

Research Article

Some Classes of Pseudosymmetric Contact Metric 3-Manifolds

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We study (a) the class of 3-dimensional pseudosymmetric contact metric manifolds with harmonic curvature and Tl constant along the direction of ξ and (b) the class of (κ, μ, ν) -contact metric pseudosymmetric 3-manifolds of type constant in the direction of ξ .

1. Introduction

A Riemannian manifold (M^m, g) is said to be pseudosymmetric according to Deszcz [1] if its curvature tensor R satisfies the condition $R(X, Y) \cdot R = L\{(X \wedge Y) \cdot R\}$, where the dot means that $R(X, Y)$ acts as a derivation on R , L is a smooth function and the endomorphism field $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad (1.1)$$

for all vectors fields X, Y, Z on M and it similarly acts as a derivation on R .

If L is constant, M is called a pseudosymmetric manifold of constant type and if particularly $L = 0$ then M is called a semisymmetric manifold first studied by E. Cartan. Semisymmetric spaces [2, 3] are a generalization of locally symmetric spaces ($\nabla R = 0$, [4]) while pseudosymmetric spaces are a natural generalization of semisymmetric spaces. There are many details and examples on pseudosymmetric manifolds in [1, 5]. We remark that in dimension three, the pseudosymmetry is equivalent to the condition: the eigenvalues ρ_1, ρ_2, ρ_3 of the Ricci tensor satisfy $\rho_1 = \rho_2$ (up to numeration) and the last one is constant [6, 7].

Kowalski and Sekizawa have studied [7–10] 3-dimensional pseudosymmetric spaces of constant type. Hashimoto and Sekizawa classified 3-dimensional conformally flat pseudosymmetric spaces of constant type [11] and finally Calvaruso [12] gave the complete classification of conformally flat pseudosymmetric spaces of constant type for dimensions >2 . Cho and Inoguchi [13] studied pseudosymmetric contact homogeneous 3-manifolds while Cho et al. [14] give the conditions so as 3-dimensional trans-Sasakians, quasi-Sasakians, non-Sasakian generalized (κ, μ) -spaces to be pseudosymmetric. Belkhef et al. [15] studied pseudosymmetric Sasakian space forms of any dimension. Finally Gouli-Andreou and Moutafi in [16, 17] have studied some classes of pseudosymmetric contact metric 3-manifolds.

The aim of this paper is the study of the 3-dimension pseudosymmetric contact metric manifolds. The paper is organized in the following way: in Section 2 we will give some preliminaries on pseudosymmetric manifolds and contact manifolds as well and in the next sections we will study 3-dimensional manifolds which satisfy one of the following conditions.

- (i) M is a pseudosymmetric contact metric manifold with harmonic curvature and Trl constant along the direction of ξ .
- (ii) M is a (κ, μ, ν) -contact metric pseudosymmetric manifold of type constant in the direction of ξ .

2. Preliminaries

Let (M^m, g) , $m \geq 3$ be a connected Riemannian smooth manifold. We denote by R its Riemannian curvature tensor given by the equation $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ for any $X, Y, Z \in \mathfrak{X}(M)$ and where ∇ is the Levi-Civita connection of M^m .

Definition 2.1. A Riemannian manifold (M^m, g) , $m \geq 3$, is called *pseudosymmetric* in the sense of Deszcz [1] if at every point of M the curvature tensor satisfies the condition:

$$(R(X, Y) \cdot R)(X_1, X_2, X_3) = L\{((X \wedge Y) \cdot R)(X_1, X_2, X_3)\} \quad (2.1)$$

or more explicitly:

$$\begin{aligned} & R(X, Y)(R(X_1, X_2)X_3) - R(R(X, Y)X_1, X_2)X_3 - R(X_1, R(X, Y)X_2)X_3 - R(X_1, X_2)(R(X, Y)X_3) \\ &= L\{(X \wedge Y)(R(X_1, X_2)X_3) - R((X \wedge Y)X_1, X_2)X_3 \\ &\quad - R(X_1, (X \wedge Y)X_2)X_3 - R(X_1, X_2)((X \wedge Y)X_3)\}, \end{aligned} \quad (2.2)$$

for any $X, Y, X_1, X_2, X_3 \in \mathfrak{X}(M)$, $X \wedge Y$ is given by (1.1) and L is a smooth function.

Definition 2.2. A differentiable manifold M^{2n+1} endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M is called a *contact* manifold.

Given a contact manifold (M, η) , there is an underlying contact metric structure (η, ξ, ϕ, g) where g is a Riemannian metric (the *associated metric*), ϕ a global tensor of type $(1,1)$, and ξ a unique global vector field (the *characteristic* or *Reeb vector field*). A differentiable $(2n + 1)$ -dimensional manifold endowed with a contact metric structure (η, ξ, ϕ, g) is called a *contact metric* (Riemannian) manifold denoted by $M(\eta, \xi, \phi, g)$. The structure tensors η, ξ, ϕ , and g satisfy the equations:

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, & \eta(X) &= g(X, \xi), & \eta(\xi) &= 1, \\ d\eta(X, Y) &= g(X, \phi Y), & g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y). \end{aligned} \quad (2.3)$$

The associated metrics can be constructed polarizing $d\eta$ on the contact subbundle D defined by $\eta = 0$. Denoting by L the Lie differentiation and R the curvature tensor, respectively, we define the tensor fields h, l , and τ by

$$hX = \frac{1}{2}(L_\xi \phi)X, \quad lX = R(X, \xi)\xi, \quad \tau(X, Y) = (L_\xi g)(X, Y). \quad (2.4)$$

These tensors also satisfy the following formulas:

$$\phi\xi = h\xi = l\xi = 0, \quad \eta \circ \phi = \eta \circ h = 0, \quad d\eta(\xi, X) = 0, \quad (2.5)$$

$$Trh = Trh\phi = 0, \quad \nabla_X \xi = -\phi X - \phi hX, \quad h\phi = -\phi h, \quad (2.6)$$

$$hX = \lambda X \implies h\phi X = -\lambda\phi X, \quad (2.7)$$

$$\nabla_\xi h = \phi - \phi l - \phi h^2, \quad \phi l\phi - l = 2(\phi^2 + h^2), \quad (2.8)$$

$$\nabla_\xi \phi = 0, \quad Trl = g(Q\xi, \xi) = 2n - Trh^2. \quad (2.9)$$

$h = 0$ (or equivalently $\tau = 0$) if and only if ξ is Killing and M is called *K-contact*. A contact structure on M implies an almost complex structure on the product manifold $M^{2n+1} \times \mathbb{R}$. If this structure is integrable, then the contact metric manifold is said to be *Sasakian*. A *K-contact* structure is Sasakian only in dimension 3, and this fails in higher dimensions. More details on contact manifolds we can find in [18, 19].

Let (M, ϕ, ξ, η, g) be a 3-dimensional contact metric manifold and U the open subset of points $p \in M$ where $h \neq 0$ in a neighborhood of p and U_0 the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p . Because h is a smooth function on M then $U \cup U_0$ is an open and dense subset of M so if a property is satisfied in $U_0 \cup U$ then this property will be satisfied in M . For any point $p \in U \cup U_0$ there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (a ϕ -basis). On U , we put $he = \lambda e$, where λ is a non vanishing smooth function which is supposed positive. From (2.7), we have $h\phi e = -\lambda\phi e$. We recall the following.

