

Research Article

θ - \mathcal{D}_g -Closed Sets

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We define θ - \mathcal{D}_g -Closed sets and discuss their properties. Using these sets, we characterize $\mathcal{T}_{1/2}$ -spaces and $\mathcal{T}_{\mathcal{D}}$ -Spaces.

1. Introduction and Preliminaries

An ideal \mathcal{D} on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{D}$ and $B \subset A$ implies $B \in \mathcal{D}$ and (ii) $A, B \in \mathcal{D}$ implies $A \cup B \in \mathcal{D}$. Given a topological space (X, τ) with an ideal \mathcal{D} on X and if $\wp(X)$ is the set of all subsets of X , a set operator $(\cdot)^* : \wp(X) \rightarrow \wp(X)$ called a local function [1] of A with respect to τ and \mathcal{D} is defined as follows: for $A \subset X$, $A^*(X, \tau) = \{x \in X \mid U \cap A \notin \mathcal{D}, \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $\text{cl}^*(\cdot)$ for a topology $\tau^*(\mathcal{D}, \tau)$ called the \star -topology, finer than τ , is defined by $\text{cl}^*(A) = A \cup A^*(\mathcal{D}, \tau)$ [2]. When there is no confusion we will simply write A^* for $A^*(\mathcal{D}, \tau)$ and τ^* for $\tau^*(\mathcal{D}, \tau)$. If \mathcal{D} is an ideal on X , then (X, τ, \mathcal{D}) is called an ideal space. A subset A of an ideal space (X, τ, \mathcal{D}) is said to be \star -closed [3] if $A^* \subset A$. A subset A of an ideal space (X, τ, \mathcal{D}) is said to be an \mathcal{D}_g -closed [4] if $A^* \subset U$ whenever $A \subset U$ and U is open. A subset A of an ideal space (X, τ, \mathcal{D}) is said to be \mathcal{D}_g -open if $X - A$ is \mathcal{D}_g -closed. An ideal space (X, τ, \mathcal{D}) is said to be a $\mathcal{T}_{\mathcal{D}}$ -space [4] if every \mathcal{D}_g -closed set is \star -closed. A subset A of an ideal space (X, τ, \mathcal{D}) is said to be \mathcal{D} -locally \star -closed [5] if there exist an open set U and a \star -closed set F such that $A = U \cap F$. If $\mathcal{D} = \{\emptyset\}$, then \mathcal{D} -locally \star -closed sets coincide with locally closed sets.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $\text{cl}(A)$ and $\text{int}(A)$ will, respectively, denote the closure and interior of A in (X, τ) and $\text{int}^*(A)$ will denote the interior of A in (X, τ^*) . A subset A of a topological space (X, τ) is said to be a g -closed set [6] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and U is open. A subset A

of a topological space (X, τ) is said to be a g -open set if $X - A$ is a g -closed set. A space (X, τ) is said to be a $\mathcal{T}_{1/2}$ -space [6] if every g -closed set is a closed set.

For a subset A of a space (X, τ) , the θ -interior [7] of A is the union of all open sets of X whose closures contained in A and is denoted by $\text{int}_\theta(A)$. The subset A is called θ -open if $A = \text{int}_\theta(A)$. The complement of a θ -open set is called a θ -closed set. Equivalently, $A \subset X$ is called θ -closed [7] if $A = \text{cl}_\theta(A)$, where $\text{cl}_\theta(A) = \{x \in X \mid \text{cl}(U) \cap A \neq \emptyset \text{ for all } U \in \tau(x)\}$. The family of all θ -open sets of X forms a topology [7] on X , which is coarser than τ and is denoted by τ_θ . A subset A of a topological space (X, τ) is said to be a θ - g -closed set [8] if $\text{cl}_\theta(A) \subset U$ whenever $A \subset U$ and U is open. A subset A of a space (X, τ) is said to be a θ - g -open set [8] if $X - A$ is a θ - g -closed set. A subset A of a space (X, τ) is said to be a Λ -set [9, 10] if $A = A^\Lambda$, where $A^\Lambda = \bigcap \{U \in \tau \mid A \subset U\}$.

