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Research Article

On Generalized Sasakian-Space-Forms

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The purpose of the present paper is to characterize pseudoprojectively flat and pseudoprojective semisymmetric generalized Sasakian-space-forms.

1. Introduction

Alegre et al. [1] introduced and studied the generalized Sasakian-space-forms. The authors Alegre and Carriazo [2], Somashekhara and Nagaraja [3, 4], and De and Sarkar [5, 6] studied the generalized Sasakian-space-forms. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a generalized Sasakian-space-form if there exist differentiable functions f_1, f_2, f_3 such that curvature tensor R of M is given by

$$R(X, Y)Z = f_1 R_1(X, Y)Z + f_2 R_2(X, Y)Z + f_3 R_3(X, Y)Z, \tag{1.1}$$

for any vector fields X, Y, Z on M, where

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z,)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi. \end{aligned} \tag{1.2}$$

In this paper, we study the curvature properties like flatness, symmetry, and semisymmetry properties in a generalized Sasakian-space-form by considering a pseudoprojective curvature tensor.

The paper is organized as follows. Section 2 of this paper contains some preliminary results on the generalized Sasakian-space-forms. In Section 3, we study the pseudoprojectively flat generalized Sasakian-space-form and obtain necessary and sufficient conditions for a generalized Sasakian-space-form to be pseudoprojectively flat. In the next section, we deal with pseudoprojectively semisymmetric generalized Sasakian-space-forms, and it is proved that a generalized Sasakian-space-form is pseudoprojectively semisymmetric if and only if the space form is pseudoprojectively flat and $f_1 = f_3$. The last section is devoted to the study of τ-flat and $\tau-\phi$-semi symmetric generalized Sasakian-space-forms. In this section, we prove that the associated functions f_1, f_2, f_3 are linearly dependent.

In a $(2n + 1)$-dimensional almost contact metric manifold, the pseudoprojective curvature tensor $\widetilde{P}$ [7] is defined by

$$\begin{aligned}\widetilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad - \frac{r}{2n+1}\left(\frac{a}{2n} + b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.3}$$

where a and b are constants and R, S, and r are the Riemannian curvature tensor of type $(0,2)$, the Ricci tensor, and the scalar curvature of the manifold, respectively. If $a = 1$, $b = -(1/2n)$, then (1.3) takes the form

$$\widetilde{P}(X,Y)Z = P(X,Y)Z, \tag{1.4}$$

where P is the projective curvature tensor. A manifold (M,ϕ,ξ,η,g) shall be called pseudoprojectively flat if the pseudoprojective curvature tensor $\widetilde{P} = 0$. It is known that the pseudoprojectively flat manifold is either projectively flat (if $a \neq 0$) or Einstein (if $a = 0$ and $b \neq 0$).

2. Preliminaries

A $(2n+1)$-dimensional C^∞-differentiable manifold M is said to admit an almost contact metric structure (ϕ,ξ,η,g) if it satisfies the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X,Y) = -g(X,\phi Y), \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.5}$$

where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form, and g is a Riemannian metric on M. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.6}$$

for all vector fields X and Y.

In a generalized Sasakian-space-form, the following hold:

$$\begin{aligned} R(X,Y)Z = {} & f_1[g(Y,Z)X - g(X,Z)Y] + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ & + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi], \end{aligned} \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.8}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.9}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.10}$$

3. Pseudoprojectively Flat Generalized Sasakian-Space-Forms

If the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ under consideration is pseudoprojectively flat, then from (1.3) we have

$$\begin{aligned} {}'R(X,Y,Z,W) = {} & \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)] \\ & + \frac{r}{(2n+1)a}\left(\frac{a}{2n} + b\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{3.1}$$

where a and b are constants and ${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Now taking $Z = \xi$ in (3.1) and using (2.1), (2.2), (2.7), and (2.9), we get

$$\begin{aligned} (f_1 - f_3)[\eta(Y)g(X,W) - \eta(X)g(Y,W)] = {} & \frac{2nb}{a}(f_1 - f_3)(\eta(Y)g(X,W) - \eta(X)g(Y,W)) \\ & + \frac{r}{(2n+1)a}\left(\frac{a}{2n} + b\right) \\ & \times (\eta(Y)g(X,W) - \eta(X)g(Y,W)). \end{aligned} \tag{3.2}$$