Lemma 2.3 (see [20]). *On U , one has*

$$\begin{aligned} \nabla_{\xi}e &= a\phi e, & \nabla_e e &= b\phi e, & \nabla_{\phi e}e &= -c\phi e + (\lambda - 1)\xi, \\ \nabla_{\xi}\phi e &= -ae, & \nabla_e\phi e &= -be + (1 + \lambda)\xi, & \nabla_{\phi e}\phi e &= ce, \\ \nabla_{\xi}\xi &= 0, & \nabla_e\xi &= -(1 + \lambda)\phi e, & \nabla_{\phi e}\xi &= (1 - \lambda)e, \end{aligned} \quad (2.10)$$

where a is a smooth function and

$$\begin{aligned} b &= \frac{1}{2\lambda} [(\phi e \cdot \lambda) + A] \quad \text{with } A = S(\xi, e), \\ c &= \frac{1}{2\lambda} [(e \cdot \lambda) + B] \quad \text{with } B = S(\xi, \phi e). \end{aligned} \quad (2.11)$$

From Lemma 2.3 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$ we can prove that

$$\begin{aligned} [e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \\ [e, \xi] &= \nabla_e \xi - \nabla_{\xi} e = -(a + \lambda + 1)\phi e, \\ [\phi e, \xi] &= \nabla_{\phi e} \xi - \nabla_{\xi} \phi e = (a - \lambda + 1)e, \end{aligned} \quad (2.12)$$

and from (1.1) we estimate

$$\begin{aligned} (e \wedge \phi e)e &= -\phi e, & (e \wedge \xi)e &= -\xi, & (\phi e \wedge \xi)\xi &= \phi e, \\ (e \wedge \phi e)\phi e &= e, & (e \wedge \xi)\xi &= e, & (\phi e \wedge \xi)\phi e &= -\xi, \end{aligned} \quad (2.13)$$

while $(X \wedge Y)Z = 0$, whenever $X \neq Y \neq Z \neq X$ and $X, Y, Z \in \{e, \phi e, \xi\}$.

By direct computations we calculate the non vanishing independent components of the Riemannian curvature tensor field R (1,3):

$$\begin{aligned} R(\xi, e)\xi &= -Ie - Z\phi e, & R(e, \phi e)e &= -C\phi e - B\xi, \\ R(\xi, \phi e)\xi &= -Ze - D\phi e, & R(\xi, e)\phi e &= -Ke + Z\xi, \\ R(e, \phi e)\xi &= Be - A\phi e, & R(\xi, \phi e)\phi e &= He + D\xi, \\ R(\xi, e)e &= K\phi e + I\xi, & R(e, \phi e)\phi e &= Ce + A\xi, \\ R(\xi, \phi e)e &= -H\phi e + Z\xi, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned}
C &= -b^2 - c^2 + \lambda^2 - 1 + 2a + (e \cdot c) + (\phi e \cdot b), \\
H &= b(\lambda - a - 1) + (\xi \cdot c) + (\phi e \cdot a), \\
K &= c(\lambda + a + 1) + (\xi \cdot b) - (e \cdot a), \\
I &= -2a\lambda - \lambda^2 + 1, \\
D &= 2a\lambda - \lambda^2 + 1, \\
Z &= \xi \cdot \lambda.
\end{aligned} \tag{2.15}$$

Setting $X = e$, $Y = \phi e$, $Z = \xi$ in the Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ and using (2.12), we get

$$\begin{aligned}
b(a + \lambda + 1) - (\xi \cdot c) - (\phi e \cdot \lambda) - (\phi e \cdot a) &= 0, \\
c(a - \lambda + 1) + (\xi \cdot b) + (e \cdot \lambda) - (e \cdot a) &= 0
\end{aligned} \tag{2.16}$$

or equivalently: $A = H$ and $B = K$.

We give the components of the Ricci operator Q with respect to a ϕ -basis:

$$\begin{aligned}
Qe &= \left(\frac{r}{2} - 1 + \lambda^2 - 2a\lambda\right)e + Z\phi e + A\xi, \\
Q\phi e &= Ze + \left(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda\right)\phi e + B\xi, \\
Q\xi &= Ae + B\phi e + 2(1 - \lambda^2)\xi,
\end{aligned} \tag{2.17}$$

where

$$r = \text{Tr}Q = 2\left[1 - \lambda^2 - b^2 - c^2 + 2a + (e \cdot c) + (\phi e \cdot b)\right] \tag{2.18}$$

is the scalar curvature. The relations (2.15) and (2.18) yield

$$C = -b^2 - c^2 + \lambda^2 - 1 + 2a + (e \cdot c) + (\phi e \cdot b) = 2\lambda^2 - 2 + \frac{r}{2}, \tag{2.19}$$

and the relation (2.9):

$$\text{Tr}l = 2(1 - \lambda^2). \tag{2.20}$$

Remark 2.4. If $M^3 = U_0$ (see [21]), Lemma 2.3 is expressed in a similar form with $\lambda = 0$, e is a unit vector field belonging to the contact distribution and for the functions A, B, D, H, I, K and Z we have: $A = B = Z = H = K = 0, I = D = 1$ and $C = r/2 - 2$.

Definition 2.5. An M^m Riemannian manifold has harmonic curvature if the Ricci operator Q satisfies the condition:

$$(\nabla_X Q)Y = (\nabla_Y Q)X, \quad \forall X, Y \in \mathfrak{X}(M). \quad (2.21)$$

From now on we shall work on a $(M^3, \phi, \xi, \eta, g)$ contact metric 3-manifold concerning a ϕ -basis $\{e, \phi e, \xi\}$ at any point $p \in M$. First from the equation $(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y)$, Lemma 2.3 and the relations (2.17), we get the equations:

$$\begin{aligned} (\nabla_e Q)\phi e &= [(e \cdot \xi \cdot \lambda) - 2b\lambda(2a + \lambda + 1) + (1 + \lambda)(\phi e \cdot \lambda)]e \\ &\quad + \left[\frac{(e \cdot r)}{2} + 2b(\xi \cdot \lambda) + 2\lambda(e \cdot a) + (4\lambda + 2a + 2)(e \cdot \lambda) - 4c\lambda(1 + \lambda) \right] \phi e \\ &\quad + \left[\frac{r}{2}(\lambda + 1) + 2\lambda(e \cdot c) + 2c(e \cdot \lambda) - (e \cdot e \cdot \lambda) + 3(\lambda^2 - 1)(\lambda + 1) \right. \\ &\quad \left. - b(\phi e \cdot \lambda) + 2\lambda(b^2 + a\lambda + a) \right] \xi, \\ (\nabla_e Q)\xi &= [3b(e \cdot \lambda) + 2\lambda(e \cdot b) - (e \cdot \phi e \cdot \lambda) - 2bc\lambda + (1 + \lambda)(\xi \cdot \lambda)]e \\ &\quad + \left[\frac{r}{2}(\lambda + 1) + 2c(e \cdot \lambda) + 2\lambda(e \cdot c) - (e \cdot e \cdot \lambda) + 3(\lambda^2 - 1)(\lambda + 1) \right. \\ &\quad \left. - b(\phi e \cdot \lambda) + 2\lambda(b^2 + a\lambda + a) \right] \phi e \\ &\quad + [4c\lambda(1 + \lambda) - (2 + 6\lambda)(e \cdot \lambda)]\xi, \\ (\nabla_{\phi e} Q)e &= \left[\frac{(\phi e \cdot r)}{2} + (4\lambda - 2a - 2)(\phi e \cdot \lambda) - 2\lambda(\phi e \cdot a) + 4b\lambda(1 - \lambda) + 2c(\xi \cdot \lambda) \right] e \\ &\quad + [4ac\lambda + (\phi e \cdot \xi \cdot \lambda) + 2c\lambda(1 - \lambda) + (\lambda - 1)(e \cdot \lambda)] \phi e \\ &\quad + \left[\frac{r}{2}(\lambda - 1) + 2\lambda(\phi e \cdot b) - c(e \cdot \lambda) - (\phi e \cdot \phi e \cdot \lambda) + 3(\lambda^2 - 1)(\lambda - 1) \right. \\ &\quad \left. + 2b(\phi e \cdot \lambda) + 2\lambda(c^2 - a\lambda + a) \right] \xi, \\ (\nabla_{\phi e} Q)\xi &= \left[\frac{r}{2}(\lambda - 1) + 2\lambda(\phi e \cdot b) - (\phi e \cdot \phi e \cdot \lambda) - c(e \cdot \lambda) + 3(\lambda^2 - 1)(\lambda - 1) \right. \\ &\quad \left. + 2b(\phi e \cdot \lambda) + 2\lambda(c^2 - a\lambda + a) \right] e \\ &\quad + [-2bc\lambda + 3c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c) - (\phi e \cdot e \cdot \lambda) + (\lambda - 1)(\xi \cdot \lambda)] \phi e \\ &\quad + [4b\lambda(\lambda - 1) - (6\lambda - 2)(\phi e \cdot \lambda)] \xi, \end{aligned}$$

$$\begin{aligned}
(\nabla_{\xi}Q)e &= \left[\frac{(\xi \cdot r)}{2} + (2\lambda - 4a)(\xi \cdot \lambda) - 2\lambda(\xi \cdot a) \right] e \\
&+ \left[-4a^2\lambda + (\xi \cdot \xi \cdot \lambda) \right] \phi e \\
&+ \left[-2ac\lambda + 2b(\xi \cdot \lambda) + 2\lambda(\xi \cdot b) - (\xi \cdot \phi e \cdot \lambda) + a(e \cdot \lambda) \right], \\
(\nabla_{\xi}Q)\phi e &= \left[(\xi \cdot \xi \cdot \lambda) - 4a^2\lambda \right] e \\
&+ \left[\frac{(\xi \cdot r)}{2} + (2\lambda + 4a)(\xi \cdot \lambda) + 2\lambda(\xi \cdot a) \right] \phi e \\
&+ \left[2ab\lambda + 2c(\xi \cdot \lambda) + 2\lambda(\xi \cdot c) - (\xi \cdot e \cdot \lambda) - a(\phi e \cdot \lambda) \right] \xi.
\end{aligned} \tag{2.22}$$

Applying (2.21) to the vectors fields of the ϕ -basis of the contact metric manifold M^3 we have: $(\nabla_e Q)\phi e = (\nabla_{\phi e} Q)e$, $(\nabla_e Q)\xi = (\nabla_{\xi} Q)e$ and $(\nabla_{\phi e} Q)\xi = (\nabla_{\xi} Q)\phi e$. We use the previous relations and we get the following nine (9) conditions for a contact metric 3-manifold to have harmonic curvature:

$$\begin{aligned}
&(e \cdot \xi \cdot \lambda) + (3 - 3\lambda + 2a)(\phi e \cdot \lambda) - \frac{(\phi e \cdot r)}{2} - 2c(\xi \cdot \lambda) \\
&\quad + 2\lambda(\phi e \cdot a) - 2b\lambda(2a + 3 - \lambda) = 0, \\
&2b(\xi \cdot \lambda) + 2\lambda(e \cdot a) + (3\lambda + 2a + 3)(e \cdot \lambda) + \frac{(e \cdot r)}{2} \\
&\quad - (\phi e \cdot \xi \cdot \lambda) - 2c\lambda(3 + 2a + \lambda) = 0, \\
&r + 2\lambda(e \cdot c) - 2\lambda(\phi e \cdot b) + 3c(e \cdot \lambda) - 3b(\phi e \cdot \lambda) - (e \cdot e \cdot \lambda) \\
&\quad + 6(\lambda^2 - 1) + (\phi e \cdot \phi e \cdot \lambda) + 2\lambda(b^2 - c^2 + 2a\lambda) = 0, \\
&3b(e \cdot \lambda) + 2\lambda(e \cdot b) - (e \cdot \phi e \cdot \lambda) + 2\lambda(\xi \cdot a) - \frac{(\xi \cdot r)}{2} - 2bc\lambda + (4a + 1 - \lambda)(\xi \cdot \lambda) = 0, \\
&\quad - (2 + 6\lambda + a)(e \cdot \lambda) - 2b(\xi \cdot \lambda) - 2\lambda(\xi \cdot b) + 2\lambda c(a + 2 + 2\lambda) + (\xi \cdot \phi e \cdot \lambda) = 0, \\
&(a - 6\lambda + 2)(\phi e \cdot \lambda) - 2c(\xi \cdot \lambda) - 2\lambda(\xi \cdot c) + 2b\lambda(2\lambda - 2 - a) + (\xi \cdot e \cdot \lambda) = 0, \\
&3c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c) - (\phi e \cdot e \cdot \lambda) - 2\lambda(\xi \cdot a) - \frac{(\xi \cdot r)}{2} - 2bc\lambda - (\lambda + 1 + 4a)(\xi \cdot \lambda) = 0, \\
&\frac{r}{2}(\lambda + 1) - b(\phi e \cdot \lambda) + 2c(e \cdot \lambda) + 2\lambda(e \cdot c) - (e \cdot e \cdot \lambda) \\
&\quad - (\xi \cdot \xi \cdot \lambda) + 3(\lambda^2 - 1)(\lambda + 1) + 2\lambda(2a^2 + b^2 + a\lambda + a) = 0,
\end{aligned}$$

$$\begin{aligned} & \frac{r}{2}(\lambda - 1) + 2b(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot b) - c(e \cdot \lambda) - (\phi e \cdot \phi e \cdot \lambda) - (\xi \cdot \xi \cdot \lambda) \\ & + 3(\lambda^2 - 1)(\lambda - 1) + 2\lambda(2a^2 + c^2 - a\lambda + a) = 0. \end{aligned} \quad (2.23)$$

Remark 2.6. From these nine conditions, we can derive some useful results: (a) by subtracting the ninth equation from the first and using (2.16), we get $\phi e \cdot r = 0$, (b) by adding the equations two and six and using similarly (2.16) we have $e \cdot r = 0$. From the relations $\phi e \cdot r = 0$, $e \cdot r = 0$, and (2.12), we can conclude $\xi \cdot r = 0$ and hence we are led to the known result that the scalar curvature r is constant in a contact metric 3-manifold with harmonic curvature. Later for our study, we will use the r as a constant and we will give to these equations a more convenient form.