A subset A of an ideal space (X, τ, \mathcal{I}) is said to be θ - \mathcal{I} -closed [11] if $\text{cl}_\theta^*(A) = A$, where $\text{cl}_\theta^*(A) = \{x \in X \mid A \cap \text{cl}^*(U) \neq \emptyset \text{ for all } U \in \tau(x)\}$. A is said to be θ - \mathcal{I} -open if $X - A$ is θ - \mathcal{I} -closed. If $\mathcal{I} = \{\emptyset\}$, $\text{cl}_\theta^*(A) = \text{cl}_\theta(A)$. If $\mathcal{I} = \wp(X)$, $\text{cl}_\theta^*(A) = \text{cl}(A)$. For a subset A of X , $\text{int}_\theta I(A) = \bigcup \{U \in \tau \mid \text{cl}^*(U) \subset A\}$ [11]. We denote this $\text{int}_\theta I(A)$ by $\text{int}_\theta^*(A)$. The family of all θ - \mathcal{I} -open sets of (X, τ, \mathcal{I}) is a topology and it is denoted by $\tau_{\theta, \mathcal{I}}$ (see [11, Theorem 1]).

Lemma 1.1 (see [11, Corollary 4 if Theorem 2]). $\tau_\theta \subset \tau_{\theta, \mathcal{I}} \subset \tau$.

Lemma 1.2 (see [11, Proposition 3]). Let (X, τ, \mathcal{I}) be an ideal space. Then, we have

(1) if $\mathcal{I} = \{\emptyset\}$ or $\mathcal{I} = \mathcal{N}$, where \mathcal{N} is the ideal of nowhere dense sets of (X, τ) , then $\tau_{\theta, \mathcal{I}} = \tau_\theta$,

(2) if $\mathcal{I} = \{\emptyset\}$, then $\tau_{\theta, \mathcal{I}} = \tau$.

Lemma 1.3 (see [5, Theorem 2.13]). Let (X, τ, \mathcal{I}) be an ideal space. Then every subset of X is \mathcal{I} - g -closed if and only if every open set is \star -closed.

Lemma 1.4 (see [11, Proposition 1]). Let (X, τ, \mathcal{I}) be an ideal space and A a subset of X . Then A is θ - \mathcal{I} -open if and only if $\text{int}_\theta^*(A) = A$.

Lemma 1.5. Let (X, τ, \mathcal{I}) be an ideal space and A a subset of X . Then $\text{cl}_\theta^*(A) = \{x \in X \mid U \cap \text{cl}^*(A) \neq \emptyset \text{ for all } U \in \tau(x)\}$ is closed.

Proof. If $x \in \text{cl}(\text{cl}_\theta^*(A))$ and $U \in \tau(x)$, then $U \cap \text{cl}_\theta^*(A) \neq \emptyset$. Then, $y \in U \cap \text{cl}_\theta^*(A)$ for some $y \in X$. Since $U \in \tau(y)$ and $y \in \text{cl}_\theta^*(A)$, from the definition of $\text{cl}_\theta^*(A)$ we have $A \cap \text{cl}^*(U) \neq \emptyset$. Therefore, $x \in \text{cl}_\theta^*(A)$. So $\text{cl}(\text{cl}_\theta^*(A)) \subset \text{cl}_\theta^*(A)$ and hence $\text{cl}_\theta^*(A)$ is closed. \square

Lemma 1.6. Let (X, τ, \mathcal{I}) be an ideal space and A a subset of X . Then, $X - \text{cl}_\theta^*(X - A) = \text{int}_\theta^*(A)$.

Proof. $x \in X - \text{cl}_\theta^*(X - A)$ if and only if $x \notin \text{cl}_\theta^*(X - A)$ if and only if there exist $U \in \tau(x)$ such that $(X - A) \cap \text{cl}^*(U) = \emptyset$ if and only if $x \in U$ and, $\text{cl}^*(U) \subset A$ if and only if $x \in U \subset \text{int}_\theta^*(A)$. \square

2. θ - \mathcal{I} - g -Closed Sets

A subset A of an ideal space (X, τ, \mathcal{I}) is said to be a θ - \mathcal{I} - g -closed set if $\text{cl}_\theta^*(A) \subset U$ whenever $A \subset U$ and U is open. Every θ - \mathcal{I} -closed set is a θ - \mathcal{I} - g -closed set. If $\mathcal{I} = \{\emptyset\}$, then $\text{cl}_\theta^*(A) = \text{cl}_\theta(A)$ and hence θ - \mathcal{I} - g -closed sets coincide with θ - g -closed sets. If $\mathcal{I} = \wp(X)$, then

$\text{cl}_\theta^*(A) = \text{cl}(A)$ and hence θ - \mathcal{D}_g -closed sets coincide with g -closed sets. Since $\text{cl}^*(A) \subset \text{cl}(A) \subset \text{cl}_\theta^*(A) \subset \text{cl}_\theta(A)$, we have the following inclusion diagram:

$$\theta\text{-}g\text{-closed} \longrightarrow \theta\text{-}\mathcal{D}_g\text{-closed} \longrightarrow g\text{-closed} \longrightarrow \mathcal{D}_g\text{-closed}. \quad (2.1)$$

Example 2.1. shows that a g -closed set needs not to be θ - \mathcal{D}_g -closed, and Example 2.2 shows that θ - \mathcal{D}_g -closed set needs not to be a θ - g -closed set.