Again putting $X = \xi$ in (3.2), we get

$$\left[\left(\frac{a+2nb}{a}\right)\left(\frac{2n(2n+1)(f_1 - f_3) - r}{2n(2n+1)}\right)\right][\eta(Y)\eta(W) - g(Y,W)] = 0. \tag{3.3}$$

The aforementioned equation implies

$$\left(\frac{a+2nb}{a}\right)\left[\frac{2n(2n+1)(f_1-f_3)-r}{2n(2n+1)}\right]=0. \tag{3.4}$$

That is, either

$$(a+2nb)=0 \tag{3.5}$$

or

$$r=2n(2n+1)(f_1-f_3). \tag{3.6}$$

If $a+2nb=0$, $a\neq 0$ and $b\neq 0$, then, from (1.3), it follows that $\widetilde{P}(X,Y)Z=aP(X,Y)Z$. Thus in this case pseudoprojective flatness and projective flatness are equivalent.

If $a+2nb\neq 0$, $a\neq 0$ and $b\neq 0$, then comparing (2.10) and (3.6), we get

$$3f_2+(2n-1)f_3=0. \tag{3.7}$$

Using (3.7) in (2.9), we get

$$S(X,Y)=2n(f_1-f_3)g(X,Y). \tag{3.8}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Taking $X=Y=e_i$ and summing over $1\leq i\leq 2n+1$, we obtain

$$r=2n(2n+1)(f_1-f_3). \tag{3.9}$$

This shows that $M(f_1,f_2,f_3)$ is Einstein with a scalar curvature $r=2n(2n+1)(f_1-f_3)$. Thus we state the following.

Theorem 3.1. *A pseudoprojectively flat generalized Sasakian-space-form is either projectively flat or an Einstein manifold with a scalar curvature $r=[2n(2n+1)(f_1-f_3)]$.*

Suppose that (3.7) holds. Then in view of (2.7) and (2.9), we can write (1.3) as

$$\begin{aligned}
{}'\widetilde{P}(X,Y,Z,W) = {} & af_1(g(Y,Z)g(X,W)-g(X,Z)g(Y,W)) \\
& + af_2[g(X,\phi Z)g(\phi Y,W)-g(Y,\phi Z)g(\phi X,W)+2g(X,\phi Y)g(\phi Z,W)] \\
& + af_3[\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)+\eta(Y)g(X,Z)g(\xi,W) \\
& \quad -\eta(X)g(Y,Z)g(\xi,W)]+b[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)] \\
& -\left(\frac{r}{2n+1}\right)\left(\frac{a}{2n}+b\right)[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)],
\end{aligned} \tag{3.10}$$

where

$$'\widetilde{P}(X,Y,Z,W) = g\left(\widetilde{P}(X,Y)Z,W\right). \tag{3.11}$$

Replacing X by ϕX and Y by ϕY, we get

$$\begin{aligned}\widetilde{P}(\phi X,\phi Y,Z,W) = {}& af_3(g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W)) \\ & + af_2\left(g(\phi X,\phi Z)g\left(\phi^2 Y,W\right) - g(\phi Y,\phi Z)g\left(\phi^2 X,W\right)\right. \\ & \left. +2g\left(\phi X,\phi^2 Y\right)g(\phi Z,W)\right).\end{aligned} \tag{3.12}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold.

Taking $Y = W = e_i$ and summation over i, $1 \le i \le 2n+1$, we get

$$\sum_{i=1}^{2n+1}\widetilde{P}(\phi X,\phi e_i,Z,e_i) = af_3(g(\phi X,\phi Z)) + af_2\left(-g(\phi X,\phi Z)g(\phi e_i,\phi e_i) - g\left(\phi^2 X,\phi^2 Z\right)\right). \tag{3.13}$$

Again putting $X = Z = e_i$ and taking summation over i, $1 \le i \le 2n+1$, we get $f_2 = 0$ with $a \neq 0$. In view of (3.7), we get $f_3 = 0$.

Now (2.7) reduces to the form

$$R(X,Y)Z = f_1\left[g(Y,Z)X - g(X,Z)Y\right], \tag{3.14}$$

from which we have $S(X,Y) = 2nf_1 g(X,Y)$, and consequently

$$r = 2n(2n+1)f_1. \tag{3.15}$$

By using (3.14) and (3.15) in (1.3), we get $\widetilde{P}(X,Y)Z = 0$. This leads to the following.