Definition 2.7 (see [22]). Let M^3 be a 3-dimensional contact metric manifold and $h = \lambda h^+ - \lambda h^-$ the spectral decomposition of h on U . If

$$\nabla_{h^- X} h^- X = [\xi, h^+ X], \quad (2.24)$$

for all vector fields X on M^3 and all points of an open subset W of U and $h = 0$ on the points of M^3 which do not belong to W , then the manifold is said to be semi- K contact manifold. From Lemma 2.3 and the relations (2.12), the above condition for $X = e$ leads to $[\xi, e] = 0$ and for $X = \phi e$ to $\nabla_{\phi e} \phi e = 0$. Hence on a semi- K contact manifold we have $a + \lambda + 1 = c = 0$. If we apply the deformation $e \rightarrow \phi e$, $\phi e \rightarrow e$, $\xi \rightarrow -\xi$, $\lambda \rightarrow -\lambda$, $b \rightarrow c$, and $c \rightarrow b$ then the contact metric structure remains the same. Hence the condition for a 3-dimensional contact metric manifold to be semi- K contact is equivalent to $a - \lambda + 1 = b = 0$.

Definition 2.8. A (κ, μ, ν) -contact metric manifold is defined in [23] as a contact metric manifold $(M^{2n+1}, \eta, \xi, \phi, g)$ on which the curvature tensor satisfies for every $X, Y \in X(M)$ the condition:

$$\begin{aligned} R(X, Y)\xi &= \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ &+ \nu(\eta(Y)\phi hX - \eta(X)\phi hY), \end{aligned} \quad (2.25)$$

where κ, μ, ν are smooth functions on M . If $\nu = 0$ we have a generalized (κ, μ) -contact metric manifold [24] and if additionally κ, μ are constants then the manifold is a contact metric (κ, μ) -space [25, 26]. Moreover, in [23] and [24] it is proved, respectively, that for a (κ, μ, ν) or a generalized (κ, μ) -contact metric manifold M^{2n+1} of dimension greater than 3 the functions κ, μ are constants and ν is the zero function.

Now, we will give some known results concerning contact metric 3-manifolds and pseudosymmetric contact metric 3-manifolds.

Proposition 2.9 (see [16]). *In a 3-dimensional contact metric manifold one has*

$$Q\phi = \phi Q \iff (\xi \cdot \lambda = 2b\lambda - (\phi e \cdot \lambda) = 2c\lambda - (e \cdot \lambda) = a\lambda = 0). \quad (2.26)$$

Let (M, η, g, ϕ, ξ) be a contact metric 3-manifold. In case $M = U_0$, that is, (ξ, η, ϕ, g) is a Sasakian structure, then M is a pseudosymmetric space of constant type [13]. Next, assume that U is not empty and let $\{e, \phi e, \xi\}$ be a ϕ -basis as in Lemma 2.3. We have the following.

Lemma 2.10 (see [16]). *Let (M, η, g, ϕ, ξ) be a contact metric three manifold. Then M is pseudosymmetric if and only if*

$$\begin{aligned} B(\xi \cdot \lambda) + (-2a\lambda - \lambda^2 + 1)A &= LA, \\ A(\xi \cdot \lambda) + (2a\lambda - \lambda^2 + 1)B &= LB, \\ (\xi \cdot \lambda)\left(\frac{r}{2} + 2\lambda^2 - 2\right) + AB &= L(\xi \cdot \lambda), \\ A^2 - |(\xi \cdot \lambda)|^2 + (2a\lambda - \lambda^2 + 1)\left(-2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right) &= L\left(-2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right), \\ B^2 - |(\xi \cdot \lambda)|^2 + (-2a\lambda - \lambda^2 + 1)\left(2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right) &= L\left(2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right), \end{aligned} \quad (2.27)$$

where L is the function in the pseudosymmetry definition (2.2).

Using (2.15), (2.19), the system (2.27) takes a more convenient form:

$$\begin{aligned} ZB + IA &= LA, \\ ZA + DB &= LB, \\ ZC + AB &= LZ, \\ A^2 - Z^2 + D(I - C) &= L(I - C), \\ B^2 - Z^2 + I(D - C) &= L(D - C). \end{aligned} \quad (2.28)$$

Remark 2.11. If $L = 0$, the manifold is semisymmetric and the above system (2.28) is in accordance with equations (3.1)–(3.5) in [27].

Proposition 2.12 (see [16]). *Let M^3 be a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$. Then, M^3 is a pseudosymmetric space of constant type.*

3. Pseudosymmetric Contact Metric 3-Manifolds with Harmonic Curvature and Trl Constant in the Direction of ξ

Theorem 3.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold with harmonic curvature and Trl constant in the direction of ξ . Then, there are at most eight open subsets of M^3 for which their union is an open and dense subset of M^3 and each of them as an open submanifold of M^3 is either: (a) Sasakian or (b) flat or (c) locally isometric to the Lie groups $SU(2)$, $SL(2, \mathbb{R})$ equipped with a left invariant metric or (d) pseudosymmetric of constant type and with scalar curvature $r = 2(1 - \lambda^2 + 2a)$ or (e) semi-K contact with $L = -3a^2 - 4a$ ($a \neq 0$) or (f) semi-K contact with $L = a^2$ ($a \neq 0$) or (g) semi-K contact of type constant along ξ and ϕe or (h) semi-K contact of type constant along ξ and e .*

Proof. We consider the next open subsets of M :

$$\begin{aligned} U_0 &= \{p \in M : \lambda = 0 \text{ in a neighborhood of } p\}, \\ U &= \{p \in M : \lambda \neq 0 \text{ in a neighborhood of } p\}, \end{aligned} \quad (3.1)$$

where $U_0 \cup U$ is open and dense subset of M .