Example 2.1. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$, and $\mathcal{D} = \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Let $A = \{c\}$. Then A is closed and hence g -closed. But A is not θ - \mathcal{D}_g -closed because $A \subset \{b, c\}$ and $\text{cl}_\theta^*(A) = X \not\subset \{b, c\}$.

Example 2.2. Let X and τ be the same as in Example 2.1. Let $\mathcal{D} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ and $A = \{c\}$. Then A is a θ - \mathcal{D} -closed and hence θ - \mathcal{D}_g -closed. Since $A \subset \{b, c\}$ and $\text{cl}_\theta(A) = X \not\subset \{b, c\}$, A is not θ - g -closed.

Theorem 2.3. *If A is a subset of an ideal space (X, τ, \mathcal{D}) , then the following are equivalent.*

- (a) A is θ - \mathcal{D}_g -closed.
- (b) For all $x \in \text{cl}_\theta^*(A)$, $\text{cl}(\{x\}) \cap A \neq \phi$.
- (c) $\text{cl}_\theta^*(A) - A$ contains no nonempty closed set.

Proof. (a) \Rightarrow (b). Suppose $x \in \text{cl}_\theta^*(A)$. If $\text{cl}(\{x\}) \cap A = \phi$, then $A \subset X - \text{cl}(\{x\})$. Since A is θ - \mathcal{D}_g -closed, $\text{cl}_\theta^*(A) \subset X - \text{cl}(\{x\})$. It is a contradiction to the fact that $x \in \text{cl}_\theta^*(A)$. This proves (b).

(b) \Rightarrow (c). Suppose $F \subset \text{cl}_\theta^*(A) - A$, F is closed and $x \in F$. Since $F \subset X - A$ and F closed, $\text{cl}(\{x\}) \cap A \subset \text{cl}(F) \cap A = F \cap A = \phi$. Since $x \in \text{cl}_\theta^*(A)$, by (b), $\text{cl}(\{x\}) \cap A \neq \phi$, a contradiction which proves (c).

(c) \Rightarrow (a). Let U be an open set containing A . Since $\text{cl}_\theta^*(A)$ is closed, $\text{cl}_\theta^*(A) \cap (X - U)$ is closed and $\text{cl}_\theta^*(A) \cap (X - U) \subset \text{cl}_\theta^*(A) - A$. By hypothesis, $\text{cl}_\theta^*(A) \cap (X - U) = \phi$ and hence $\text{cl}_\theta^*(A) \subset U$. Thus, A is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.3, we get Corollary 2.4 which gives characterizations for θ - g -closed sets. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.3, we get Corollary 2.5 which gives characterizations for g -closed sets.

Corollary 2.4. *If A is a subset of a topological space (X, τ) , then the following are equivalent.*

- (a) A is θ - g -closed.
- (b) For all $x \in \text{cl}_\theta(A)$, $\text{cl}(\{x\}) \cap A \neq \phi$.
- (c) $\text{cl}_\theta(A) - A$ contains no nonempty closed set.

Corollary 2.5 (see [12, Theorem 2.2]). *If A is a subset of a topological space (X, τ) , then the following are equivalent.*

- (a) A is g -closed.
- (b) For all $x \in \text{cl}(A)$, $\text{cl}(\{x\}) \cap A \neq \phi$.
- (c) $\text{cl}(A) - A$ contains no nonempty closed set.

The following Corollary 2.6 shows that in \mathcal{T}_1 -space, θ - \mathcal{D}_g -closed sets are θ - \mathcal{D} -closed, the proof of which follows from Theorem 2.3(c). Corollary 2.7 gives the relation between θ - \mathcal{D}_g -closed and θ - \mathcal{D} -closed sets.

Corollary 2.6. *If (X, τ, \mathcal{D}) is a \mathcal{T}_1 -space and A is θ - \mathcal{D}_g -closed then A is a θ - \mathcal{D} -closed set.*

Corollary 2.7. *If (X, τ, \mathcal{D}) is an ideal space and A is a θ - \mathcal{D}_g -closed set, then the following are equivalent.*

- (a) A is a θ - \mathcal{D} -closed set.
- (b) $\text{cl}_\theta^*(A) - A$ is a closed set.