Theorem 3.2. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is pseudoprojectively flat if and only if $a+2nb \neq 0$, $a \neq 0$, $b \neq 0$ and $3f_2 + (2n-1)f_3 = 0$.*

Alegre and Carriazo [2] proved that any contact metric generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with a dimension greater than or equal to five is a Sasakian manifold and f_1, f_2, and f_3 must be constants.

Thus from (3.14), we have the following theorem.

Theorem 3.3. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with a dimension greater than or equal to 5 is of constant curvature f_1 if and only if $a+2nb \neq 0$, $a \neq 0$, $b \neq 0$, and $3f_2 + (2n-1)f_3 = 0$.*

4. Pseudoprojective Semisymmetric Generalized Sasakian-Space-Form

Definition 4.1. If a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ satisfies

$$R(X,Y)\cdot \widetilde{P} = 0, \tag{4.1}$$

then the manifold is said to be pseudoprojectively semisymmetric manifold.

By using (1.3), (2.1), (2.2), (2.7), and (2.9), we have

$$\begin{aligned}\eta\left(\widetilde{P}(X,Y)Z\right) = {}& \left[a(f_1 - f_3) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ & + b\left[S(Y,Z)\eta(X) - S(Y,Z)\eta(Y)\right].\end{aligned} \tag{4.2}$$

Taking $Z = \xi$ in (4.2), we get

$$\eta\left(\widetilde{P}(X,Y)\xi\right) = 0. \tag{4.3}$$

Again putting $X = \xi$ in (4.2), we get

$$\begin{aligned}\left(\eta\left(\widetilde{P}(\xi,Y)Z\right)\right) = {}& \left[a(f_1 - f_3) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right][g(Y,Z) - \eta(Y)\eta(Z)] \\ & + b\left[S(Y,Z) - 2n(f_1 - f_3)\eta(Y)\eta(Z)\right].\end{aligned} \tag{4.4}$$

From (4.1), we have

$$\begin{aligned}&\left(R(X,Y)\widetilde{P}(U,V)W\right) - \widetilde{P}(R(X,Y)U,V)W \\ &\quad - \widetilde{P}(U,R(X,Y)V)W - \widetilde{P}(U,V)R(X,Y)W = 0.\end{aligned} \tag{4.5}$$

Taking $X = \xi$ and contracting the above with respect to ξ, we get

$$\begin{aligned}(f_1 - f_3)\Big\{&{}'\overline{P}(U,V,W,Y) - \eta(Y)\eta\left(\overline{P}(U,V)W\right) + \eta(U)\eta\left(\overline{P}(Y,V)W\right) \\ & - g(Y,U)\eta\left(\overline{P}(\xi,V)W\right) + \eta(V)\eta\left(\overline{P}(U,Y)W\right) - g(Y,V)\eta\left(\overline{P}(U,\xi)W\right) \\ & + \eta(W)\eta\left(\overline{P}(U,V)Y\right) - g(Y,W)\eta\left(\overline{P}(U,V)\xi\right)\Big\} = 0.\end{aligned} \tag{4.6}$$

Putting $U = Y$ in (4.6) and with the help of (4.2) and (4.3), we get either

$$f_1 = f_3 \tag{4.7}$$

or

$$\begin{aligned} {}'\overline{P}(Y,V,W,Y) &- g(Y,Y)\eta\left(\overline{P}(\xi,V)W\right) \\ &- g(Y,V)\eta\left(\overline{P}(Y,\xi)W\right) + \eta(W)\eta\left(\overline{P}(Y,V)Y\right) = 0. \end{aligned} \tag{4.8}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold of the manifold. Putting $Y = e_i$ and taking summation over i, $1 \leq i \leq 2n+1$, and using (4.2) and (4.4), we obtain

$$S(V,W) = Ag(V,W) + B\eta(V)\eta(W), \tag{4.9}$$

where

$$\begin{aligned} A &= 2nf_1 + 3f_2 - f_3, \\ B &= (2n+1)\left[-3f_2 - (2n-1)f_3\right]. \end{aligned} \tag{4.10}$$

Now contracting (4.9), we obtain

$$r = (2n+1)A + B. \tag{4.11}$$

Using (4.10) in (4.11), we get

$$r = 2n(2n+1)\left(f_1 - f_3\right). \tag{4.12}$$

In view of (2.10), (4.12) yields

$$3f_2 + (2n-1)f_3 = 0. \tag{4.13}$$

From (2.9) and (4.13), we have

$$S(V,W) = 2n\left(f_1 - f_3\right)g(V,W). \tag{4.14}$$

Now using (4.12) and (4.14) in (4.2), we get

$$\eta\left(\widetilde{P}(U,V)W\right) = 0. \tag{4.15}$$

Plugging (4.15) in (4.6), we obtain

$$\widetilde{P}(U,V,W,Y) = 0. \tag{4.16}$$

Therefore by taking into account (4.7) and (4.16), we have either $f_1 = f_3$ or $M(f_1, f_2, f_3)$ is pseudoprojectively flat.