In case $M = U_0$, M is a pseudosymmetric space of constant type [13] and we get the (a) case of present Theorem 3.1. Next, assume that U is not empty and let $\{e, \phi e, \xi\}$ be a ϕ -basis. First we note that in the neighborhood U where $\lambda \neq 0$ we have from (2.20)

$$\xi \cdot Trl = 0 \iff \xi \cdot \lambda = 0. \quad (3.2)$$

Equations (2.23) because of (3.2) and the fact that r is constant become, respectively,

$$(3 - 3\lambda + 2a)(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot a) - 2b\lambda(2a + 3 - \lambda) = 0, \quad (3.3)$$

$$2\lambda(e \cdot a) + (3\lambda + 2a + 3)(e \cdot \lambda) - 2c\lambda(3 + 2a + \lambda) = 0, \quad (3.4)$$

$$\begin{aligned} r + 2\lambda(e \cdot c) - 2\lambda(\phi e \cdot b) + 3c(e \cdot \lambda) - 3b(\phi e \cdot \lambda) - (e \cdot e \cdot \lambda) + 6(\lambda^2 - 1) \\ + (\phi e \cdot \phi e \cdot \lambda) + 2\lambda(b^2 - c^2 + 2a\lambda) = 0, \end{aligned} \quad (3.5)$$

$$\begin{aligned} 3b(e \cdot \lambda) + 2\lambda(e \cdot b) - (e \cdot \phi e \cdot \lambda) + 2\lambda(\xi \cdot a) - 2bc\lambda = 0, \\ \text{or } 3b(e \cdot \lambda) + (e \cdot A) - 6bc\lambda + 2\lambda(\xi \cdot a) = 0, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \frac{r}{2}(\lambda + 1) - b(\phi e \cdot \lambda) + 2c(e \cdot \lambda) + 2\lambda(e \cdot c) + 2\lambda(2a^2 + b^2 + a\lambda + a) \\ - (e \cdot e \cdot \lambda) + 3(\lambda^2 - 1)(\lambda + 1) = 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} -(2 + 6\lambda + a)(e \cdot \lambda) - 2\lambda(\xi \cdot b) + 2\lambda c(a + 2 + 2\lambda) + (\xi \cdot \phi e \cdot \lambda) = 0, \\ \text{or } -(2 + 6\lambda + a)(e \cdot \lambda) + 2\lambda c(a + 2 + 2\lambda) - (\xi \cdot A) = 0, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \frac{r}{2}(\lambda - 1) + 2b(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot b) - c(e \cdot \lambda) - (\phi e \cdot \phi e \cdot \lambda) \\ + 3(\lambda^2 - 1)(\lambda - 1) + 2\lambda(2a^2 + c^2 - a\lambda + a) = 0, \end{aligned} \quad (3.9)$$

$$\begin{aligned} 3c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c) - (\phi e \cdot e \cdot \lambda) - 2\lambda(\xi \cdot a) - 2bc\lambda = 0, \\ \text{or } 3c(\phi e \cdot \lambda) + (\phi e \cdot B) - 2\lambda(\xi \cdot a) - 6bc\lambda = 0, \end{aligned} \quad (3.10)$$

$$\begin{aligned} (a - 6\lambda + 2)(\phi e \cdot \lambda) - 2\lambda(\xi \cdot c) + 2b\lambda(2\lambda - 2 - a) + (\xi \cdot e \cdot \lambda) = 0, \\ \text{or } (a - 6\lambda + 2)(\phi e \cdot \lambda) - (\xi \cdot B) + 2b\lambda(2\lambda - 2 - a) = 0, \end{aligned} \quad (3.11)$$

where for the second form of (3.6), (3.8), we also used $A = 2b\lambda - (\phi e \cdot \lambda)$ and $B = 2c\lambda - (e \cdot \lambda)$ in (3.10), (3.11).

In the neighborhood U , the system (2.28) for the pseudosymmetric contact metric 3-manifolds of Lemma 2.10 because of (3.2) becomes

$$\begin{aligned}
 (I - L)A &= 0, \\
 (D - L)B &= 0, \\
 AB &= 0, \\
 A^2 + (D - L)(I - C) &= 0, \\
 B^2 + (I - L)(D - C) &= 0,
 \end{aligned} \tag{3.12}$$

where A, B, C, D, I are given by (2.15), (2.19).

Studying the third equation, we regard the following open subsets of U :

$$\begin{aligned}
 W &= \{p \in U : A = 2b\lambda - (\phi e \cdot \lambda) = 0 \text{ in a neighborhood of } p\}, \\
 W_3 &= \{p \in U : A = 2b\lambda - (\phi e \cdot \lambda) \neq 0 \text{ in a neighborhood of } p\},
 \end{aligned} \tag{3.13}$$

where $W \cup W_3$ is open and dense in the closure of U .

In W we have

$$\begin{aligned}
 (D - L)B &= 0, \\
 (D - L)(I - C) &= 0, \\
 B^2 + (I - L)(D - C) &= 0,
 \end{aligned} \tag{3.14}$$

hence, we regard the subsets of W :

$$\begin{aligned}
 W_1 &= \{p \in W : B = 2c\lambda - (e \cdot \lambda) = 0 \text{ in a neighborhood of } p\}, \\
 W_2 &= \{p \in W : B = 2c\lambda - (e \cdot \lambda) \neq 0 \text{ in a neighborhood of } p\},
 \end{aligned} \tag{3.15}$$

where $W_1 \cup W_2$ is open and dense in the closure of W and $W_1 \cup W_2 \cup W_3$ is open and dense in the closure of U . We study the initial system at each W_i for $i = 1, 2, 3$.

In W_1 the initial system (2.28) becomes

$$\begin{aligned}
 (D - L)(I - C) &= 0, \\
 (I - L)(D - C) &= 0
 \end{aligned} \tag{3.16}$$

or more explicitly

$$\begin{aligned}
 (\phi e \cdot \lambda) &= 2b\lambda, \\
 (e \cdot \lambda) &= 2c\lambda, \\
 \xi \cdot \lambda &= 0, \\
 [2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)] \times (-2a\lambda - \lambda^2 + 1 - L) &= 0, \\
 [-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)] \times (2a\lambda - \lambda^2 + 1 - L) &= 0.
 \end{aligned} \tag{3.17}$$

We have studied this system in [17] (Theorem 4.1) and we get the cases (b), (c), (d), (e), and (f) of the present Theorem 3.1.

In W_2 the initial system (2.28) becomes

$$\begin{aligned}
 D - L &= 0, \\
 B^2 + (I - L)(D - C) &= 0.
 \end{aligned} \tag{3.18}$$

Apart from (3.2), we also have the following equations:

$$(\phi e \cdot \lambda) = 2b\lambda, \tag{3.19}$$

$$B = 2c\lambda - (e \cdot \lambda) \neq 0, \tag{3.20}$$

$$2a\lambda - \lambda^2 + 1 - L = 0, \tag{3.21}$$

$$B^2 = 4a\lambda \left(2a\lambda - 3\lambda^2 + 3 - \frac{r}{2} \right) = 4a\lambda \left(L - 2\lambda^2 + 2 - \frac{r}{2} \right) \tag{3.22}$$

while we will also use (3.3), (3.4), (3.6), and (3.11).