Proof. (a) \Rightarrow (b). If A is θ - \mathcal{D} -closed, then $\text{cl}_\theta^*(A) - A = \emptyset$ and so $\text{cl}_\theta^*(A) - (A)$ is closed.

(b) \Rightarrow (a). If $\text{cl}_\theta^*(A) - (A)$ is closed, since A is θ - \mathcal{D}_g -closed, by Theorem 2.3(c), $\text{cl}_\theta^*(A) - (A) = \emptyset$ and so A is θ - \mathcal{D} -closed. \square

If we put $\mathcal{D} = \{\emptyset\}$ in Corollary 2.7, we get Corollary 2.8. If we put $\mathcal{D} = \wp(X)$ in Corollary 2.7, we get Corollary 2.9.

Corollary 2.8. *If $(X, \tau,)$ is a topological space and A is a θ - g -closed set, then the following are equivalent.*

- (a) A is a θ -closed set.
- (b) $\text{cl}_\theta(A) - A$ is a closed set.

Corollary 2.9 (see [6, Corollary 2.3]). *If (X, τ) is an topological space and A is a g -closed set, then the following are equivalent.*

- (a) A is a closed set.
- (b) $\text{cl}(A) - A$ is a closed set.

Theorem 2.10. *If every open set of an ideal space (X, τ, \mathcal{D}) is \star -closed, then every g -closed set is θ - \mathcal{D}_g -closed.*

Proof. Since every open set is \star -closed, $\text{cl}^*(U) = U$ for every $U \in \tau$. Therefore, for every subset A of X , $\text{int}_\theta^*(A) = \cup\{U \in \tau \mid \text{cl}^*(U) \subset A\} = \cup\{U \in \tau \mid U \subset A\} = \text{int}(A)$. So $\text{cl}_\theta^*(A) = \text{cl}(A)$ for every subset A of X . This implies that every g -closed set is θ - \mathcal{D}_g -closed. \square

Corollary 2.11. *If every subset of an ideal space (X, τ, \mathcal{D}) is \mathcal{D}_g -closed, then every g -closed set is θ - \mathcal{D}_g -closed.*

The proof follows from Lemma 1.3 and Theorem 2.10.

Theorem 2.12. *Let (X, τ, \mathcal{D}) be an ideal space. Then every subset of X is θ - \mathcal{D}_g -closed if and only if every open set is θ - \mathcal{D} -closed.*

Proof. Suppose every subset of X is θ - \mathcal{D}_g -closed. If U is open, then U is θ - \mathcal{D}_g -closed and so $\text{cl}_\theta^*(U) \subset U$. Hence U is θ - \mathcal{D} -closed. Conversely, suppose $A \subset U$ and U is open. Since every open set is θ - \mathcal{D} -closed, $\text{cl}_\theta^*(A) \subset U$ and so A is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\emptyset\}$ in Theorem 2.12, we get Corollary 2.13. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.12, we get Corollary 2.14.

Corollary 2.13. *Let (X, τ) be a topological space. Then every subset of X is θ - g -closed if and only if every open set is θ -closed.*

Corollary 2.14 (see [6, Theorem 2.10]). *Let (X, τ) be a topological space. Then every subset of X is g -closed if and only if every open set is closed.*

Theorem 2.15. *If every θ - \mathcal{D}_g -closed set of an ideal space (X, τ, \mathcal{D}) is θ -closed, then (X, τ) is a \mathcal{T}_1 space.*

Proof. Suppose $\{x\}$ is not closed for some $x \in X$. Then, $B = X - \{x\}$ is not open. So B is θ - \mathcal{D}_g -closed. By hypothesis, B is θ -closed. Therefore, $\{x\}$ is θ -open. So $\{x\}$ is both open and closed, a contradiction. Hence, (X, τ) is a \mathcal{T}_1 -space. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.15, we get Corollary 2.16.