Conversely, suppose that $f_1 = f_3$. Then, from (2.1), (2.2) and (2.7), we have $R(\xi, X)Y = 0$. Hence $R(\xi, U) \cdot \widetilde{P} = 0$. If the space-form is pseudoprojectively flat then clearly it is pseudo-projectively semisymmetric. Hence we can state the following.

Theorem 4.2. *A $2n+1$-dimensional generalized Sasakian-space-form is pseudoprojectively semisymmetric if and only if the space form is either pseudoprojectively flat or $f_1 = f_3$.*

By combining Theorems 3.2 and 4.2, we have the following.

Corollary 4.3. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is pseudoprojectively flat if and only if $f_1 = f_3$ or $a + 2nb \neq 0$ and $3f_2 + (2n-1)f_3 = 0$.*

5. τ-Curvature Tensor in a Generalized Sasakian-Space-Form

In a $(2n+1)$-dimensional Riemannian manifold M, the τ-curvature tensor is given by [8]

$$\begin{aligned}
\tau(X,Y)Z = {} & a_0 R(X,Y)Z + a_1 S(Y,Z)X + a_2 S(X,Z)Y + a_3 S(X,Y)Z \\
& + a_4 g(Y,Z)QX + a_5 g(X,Z)QY + a_6 g(X,Y)QZ \\
& + a_7 r\big(g(Y,Z)X - g(X,Z)Y\big),
\end{aligned} \tag{5.1}$$

where $a_0, \ldots, a_7$ are some smooth functions on M. For different values of $a_0, \ldots, a_7$, the τ-curvature tensor reduces to the curvature tensor, quasiconformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor, projective curvature tensor, M-projective curvature tensor, W_i-curvature tensors $(i = 0, \ldots 9)$, and W_j^*-curvature tensors $(j = 0, 1)$.

Suppose that $M(f_1, f_2, f_3)$ is τ-flat. Then from (5.1), we have

$$\begin{aligned}
-a_0 R(X,Y)Z = {} & a_1 S(Y,Z)X + a_2 S(X,Z)Y + a_3 S(X,Y)Z \\
& + a_4 g(Y,Z)QX + a_5 g(X,Z)QY + a_6 g(X,Y)QZ \\
& + a_7 r\big(g(Y,Z)X - g(X,Z)Y\big).
\end{aligned} \tag{5.2}$$

In view of (2.7), (2.8), and (2.9) in (5.2), we have

$$\begin{aligned}
& - a_0 \big\{ f_1 \big[g(Y,Z)X - g(X,Z)Y\big] + f_2 \big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\big] \\
& \quad + f_3 \big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\big] \big\} \\
& \qquad = a_1 \big[(2nf_1 + 3f_2 - f_3) g(Y,Z) - (3f_2 + (2n-1)f_3)\eta(Y)\eta(Z)\big] X \\
& \qquad\quad + a_2 \big[(2nf_1 + 3f_2 - f_3) g(X,Z) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Z)\big] Y \\
& \qquad\quad + a_3 \big[(2nf_1 + 3f_2 - f_3) g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y)\big] Z
\end{aligned}$$

$$
\begin{aligned}
&+ a_4 g(Y,Z)\left[(2nf_1+3f_2-f_3)X-(3f_2+(2n-1)f_3)\eta(X)\xi\right] \\
&+ a_5 g(X,Z)\left[(2nf_1+3f_2-f_3)Y-(3f_2+(2n-1)f_3)\eta(Y)\xi\right] \\
&+ a_6 g(X,Y)\left[(2nf_1+3f_2-f_3)Z-(3f_2+(2n-1)f_3)\eta(Z)\xi\right] \\
&+ a_7 r\left[g(Y,Z)X-g(X,Z)Y\right].
\end{aligned}
\tag{5.3}
$$