Differentiating (3.21) with respect to ξ , ϕe we get, respectively,

$$2\lambda(\xi \cdot a) = \xi \cdot L, \tag{3.23}$$

$$2\lambda(\phi e \cdot a) + 4ab\lambda - 4b\lambda^2 = \phi e \cdot L \tag{3.24}$$

(the derivative $e \cdot L = 2\lambda(e \cdot a) + 2(a - \lambda)(e \cdot \lambda)$ can not be estimated any further). Equations (3.3), (3.11) because of (3.19) yield, respectively,

$$\phi e \cdot a = \phi e \cdot \lambda = 2b\lambda, \tag{3.25}$$

$$\xi \cdot B = -8b\lambda^2. \tag{3.26}$$

Differentiating (3.22) with respect to ξ and using (3.2), (3.26) and the fact that r is constant, we get

$$-4b\lambda B = \left(4a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right)(\xi \cdot a). \quad (3.27)$$

The first form of (3.6) and (3.19) give

$$bB = 2\lambda(\xi \cdot a), \quad (3.28)$$

hence (3.27), (3.28) yield

$$\left(4a\lambda + 5\lambda^2 + 3 - \frac{r}{2}\right)(\xi \cdot a) = 0. \quad (3.29)$$

We suppose that there is a point $p \in W_2$ where $\xi \cdot a \neq 0$. Because of the continuity of the function $\xi \cdot a$, there is a neighborhood of this point $S \subset W_2 \subset U$: $S = \{q \in W_2 : \xi \cdot a \neq 0\}$. In S , we have $4a\lambda + 5\lambda^2 + 3 - r/2 = 0$. Differentiating this equation with respect to ξ and using (3.2), the constancy of r and the fact that we work in U where $\lambda \neq 0$ we conclude that $\xi \cdot a = 0$ in S , which is a contradiction. Hence

$$\xi \cdot a = 0, \quad (3.30)$$

everywhere in W_2 . Because of (3.30) the equations (3.20), (3.27) give

$$b = 0. \quad (3.31)$$

Differentiating (3.2) with respect to ϕe , (3.19) with respect to ξ , subtracting and using (2.12), we get

$$(a - \lambda + 1)(e \cdot \lambda) = 0. \quad (3.32)$$

Let's suppose that there is a point p in W_2 where $a - \lambda + 1 \neq 0$. The function $a - \lambda + 1$ is continuous, hence there is an open neighborhood V of p , $V \subset W_2$, where $a - \lambda + 1 \neq 0$ everywhere in V , hence

$$e \cdot \lambda = 0. \quad (3.33)$$

From (3.20) and (3.33) we have in $V \subset W_2 \subset U$:

$$c \neq 0. \quad (3.34)$$

Equation (3.4) because of (3.33) gives $e \cdot a = c(2a + \lambda + 3)$. From the second of (2.16) and because of (3.31), (3.33), (3.34), we get in V : $e \cdot a = c(a - \lambda + 1) \neq 0$. By equalizing these two

results and because of (3.34), we get: $a + 2\lambda + 2 = 0$. We differentiate this equation with respect to e and because of (3.33), we get $e \cdot a = 0$, which is a contradiction in V . Hence $a - \lambda + 1 = 0$ everywhere in W_2 and because of (3.31), we can conclude according to Definition 2.7 that the structure is semi- K contact and pseudosymmetric with L constant along the directions of ξ and ϕe because of (3.23), (3.24) and (3.25), (3.30), (3.31).

In W_3 the initial system (2.28) becomes

$$\begin{aligned} I - L &= 0, \\ A^2 + (D - L)(I - C) &= 0. \end{aligned} \tag{3.35}$$

We have the following equations and (3.2):

$$(e \cdot \lambda) = 2c\lambda, \tag{3.36}$$

$$A = 2b\lambda - (\phi e \cdot \lambda) \neq 0, \tag{3.37}$$

$$-2a\lambda - \lambda^2 + 1 - L = 0, \tag{3.38}$$

$$A^2 = -4a\lambda \left(-2a\lambda - 3\lambda^2 + 3 - \frac{r}{2} \right) = -4a\lambda \left(L - 2\lambda^2 + 2 - \frac{r}{2} \right) \tag{3.39}$$

while we will also use (3.3), (3.4), (3.6), (3.8).

Differentiating (3.38) with respect to ξ , e we get, respectively,

$$\begin{aligned} -2\lambda(\xi \cdot a) &= \xi \cdot L, \\ -2\lambda(e \cdot a) - 4ac\lambda - 4c\lambda^2 &= e \cdot L, \end{aligned} \tag{3.40}$$

(we neglect the derivative $\phi e \cdot L$ because we can not estimate it). Equations (3.4), (3.8) because of (3.36) yield, respectively,

$$e \cdot a = -e \cdot \lambda = -2c\lambda, \tag{3.41}$$

$$\xi \cdot A = -8c\lambda^2. \tag{3.42}$$

Differentiating (3.39) with respect to ξ and using (3.2), (3.42) and the fact that r is constant, we get

$$-4c\lambda A = \left(4a\lambda + 3\lambda^2 - 3 + \frac{r}{2} \right) (\xi \cdot a). \tag{3.43}$$

The first form of (3.10) and (3.36) give

$$-cA = 2\lambda(\xi \cdot a), \tag{3.44}$$

hence (3.43), (3.44) yield

$$\left(4a\lambda - 5\lambda^2 - 3 + \frac{r}{2}\right)(\xi \cdot a) = 0. \quad (3.45)$$

We suppose that there is a point $p \in W_3$ where $\xi \cdot a \neq 0$. Because of the continuity of this function, there is a neighborhood of p $S \subset W_3 \subset U$: $S = \{q \in W_3 : \xi \cdot a \neq 0\}$. In S , we have $4a\lambda - 5\lambda^2 - 3 + r/2 = 0$. Differentiating this equation with respect to ξ and using (3.2), the constancy of r and the fact that we work in U where $\lambda \neq 0$ we conclude that $\xi \cdot a = 0$ in S , which is a contradiction. Hence

$$\xi \cdot a = 0, \quad (3.46)$$

everywhere in W_3 . Because of (3.46) the equations (3.37), (3.43) give

$$c = 0. \quad (3.47)$$

Differentiating (3.2) with respect to e , (3.36) with respect to ξ , subtracting and using (2.12), we get

$$-(a + \lambda + 1)(\phi e \cdot \lambda) = 0. \quad (3.48)$$

Let's suppose that there is a point p in W_3 where $a + \lambda + 1 \neq 0$. This function is smooth, then because of its continuity, there is an open neighborhood V of p , $V \subset W_3$, where $a + \lambda + 1 \neq 0$ everywhere in V , hence

$$\phi e \cdot \lambda = 0. \quad (3.49)$$

From (3.37) and (3.49) we have in $V \subset W_3 \subset U$:

$$b \neq 0. \quad (3.50)$$

From (3.3) and (3.49), we get $\phi e \cdot a = b(2a - \lambda + 3)$. From the first of (2.16) and because of (3.47), (3.49), (3.50), we get in V : $\phi e \cdot a = b(a + \lambda + 1) \neq 0$. By equalizing these two results and because of (3.50), we get: $a - 2\lambda + 2 = 0$. We differentiate this equation with respect to ϕe and because of (3.49), we get $\phi e \cdot a = 0$, which is a contradiction in V . Hence $a + \lambda + 1 = 0$ everywhere in W_3 and because of (3.47), we can conclude according to Definition 2.7 that the structure is semi- K contact and pseudosymmetric with L constant along the directions of ξ and e because of (3.40) and (3.41), (3.46), (3.47).

Finally, we remark that the cases (g) and (h) of the present Theorem 3.1 that result from the structures studied in the sets W_2 and W_3 , respectively. \square

Remark 3.2. (i) The conditions of harmonic curvature help us to the systems in the neighborhoods W_2 and W_3 where we had equations of the type $A^2 = -4a\lambda(-2a\lambda - 3\lambda^2 + 3 - r/2)$ and which we could not handle in our previous articles [16, 17].