Corollary 2.16. *If every θ - g -closed set of a space (X, τ) is θ -closed, then (X, τ) is a \mathcal{T}_1 space.*

Theorem 2.17. *Intersection of a θ - \mathcal{D}_g -closed set and a θ - \mathcal{D} -closed set is always θ - \mathcal{D}_g -closed.*

Proof. Let A be a θ - \mathcal{D}_g -closed set and F a θ - \mathcal{D} -closed set of an ideal space (X, τ, \mathcal{D}) . Suppose $A \cap F \subset U$ and U is open in X . Then, $A \subset U \cup (X - F)$. Now $X - F$ is θ - \mathcal{D} -open and hence open. So $U \cup (X - F)$ is an open set containing A . Since A is θ - \mathcal{D}_g -closed, $\text{cl}_\theta^*(A) \subset U \cup (X - F)$. Therefore, $\text{cl}_\theta^*(A) \cap F \subset U$ which implies that $\text{cl}_\theta^*(A \cap F) \subset U$. So $A \cap F$ is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.17, we get Corollary 2.18. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.17, we get Corollary 2.19.

Corollary 2.18 (see [8, Proposition 3.11]). *Intersection of a θ - g -closed set and a θ -closed set is always θ - g -closed.*

Corollary 2.19 (see [6, Corollary 2.7]). *Intersection of a g -closed set and a closed set is always a g -closed set.*

Theorem 2.20. *A subset A of an ideal space (X, τ, \mathcal{D}) is θ - \mathcal{D}_g -closed if and only if $\text{cl}_\theta^*(A) \subset A^\Delta$.*

Proof. Suppose A is θ - \mathcal{D}_g -closed and $x \in \text{cl}_\theta^*(A)$. If $x \notin A^\Delta$, then there exists an open set U such that $A \subset U$, but $x \notin U$. Since A is θ - \mathcal{D}_g -closed, $\text{cl}_\theta^*(A) \subset U$ and so $x \in \text{cl}_\theta^*(A)$, a contradiction. Therefore, $\text{cl}_\theta^*(A) \subset A^\Delta$. Conversely, suppose that $\text{cl}_\theta^*(A) \subset A^\Delta$. If $A \subset U$ and U is open, then $A^\Delta \subset U$ and so $\text{cl}_\theta^*(A) \subset U$. Therefore, A is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.20, we get Corollary 2.21. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.20, we get Corollary 2.22.

Corollary 2.21. *A subset A of a space (X, τ) is θ - g -closed if and only if $\text{cl}_\theta(A) \subset A^\Delta$.*

Corollary 2.22. *A subset A of a space (X, τ) is g -closed if and only if $\text{cl}(A) \subset A^\Delta$.*

Theorem 2.23. *Let A be a Δ -set of an ideal space (X, τ, \mathcal{D}) . Then A is θ - \mathcal{D}_g -closed if and only if A is θ - \mathcal{D} -closed.*

Proof. Suppose A is θ - \mathcal{D}_g -closed. By Theorem 2.20, $\text{cl}_\theta^*(A) \subset A^\Delta = A$, since A is a Δ -set. Therefore, A is θ - \mathcal{D} -closed. Converse follows from the fact that every θ - \mathcal{D} -closed is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.23, we get Corollary 2.24. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.23, we get Corollary 2.25.

Corollary 2.24. *Let A be a Λ -set of a space (X, τ) . Then A is θ - g -closed if and only if A is θ -closed.*

Corollary 2.25. *Let A be a Λ -set of a space (X, τ) . Then A is g -closed if and only if A is closed.*

Theorem 2.26. *Let (X, τ, \mathcal{D}) be an ideal space and $A \subset X$. If A^\wedge is θ - \mathcal{D}_g -closed, then A is also θ - \mathcal{D}_g -closed.*

Proof. Suppose that A^\wedge is a θ - \mathcal{D}_g -closed set. If $A \subset U$ and U is open, then $A^\wedge \subset U$. Since A^\wedge is θ - \mathcal{D}_g -closed, $\text{cl}_\theta^*(A^\wedge) \subset U$. But, $\text{cl}_\theta^*(A) \subset \text{cl}_\theta^*(A^\wedge)$. Therefore, A is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.26, we get Corollary 2.27. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.26, we get Corollary 2.28.

Corollary 2.27. *Let (X, τ) be a topological space and $A \subset X$. If A^\wedge is θ - g -closed, then A is also θ - g -closed.*

Corollary 2.28. *Let (X, τ) be a space and $A \subset X$. If A^\wedge is g -closed set, then A is also g -closed.*

Theorem 2.29. *For an ideal space (X, τ, \mathcal{D}) , the following are equivalent.*

- (a) *Every θ - \mathcal{D}_g -closed set is θ - \mathcal{D} -closed.*
- (b) *Every singleton of X is closed or θ - \mathcal{D} -open.*

Proof. (a) \Rightarrow (b). Let $x \in X$. If $\{x\}$ is not closed, then $A = X - \{x\} \notin \tau$ and then A is trivially θ - \mathcal{D}_g -closed. By (a), A is θ - \mathcal{D} -closed. Hence $\{x\}$ is θ - \mathcal{D} -open.