Putting $X = \phi Y$ in (5.3), we get

$$
\begin{aligned}
&- a_0\Big\{ f_1\left[g(Y,Z)\phi Y - g(\phi Y,Z)Y\right] + f_2\Big[g(\phi Y,\phi Z)\phi Y - g(Y,\phi Z,)\phi^2 Y + 2g(\phi Y,\phi Y)\phi Z\Big] \\
&\quad + f_3\left[-\eta(Y)\eta(Z)\phi Y + g(\phi Y,Z)\eta(Y)\xi\right]\Big\} \\
&\qquad = a_1\left[(2nf_1+3f_2-f_3)g(Y,Z)-(3f_2+(2n-1)f_3)\eta(Y)\eta(Z)\right]\phi Y \\
&\qquad\quad + a_2(2nf_1+3f_2-f_3)g(\phi Y,Z)Y + a_4(2nf_1+3f_2-f_3)g(Y,Z)\phi Y \\
&\qquad\quad + a_5 g(\phi Y,Z)\left[(2nf_1+3f_2-f_3)Y-(3f_2+(2n-1)f_3)\eta(Y)\xi\right] \\
&\qquad\quad + a_7 r\left[g(Y,Z)\phi Y - g(\phi Y,Z)Y\right].
\end{aligned}
\tag{5.4}
$$

If we choose a unit vector U orthogonal to ξ and taking $Y = U$, then making use of (2.1) and (2.3) in (5.4), we obtain

$$
\begin{aligned}
&\left[-a_0 f_1 + (a_2+a_5)(2nf_1+3f_2-f_3) - a_7 r + f_2\right] g(\phi U,Z)U \\
&\quad + \left[a_0(f_1+f_2) + (a_1+a_4)(2nf_1+3f_2-f_3) + a_7 r\right] g(U,Z)\phi U \\
&\quad + 2a_0 f_2 g(U,U)\phi Z = 0.
\end{aligned}
\tag{5.5}
$$

Putting $Z = U$ in (5.5), we have

$$
\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0, \tag{5.6}
$$

where

$$
\begin{aligned}
\lambda_1 &= a_0 + 2n(a_1+a_4) + 2n(2n+1)a_7, \\
\lambda_2 &= 3(a_0 + a_1 + a_4 + 2na_7), \\
\lambda_3 &= -(a_1 + a_4 + 4na_7).
\end{aligned}
\tag{5.7}
$$

Thus we have the following.

Theorem 5.1. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is τ-flat, then* (5.6) *holds.*

From the above theorem, we discuss the following cases.

Case (i). (1) If $M(f_1, f_2, f_3)$ is quasiconformally flat, then $a_1 = -a_2 = a_4 = -a_5$, $a_3 = a_6 = 0$, $a_7 = (-1/(2n+1))(a_0/2n + 2a_1)$. Putting these in (5.7), we obtain $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

(2) If $M(f_1, f_2, f_3)$ is conharmonically flat, then $a_0 = 1$, $a_1 = -a_2 = a_4 = -a_5 = -(1/(2n-1))$, $a_3 = a_6 = 0$, $a_7 = 0$. Putting these in (5.7), we get $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Similarly for W_0^*-flat, W_1-flat, W_3-flat, W_9-flat spaces, (5.7) gives $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Case (ii). If $M(f_1, f_2, f_3)$ is conformally flat, then $a_0 = 1$, $a_1 = -a_2 = a_4 = -a_5 = -(1/(2n-1))$, $a_3 = a_6 = 0$, $a_7 = 1/2n(2n-1)$.

Putting these in (5.7), we obtain $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$. Hence $f_2 = 0$.

Case (iii). (a) If $M(f_1, f_2, f_3)$ is pseudoprojectively flat, then $a_1 = -a_2, a_3 = a_4 = a_5 = a_6 = 0$, $a_7 = -(1/(2n+1))(a_0/2n + a_1)$.

By putting these values in (5.7), we have $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

(b) If $M(f_1, f_2, f_3)$ is projectively flat, then $a_0 = 1$, $a_1 = -a_2 = -(1/2n)$, $a_3 = a_4 = a_5 = a_6 = a_7 = 0$.

Making use of the above functional values in (5.7), we get $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Similarly for concircularly flat, M-projectively flat, W_0-flat, W_1^*-flat, W_2-flat, W_6-flat, and W_8-flat spaces, (5.7) gives $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$.

Case (iv). (a) If $M(f_1, f_2, f_3)$ is W_4-flat, then $a_0 = 1$, $a_5 = -a_6 = 1/2n$, $a_1 = a_2 = a_3 = a_4 = a_7 = 0$.