(ii) In case (d) where L is constant, we can also use the classification of [11] to improve our results as the manifolds with harmonic curvature are a special case of conformally flat manifolds in dimension 3.

4. Pseudosymmetric (κ, μ, ν) -Contact Metric 3-Manifolds of Type Constant in the Direction of ξ

Theorem 4.1. *Let M^3 be a pseudosymmetric (κ, μ, ν) -contact metric 3-manifold of type constant along the direction ξ . Then, there are at most five open subsets of M^3 for which their union is an open and dense subset of M^3 and each of them as an open submanifold of M^3 is either (a) Sasakian or (b) flat or (c) pseudosymmetric of constant type $L = \kappa = 1/2(\text{Tr}l)$, $\mu = \nu = 0$ and of constant scalar curvature $r = 2\kappa$ or (d) pseudosymmetric generalized (κ, μ) -contact metric manifold of type $L = \kappa - \mu\lambda$, of scalar curvature $r = 2(3\kappa - \mu\lambda)$ and $\xi \cdot \mu = \xi \cdot \kappa = 0$ or (e) pseudosymmetric generalized (κ, μ) -contact metric manifold of type $L = \kappa + \mu\lambda$, of scalar curvature $r = 2(3\kappa + \mu\lambda)$ and $\xi \cdot \mu = \xi \cdot \kappa = 0$.*

Proof. We study pseudosymmetric (κ, μ, ν) -contact metric 3-manifolds with

$$\xi \cdot L = 0, \quad (4.1)$$

where L is the function in (2.2). We consider the next open subsets of M ,

$$\begin{aligned} U_0 &= \{p \in M : \lambda = 0 \text{ in a neighborhood of } p\}, \\ U &= \{p \in M : \lambda \neq 0 \text{ in a neighborhood of } p\}, \end{aligned} \quad (4.2)$$

where $U_0 \cup U$ is open and dense subset of M .

In case $M = U_0$, (M, ξ, η, ϕ, g) is a Sasakian structure which is a pseudosymmetric space of constant type [13] with $\kappa = 1$, $\mu, \nu \in \mathbb{R}$ and $h = 0$ and we get the (a) case of present Theorem 4.1. Next, assume that U is not empty and let $\{e, \phi e, \xi\}$ be a ϕ -basis. From (2.25), we can calculate the following components of the Riemannian curvature tensor:

$$\begin{aligned} R(\xi, e)\xi &= -(\kappa + \lambda\mu)e - \lambda\nu\phi e, & R(e, \phi e)\xi &= 0, \\ R(\xi, \phi e)\xi &= -\lambda\nu e - (\kappa - \lambda\mu)\phi e. \end{aligned} \quad (4.3)$$

By virtue of (2.14) we can conclude that

$$\begin{aligned} A &= 2b\lambda - (\phi e \cdot \lambda) = 0, & B &= 2c\lambda - (e \cdot \lambda) = 0, & Z &= \xi \cdot \lambda = \lambda\nu, \\ D &= 2a\lambda - \lambda^2 + 1 = \kappa - \lambda\mu, & I &= -2a\lambda - \lambda^2 + 1 = \kappa + \lambda\mu, \end{aligned} \quad (4.4)$$

and hence the system (2.28) becomes

$$\begin{aligned} Z(C - L) &= 0, \\ -Z^2 + (D - L)(I - C) &= 0, \\ -Z^2 + (I - L)(D - C) &= 0, \end{aligned} \quad (*)$$

where A, B, C, D, I, Z are given by (2.19) and (4.4). Substituting from (4.4) $(\phi e \cdot \lambda)$, $(e \cdot \lambda)$ in (2.16) we also have

$$\xi \cdot c = -(\phi e \cdot a) + b(a - \lambda + 1), \quad (4.5)$$

$$\xi \cdot b = (e \cdot a) - c(\lambda + a + 1). \quad (4.6)$$

First we will prove that $Z = \xi \cdot \lambda = 0$ (equivalently $\nu = 0$ as we work in U where $\lambda \neq 0$). We suppose that there is a point $p \in U$ where $\xi \cdot \lambda \neq 0$. By the continuity of this function, we can consider that there is a neighborhood V of p , where $\xi \cdot \lambda \neq 0$ everywhere in $V \subset U$. We work in V . Then the first equation of (*) becomes $C - L = 0$ or equivalently:

$$(e \cdot c) + (\phi e \cdot b) = L + b^2 + c^2 - \lambda^2 + 1 - 2a. \quad (4.7)$$

We differentiate this equation with respect to ξ and by virtue of (4.1) we get

$$\xi \cdot e \cdot c + \xi \cdot \phi e \cdot b = 2b(\xi \cdot b) + 2c(\xi \cdot c) - 2\lambda(\xi \cdot \lambda) - 2(\xi \cdot a), \quad (4.8)$$

which because of (4.5), (4.6) becomes

$$\xi \cdot e \cdot c + \xi \cdot \phi e \cdot b = 2b(e \cdot a) - 2c(\phi e \cdot a) - 2\lambda(\xi \cdot \lambda) - 2(\xi \cdot a) - 4bc\lambda. \quad (4.9)$$

Next, we differentiate (4.5) and (4.6) with respect to e and ϕe , respectively, and adding we have

$$\begin{aligned} e \cdot \xi \cdot c + \phi e \cdot \xi \cdot b &= -[e, \phi e]a - (a + \lambda + 1)(\phi e \cdot c) + (a - \lambda + 1)(e \cdot b) - c(\phi e \cdot a) \\ &\quad + b(e \cdot a) - 4bc\lambda. \end{aligned} \quad (4.10)$$

We subtract this last equation from (4.9) and we get

$$\begin{aligned} [\xi, e]c + [\xi, \phi e]b &= b(e \cdot a) - c(\phi e \cdot a) - 2(\xi \cdot a) - 2\lambda(\xi \cdot \lambda) + [e, \phi e]a \\ &\quad + (a + \lambda + 1)(\phi e \cdot c) - (a - \lambda + 1)(e \cdot b) \end{aligned} \quad (4.11)$$

or because of (2.12)

$$\begin{aligned}
& (a + \lambda + 1)(\phi e \cdot c) + (\lambda - a - 1)(e \cdot b) \\
&= b(e \cdot a) - c(\phi e \cdot a) - 2(\xi \cdot a) - 2\lambda(\xi \cdot \lambda) \\
&\quad - b(e \cdot a) + c(\phi e \cdot a) + 2(\xi \cdot a) + (\lambda + a + 1)(\phi e \cdot c) + (\lambda - a - 1)(e \cdot b)
\end{aligned} \tag{4.12}$$

or equivalently: $\lambda(\xi \cdot \lambda) = 0$ and because we work in $V \subset U$, we have $\xi \cdot \lambda = 0$, which is a contradiction. Hence, we can deduce everywhere in U :

$$\xi \cdot \lambda = 0 \iff \nu = 0. \tag{4.13}$$

Next we will derive some useful relations. From (4.4) we have:

$$\begin{aligned}
\phi e \cdot \lambda &= 2b\lambda, \\
e \cdot \lambda &= 2c\lambda.
\end{aligned} \tag{4.14}$$