(b) \Rightarrow (a). Let A be a θ - \mathcal{D}_g -closed set and let $x \in \text{cl}_\theta^*(A)$. We have the following cases.

Case 1. $\{x\}$ is closed. By Theorem 2.3, $\text{cl}_\theta^*(A) - A$ does not contain a nonempty closed subset. This shows $\{x\} \in A$.

Case 2. $\{x\}$ is θ - \mathcal{D} -open. Then, $\{x\} \cap A \neq \phi$. Hence, $x \in A$.

Thus in both cases $x \in A$ and so $A = \text{cl}_\theta^*(A)$, that is, A is θ - \mathcal{D} -closed, which proves (a). \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.29, we get Corollary 2.30. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.29, we get Corollary 2.31.

Corollary 2.30. *For an ideal space (X, τ) , the following are equivalent.*

- (a) *Every θ - g -closed set is θ -closed.*
- (b) *Every singleton of X is closed or θ -open.*

Corollary 2.31 (see [13, Theorem 2.5]). *For an ideal space (X, τ) , the following are equivalent.*

- (a) *Every g -closed set is closed.*
- (b) *Every singleton of X is closed or open.*

Theorem 2.32. *Let (X, τ, \mathcal{D}) be an ideal space and $A \subset X$. Then A is θ - \mathcal{D}_g -closed if and only if $A = F - N$, where F is θ - \mathcal{D} -closed and N contains no nonempty closed set.*

Proof. If A is θ - \mathcal{D}_g -closed, then by Theorem 2.3, $N = \text{cl}_\theta^*(A) - A$ contains no nonempty closed set. If $F = \text{cl}_\theta^*(A)$, then F is θ - \mathcal{D} -closed such that $F - N = \text{cl}_\theta^*(A) - (\text{cl}_\theta^*(A) - A) = \text{cl}_\theta^*(A) \cap ((X - \text{cl}_\theta^*(A)) \cup A) = A$. Conversely, suppose $A = F - N$, where F is θ - \mathcal{D} -closed and N contains no nonempty closed set. Let U be an open set such that $A \subset U$. Then, $F - N \subset U$ which implies that $F \cap (X - U) \subset N$. Now, $A \subset F$ and F is θ - \mathcal{D} -closed implies that $\text{cl}_\theta^*(A) \cap (X - U) \subset \text{cl}_\theta^*(F) \cap (X - U) \subset F \cap (X - U) \subset N$. Since θ - \mathcal{D} -closed sets are closed, $\text{cl}_\theta^*(A) \cap (X - U)$ is closed. By hypothesis, $\text{cl}_\theta^*(A) \cap (X - U) = \emptyset$ and so $\text{cl}_\theta^*(A) \subset U$, which implies that A is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.32, we get Corollary 2.33. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.32, we get Corollary 2.34.

Corollary 2.33. *Let (X, τ) be a space and $A \subset X$. Then A is θ - g -closed subset of X if and only if $A = F - N$, where F is θ -closed and N contains no nonempty closed set.*

Corollary 2.34 (see [12, Corollary 2.3]). *Let (X, τ) be a space and $A \subset X$. Then A is g -closed if and only if $A = F - N$, where F is closed and N contains no nonempty closed set.*

Theorem 2.35. *Let (X, τ, \mathcal{D}) be an ideal space. If A is a θ - \mathcal{D}_g -closed subset of X and $A \subset B \subset \text{cl}_\theta^*(A)$, then B is also θ - \mathcal{D}_g -closed.*

Proof. $\text{cl}_\theta^*(B) - B \subset \text{cl}_\theta^*(A) - A$, and since $\text{cl}_\theta^*(A) - A$ has no nonempty closed subset, neither does $\text{cl}_\theta^*(B) - B$. By Theorem 2.3, B is θ - \mathcal{D}_g -closed. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.35, we get Corollary 2.36. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.35, we get Corollary 2.37.

Corollary 2.36. *Let (X, τ) be a space. If A is a θ - g -closed subset of X and $A \subset B \subset \text{cl}_\theta(A)$, then B is also θ - g -closed.*

Corollary 2.37 (see [6, Theorem 2.8]). *Let (X, τ) be a space. If A is a g -closed subset of X and $A \subset B \subset \text{cl}(A)$, then B is also g -closed.*

A subset A of an ideal space (X, τ, \mathcal{D}) is said to be θ - \mathcal{D}_g -open if $X - A$ is θ - \mathcal{D}_g -closed.