Putting these in (5.7), we obtain that $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

(b) If $M(f_1, f_2, f_3)$ is W_5-flat, then $a_0 = 1$, $a_2 = -a_5 = -(1/2n)$, $a_1 = a_3 = a_4 = a_6 = a_7 = 0$. Putting these in (5.7), we have $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

Similarly, for a W_7-flat space, (5.7) gives $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

Summarizing the above cases, we have the following corollaries.

Corollary 5.2. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either quasiconformally flat, conharmonically flat, W_0^*-flat, W_1-flat, W_3-flat, or W_9-flat, then f_1, f_2, and f_3 are linearly dependent.*

Corollary 5.3. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is conformally flat, then $f_2 = 0$.*

The above corollary was already proved by Kim [9] and Sarkar and De [10].

Corollary 5.4. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either pseudoprojectively flat, projectively flat, concircularly flat, M-projectively flat, W_0-flat, W_1^*-flat, W_2-flat, W_6-flat, or W_8-flat, then f_2 and f_3 are linearly dependent.*

Corollary 5.5. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either W_4-flat, W_5-flat, or W_7-flat, then f_1 and f_2 are linearly dependent.*

5.1. $\tau - \phi$-Semisymmetric Generalized Sasakian-Space-Form

Definition 5.6. $M(f_1, f_2, f_3)$ is $\tau - \phi$-semisymmetric if

$$\tau(X, Y) \cdot \phi = 0 \tag{5.8}$$

holds in $M(f_1, f_2, f_3)$.

We know that

$$(\tau(X,Y)\cdot\phi)Z = \tau(X,Y)\phi Z - \phi(\tau(X,Y)Z). \tag{5.9}$$

From (5.8) and (5.9), we have

$$\tau(X,Y)\phi Z - \phi(\tau(X,Y)Z) = 0. \tag{5.10}$$

By using (5.1) in (5.10), we have

$$\begin{aligned}
&a_0 R(X,Y)\phi Z + a_1 S(Y,\phi Z)X + a_2 S(X,\phi Z)Y + a_3 S(X,Y)\phi Z \\
&\quad + a_4 g(Y,\phi Z)QX + a_5 g(X,\phi Z)QY + a_6 g(X,Y)Q(\phi Z) \\
&\quad + a_7 r[g(Y,\phi Z)X - g(X,\phi Z)Y] - a_7 r[g(Y,Z)\phi X - g(X,Z)\phi Y] \\
&\quad - \{a_0\phi(R(X,Y)\phi Z) + a_1 S(Y,Z)\phi X + a_2 S(X,Z)\phi Y + a_3 S(X,Y)\phi Z \\
&\qquad + a_4 g(Y,Z)\phi(QX) + a_5 g(X,Z)\phi(QY) + a_6 g(X,Y)\phi(QZ)\} = 0.
\end{aligned} \tag{5.11}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Contracting (5.11) with respect to W and putting $Y = W = e_i$, also taking summation over i, $1 \le i \le 2n+1$, and making use of (2.1), (2.4), (2.7), (2.9), and (2.8), we have

$$[2a_1 + (2n+1)a_2]S(X,\phi Z) = Ag(X,\phi Z), \tag{5.12}$$

where

$$\begin{aligned}
A = &\left[-(2n-1)a_0 + 4na_4 + 2n(2n+1)a_5 - 2n\left(4n^2-1\right)a_7\right]f_1 \\
&+ [2(n-1)a_0 + 6a_4 + 6na_5 - 6n(2n-1)a_7]f_2 \\
&+ [-2a_4 - 4na_5 + 4n(2n-1)a_7]f_3.
\end{aligned} \tag{5.13}$$

Changing Z to ϕZ in (5.12) and also in view of (2.1) and (2.2), (2.9) yields

$$\begin{aligned}
S(X,Z) = &\left[\frac{A}{(2a_1 + (2n+1)a_2)}\right]g(X,Z) \\
&+ \left[\frac{2n(2a_1 + (2n+1)a_2)(f_1 - f_3) - A}{(2a_1 + (2n+1)a_2)}\right]\eta(X)\eta(Z).
\end{aligned} \tag{5.14}$$

Thus we can state the following.

Theorem 5.7. *A $\tau-\phi$-semisymmetric generalized Sasakian-space-form is η-Einstein provided* $(2a_1 + (2n+1)a_2) \neq 0$.

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