We differentiate these equations with respect to e and ϕe , respectively, we subtract, we use the relations (2.12), (4.4) and we get

$$\xi \cdot \lambda = \lambda[(e \cdot b) - (\phi e \cdot c)] \tag{4.15}$$

or because of (4.13)

$$e \cdot b = \phi e \cdot c. \tag{4.16}$$

We differentiate the relations $\phi e \cdot \lambda = 2b\lambda$ and (4.13) with respect to ξ and ϕe , respectively, and subtracting we obtain: $[\xi, \phi e]\lambda = 2\lambda(\xi \cdot b)$ or because of (2.12), (4.4), (4.6)

$$e \cdot a = 2c\lambda. \tag{4.17}$$

We differentiate the relations $e \cdot \lambda = 2c\lambda$ and (4.13) with respect to ξ and e , respectively, and subtracting we obtain: $[\xi, e]\lambda = 2\lambda(\xi \cdot c)$ or because of (2.12), (4.4), (4.5)

$$\phi e \cdot a = -2b\lambda. \tag{4.18}$$

Finally, after substituting D, I, Z from (4.4), (4.13) the final form of the system (*) is

$$\begin{aligned}
\mu(C - L) &= 0, \\
(\kappa - \lambda\mu - L)(\kappa + \lambda\mu - C) &= 0.
\end{aligned} \tag{4.19}$$

In order to study this system we regard the following open subsets of U :

$$\begin{aligned} V_1 &= \{p \in U : C - L \neq 0 \text{ in a neighborhood of } p\}, \\ V_2 &= \{p \in U : C - L = 0 \text{ in a neighborhood of } p\}, \end{aligned} \quad (4.20)$$

where $V_1 \cup V_2$ is open and dense in the closure of U .

In V_1 , we have $\mu = 0$ and hence from (4.4): $I = D = \kappa$ or $2a\lambda - \lambda^2 + 1 = -2a\lambda - \lambda^2 + 1$ or finally $a = 0$ and $\kappa = 1 - \lambda^2$. From (4.17), (4.18) we deduce that $b = c = 0$. Having also the second equation of (4.19), we regard the open subsets of V_1

$$\begin{aligned} Y_1 &= \{p \in V_1 : \kappa - C = 0 \text{ in a neighborhood of } p\}, \\ Y_2 &= \{p \in V_1 : \kappa - C \neq 0 \text{ in a neighborhood of } p\}, \end{aligned} \quad (4.21)$$

where $Y_1 \cup Y_2$ is open and dense in the closure of V_1 .

In Y_1 substituting in $\kappa - C = 0$, C from (2.15), $a = b = c = 0$, we get $\kappa = 1 - \lambda^2 = 0$ and hence the structure is flat with $\kappa = \mu = \nu$.

In Y_2 from $\mu = 0$ we have again $I = D = \kappa$, $a = 0$ and from (4.17), (4.18) $b = c = 0$ while we must also have $\kappa = L$. Hence, $L = \kappa = (1/2)Trl$ and from (2.18) of constant scalar curvature $r = 2(1 - \lambda^2)$.

In V_2 having $C = L$, the second equation of (4.19) becomes $(2a\lambda - \lambda^2 + 1 - L)(-2a\lambda - \lambda^2 + 1 - L) = 0$. Hence, we regard the open subsets of V_2

$$\begin{aligned} W_1 &= \{p \in V_2 : -2a\lambda - \lambda^2 + 1 - L \neq 0 \text{ in a neighborhood of } p\}, \\ W_2 &= \{p \in V_2 : -2a\lambda - \lambda^2 + 1 - L = 0 \text{ in a neighborhood of } p\}, \end{aligned} \quad (4.22)$$

where $W_1 \cup W_2$ is open and dense in the closure of V_2 .

In W_1 we must have $2a\lambda - \lambda^2 + 1 - L = 0$ while in W_2 we have $-2a\lambda - \lambda^2 + 1 - L = 0$. We differentiate these equations with respect to ξ and because of (4.13) we get

$$\xi \cdot a = 0. \quad (4.23)$$

By virtue of I and D in (4.4) we deduce $\mu = -2a$ and hence

$$\xi \cdot \mu = 0. \quad (4.24)$$

In W_2 we differentiate $-2a\lambda - \lambda^2 + 1 - L = 0$ with respect to ξ and similarly we also obtain (4.24). Each of W_1 and W_2 is a generalized (κ, μ) -contact metric 3-manifold with $\xi \cdot \mu = 0$ and scalar curvature $r = 2(2a\lambda - 3\lambda^2 + 3) = 2(3\kappa - \mu\lambda)$ or $r = 2(-2a\lambda - 3\lambda^2 + 3) = 2(3\kappa + \mu\lambda)$ respectively and from (4.1), (4.13) and (4.23) or (4.24) $\xi \cdot \kappa = 0$ and $\xi \cdot r = 0$.

Concluding: the structure in U_0 gives the Sasakian case, the structures in Y_1 and Y_2 give the (b) and (c) cases of the present Theorem 4.1 and the structures in W_1 and W_2 give (d) and (e) respectively. \square

Remark 4.2. The generalized (κ, μ) -contact metric manifolds in dimension 3 with $\kappa < 1$ (equivalently $\lambda \neq 0$) and $\xi \cdot \mu = 0$ have been studied by Koufogiorgos and Tsihlias [28]. They proved in their Theorem 4.1 of [28] that at any point of $P \in M$, precisely one of the following relations is valid: $\mu = 2(1 + \sqrt{1 - \kappa})$, or $\mu = 2(1 - \sqrt{1 - \kappa})$, while there exists a chart $(U, (x, y, z))$ with $P \in U \subseteq M$ such that the functions κ, μ depend only on z and the tensors fields η, ξ, ϕ, g take a suitable form. Each of our submanifolds W_1 and W_2 is such a generalized (κ, μ) -contact metric 3-manifold.

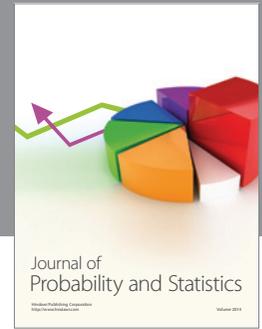
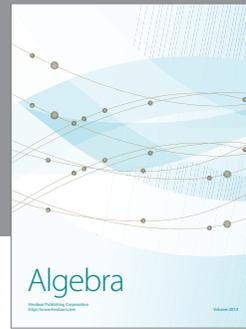
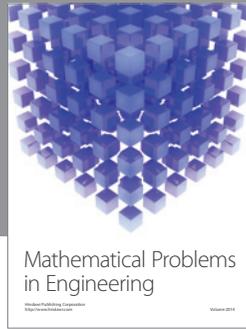
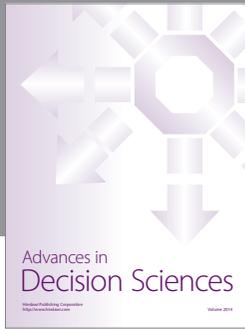
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