Theorem 2.38. *A subset A of an ideal space (X, τ, \mathcal{D}) is θ - \mathcal{D}_g -open if and only if $F \subset \text{int}_\theta^*(A)$ whenever F is closed and $F \subset A$.*

Proof. Suppose A is a θ - \mathcal{D}_g -open set and F is a closed set contained in A , then $X - A \subset X - F$ and $X - F$ is open. Since $X - A$ is θ - \mathcal{D}_g -closed, $\text{cl}_\theta^*(X - A) \subset (X - F)$ and so $F \subset X - \text{cl}_\theta^*(X - A) = \text{int}_\theta^*(A)$. Conversely, suppose $X - A \subset U$ and $X - U$ is closed. By hypothesis, $X - U \subset \text{int}_\theta^*(A)$, which implies that $\text{cl}_\theta^*(X - A) = X - \text{int}_\theta^*(A) \subset U$. Therefore, $X - A$ is θ - \mathcal{D}_g -closed and hence A is θ - \mathcal{D}_g -open. \square

If we put $\mathcal{D} = \{\phi\}$ in Theorem 2.38, we get Corollary 2.39. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.38, we get Corollary 2.40.

Corollary 2.39. *A subset A of a space (X, τ) is θ - g -open if and only if $F \subset \text{int}_\theta(A)$ whenever F is closed and $F \subset A$.*

Corollary 2.40 (see [6, Theorem 4.2]). *A subset A of a space (X, τ) is g -open if and only if $F \subset \text{int}(A)$ whenever F is closed and $F \subset A$.*

Theorem 2.41. *Let (X, τ, \mathcal{D}) be an ideal space and $A \subset U$. Then the following are equivalent.*

- (a) A is $\theta\text{-}\mathcal{D}_g$ -closed.
- (b) $A \cup (X - \text{cl}_\theta^*(A))$ is $\theta\text{-}\mathcal{D}_g$ -closed.
- (c) $\text{cl}_\theta^*(A) - A$ is $\theta\text{-}\mathcal{D}_g$ -open.

Proof. (a) \Rightarrow (b). Suppose A is $\theta\text{-}\mathcal{D}_g$ -closed. If U is any open set containing $A \cup (X - \text{cl}_\theta^*(A))$, then $X - U \subset X - (A \cup (X - \text{cl}_\theta^*(A))) = \text{cl}_\theta^*(A) - A$. Since A is $\theta\text{-}\mathcal{D}_g$ -closed, by Theorem 2.3(c), it follows that $X - U = \emptyset$ and so $X = U$. Since X is the only open set containing $A \cup (X - \text{cl}_\theta^*(A))$, $A \cup (X - \text{cl}_\theta^*(A))$ is $\theta\text{-}\mathcal{D}_g$ -closed.

(b) \Rightarrow (a). Suppose $A \cup (X - \text{cl}_\theta^*(A))$ is $\theta\text{-}\mathcal{D}_g$ -closed. If F is any closed set contained in $\text{cl}_\theta^*(A) - A$, then $A \cup (X - \text{cl}_\theta^*(A)) \subset X - F$ and $X - F$ is open. Therefore, $\text{cl}_\theta^*(A \cup (X - \text{cl}_\theta^*(A))) \subset X - F$, which implies that $\text{cl}_\theta^*(A) \cup \text{cl}_\theta^*(X - \text{cl}_\theta^*(A)) \subset X - F$ and so $X \subset X - F$; it follows that $F = \emptyset$. Hence A is $\theta\text{-}\mathcal{D}_g$ -closed.

The equivalence of (b) and (c) follows from the fact that $X - (\text{cl}_\theta^*(A) - A) = A \cup (X - \text{cl}_\theta^*(A))$. \square

If we put $\mathcal{D} = \{\emptyset\}$ in Theorem 2.41, we get Corollary 2.42. If we put $\mathcal{D} = \wp(X)$ in Theorem 2.41, we get Corollary 2.43.

Corollary 2.42. *Let (X, τ) be a space and $A \subset U$. Then the following are equivalent.*

- (a) A is θ - g -closed.
- (b) $A \cup (X - \text{cl}_\theta(A))$ is θ - g -closed.
- (c) $\text{cl}_\theta(A) - A$ is θ - g -open.

Corollary 2.43. *Let (X, τ) be an ideal space and $A \subset U$. Then the following are equivalent.*

- (a) A is g -closed.
- (b) $A \cup (X - \text{cl}(A))$ is g -closed.
- (c) $\text{cl}(A) - A$ is g -open.

