } i = j = 1, \\ -4e^{x^4} & \text{for } i = j = 2, \\ -e^{x^3+x^4} & \text{for } i = 1,\ j = 3, \\ -\dfrac{2e^{x^3+x^4}}{\sqrt{e^{x^1}}} & \text{for } i = 2,\ j = 3, \\ 0 & \text{otherwise}. \end{cases}
\end{gathered} \tag{5.4}$$

In terms of local coordinate system, the defining condition (1.3) of an $S(QE)_n$ can be written as

$$S_{ij} = ag_{ij} + bA_iA_j + c\left[A_iB_j + A_jB_i\right] + dD_{ij}, \quad i, j = 1, 2, 3, 4. \tag{5.5}$$

By virtue of (5.3) and (5.4), it can be easily shown that the relation (5.5) holds for $i, j = 1, 2, 3, 4$. Therefore, (M^4, g) is an $S(QE)_4$, which is neither quasi-Einstein nor generalized quasi-Einstein. Hence, we can state the following.

Theorem 5.2. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in* (5.1), *then (M^4, g) is an $S(QE)_4$ with nonvanishing scalar curvature which is neither quasi-Einstein nor generalized quasi-Einstein.*

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