3. Characterization of $\mathcal{T}_{1/2}$ and $\mathcal{T}_\mathcal{D}$ -Space

Theorem 3.1. *In an ideal space (X, τ, \mathcal{D}) , the following are equivalent.*

- (a) Every θ - g -closed set is closed.
- (b) (X, τ) is a $\mathcal{T}_{1/2}$ -space.
- (c) Every $\theta\text{-}\mathcal{D}_g$ -closed set is closed.

Proof. (a) \Leftrightarrow (b). Equivalence of (a) and (b) follows from Theorem 4.1 of [8].

(b) \Rightarrow (c). Let A be a $\theta\text{-}\mathcal{D}_g$ -closed set. Since every $\theta\text{-}\mathcal{D}_g$ -closed set is g -closed, A is g -closed. By hypothesis, A is closed.

(c) \Rightarrow (b). Let $x \in X$. If $\{x\}$ is not closed, then $B = X - \{x\}$ is not open. So B is $\theta\text{-}\mathcal{D}_g$ -closed. By hypothesis, B is closed and so $\{x\}$ is open. By Corollary 2.31, (X, τ) is a $\mathcal{T}_{1/2}$ -space. \square

Theorem 3.2. *In an ideal space (X, τ, \mathcal{D}) the following, are equivalent.*

- (a) Every θ - g -closed set is \star -closed.

- (b) (X, τ, \mathcal{D}) is a $\mathcal{T}_{\mathcal{D}}$ -Space.
 (c) Every θ - \mathcal{D}_g -closed set is \star -closed.

Proof. (a) \Rightarrow (b). Let $x \in X$. If $\{x\}$ is not closed, then X is the only open set containing $X - \{x\}$ and so $X - \{x\}$ is θ - g -closed. By hypothesis, $X - \{x\}$ is \star -closed. Equivalently $\{x\}$ is \star -open. Thus, every singleton set in X is either closed or \star -open. By Theorem 3.3 of [4], (X, τ, \mathcal{D}) is a $\mathcal{T}_{\mathcal{D}}$ -Space.

(b) \Rightarrow (a). The proof follows from the fact that every θ - g -closed set is \mathcal{D}_g -closed.

(b) \Rightarrow (c). The proof follows from the fact that every set is θ - \mathcal{D}_g -closed \mathcal{D}_g -closed.

(c) \Rightarrow (b). Let $x \in X$. If $\{x\}$ is not closed, then X is the only open set containing $x - \{x\}$ and so $x - \{x\}$ is θ - \mathcal{D}_g -closed. By hypothesis, $X - \{x\}$ is \star -closed. Thus, $\{x\}$ is \star -open. Therefore, every singleton set in X is either \star -open or closed. By Theorem of 3.3 [4], (X, τ, \mathcal{D}) is a $\mathcal{T}_{\mathcal{D}}$ -Space. \square

The proof of the Corollary 3.3 follows from Theorem 3.2 and Theorem 3.10 of [5].

If we put $\mathcal{D} = \{\emptyset\}$ in Corollary 3.3, we get Corollary 3.4.

Corollary 3.3. *In an ideal space (X, τ, \mathcal{D}) , the following are equivalent.*

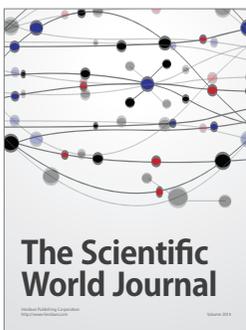
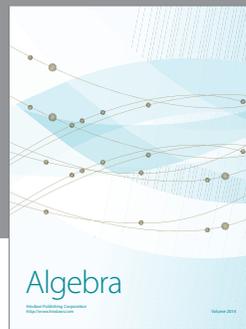
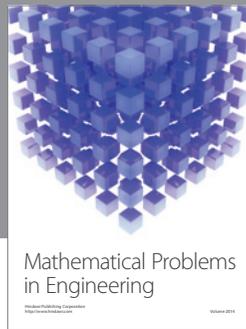
- (a) Every θ - g -closed set is \star -closed.
 (b) Every θ - \mathcal{D}_g -closed set is \star -closed.
 (c) Every \mathcal{D}_g -closed set is an \mathcal{D} -locally \star -closed set.

Corollary 3.4. *In a topological space (X, τ) , the following are equivalent.*

- (a) Every θ - g -closed set is closed.
 (b) Every g -closed set is a locally closed set.

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