International Scholarly Research Network
ISRN Geometry
Volume 2012, Article ID 757489, 12 pages
doi:10.5402/2012/757489

Research Article

Algebraic Characterization of Isometries of the Hyperbolic 4-Space

Krishnendu Gongopadhyay

Indian Institute of Science Education and Research (IISER) Mohali, Knowledge City, SAS Nagar, Sector 81, P.O. Mohali 140306, India

Correspondence should be addressed to Krishnendu Gongopadhyay, krishnendug@gmail.com

Received 20 November 2011; Accepted 8 December 2011

Academic Editor: S. Hernández

We classify isometries of the real hyperbolic 4-space by their conjugacy classes of centralizers. We use the representation of the isometries by 2×2 quaternionic matrices to obtain this characterization. Another characterization in terms of conjugacy invariants is also given.

1. Introduction

Let $\mathbf{H}^n$ denote the n-dimensional real hyperbolic space. The isometries of $\mathbf{H}^n$ are always assumed to be orientation preserving unless stated otherwise. The isometries of $\mathbf{H}^2$ can be identified with the group $PSL(2, \mathbb{R})$. This group acts by the real Möbius transformations or the linear fractional transformations $z \mapsto (az+b)/(cz+d)$ on the hyperbolic plane. Similarly, $PSL(2, \mathbb{C})$ acts on $\mathbf{H}^3$ by complex Möbius transformations. Classically the isometries of $\mathbf{H}^n$ are classified according to their fixed-point dynamics into three mutually exclusive classes: elliptic, parabolic, and hyperbolic. Recall that an isometry g is elliptic if it has a fixed point on $\mathbf{H}^n$; it is parabolic, respectively, hyperbolic if g has no fixed point on $\mathbf{H}^n$ and exactly one, respectively, two fixed points on the boundary $\partial\mathbf{H}^n$. It is well known that for $\mathbf{H}^2$ these types are characterized algebraically in terms of their traces, compare with [1]. This classification is of fundamental importance in dynamics, arithmetic, and geometry of $\mathbf{H}^2$. Analogous characterization of isometries of $\mathbf{H}^3$ in terms of their traces is also well known, compare with [1], also see [2, Appendix-A]. In higher dimensions, the isometries of the n-dimensional hyperbolic space $\mathbf{H}^n$ can be identified with 2×2 matrices over the Clifford numbers, see [3, 4]. Using the Clifford algebraic approach, a characterization of the isometries was obtained by Wada [5]. However, in higher dimensions the above trichotomy of isometries can be refined further, see [2], also see [6, 7]. For a complete understanding of the dynamics and geometry of the isometries, it is desirable to characterize these refined classes. Algebraic characterization of the refined classes using Clifford algebra is a daunting task due to the complicated nature of the Clifford numbers.

Between complex and Clifford numbers, there is an intermediate step involving the quaternions. The isometries of $\mathbf{H}^4$ and $\mathbf{H}^5$ can be considered as 2×2 matrices over the quaternions $\mathbb{H}$, where the respective isometry group acts by the quaternionic Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto (az+b)(cz+d)^{-1}. \tag{1.1}$$

It is natural to ask for algebraic characterizations of the quaternionic Möbius transformations of $\mathbf{H}^4$ and $\mathbf{H}^5$ which will generalize the classical trace identities of the real and complex Möbius transformations. Under the above action, the group of invertible quaternionic 2×2 matrices can be identified with the isometries of $\mathbf{H}^5$. Using this identification, the author has obtained an algebraic characterization of the isometries of $\mathbf{H}^5$, see [8, Theorem-1.1]. The problem of classifying the isometries of $\mathbf{H}^5$ has been attempted by many other authors as well, see [7, 9, 10]. In [9, 10], the authors obtained an algebraic characterization of the classical trichotomy of the isometries. Complete characterizations which algebraically classified the refined classes of isometries as well are obtained only in [7, 8]. However, the approaches used in [7, 8] are independent of each other, and hence, the respective characterizations are also of different flavors. Algebraic characterizations of isometries of $\mathbf{H}^4$ are also known due to the work of several authors; most notably among them is the work ofCao et al. [6]. Independently of [6], Kido [11] also provided a classification and algebraic characterization of isometries of $\mathbf{H}^4$. Kido's approach is nearly similar to that of Cao et al. However, Kido also provided a clear classification of the fixed points of the quaternionic Möbius transformations of $\mathbf{H}^4$. Unfortunately, Kido's preprint of 2005 was never published until very recently.

In all the above works, the authors obtained their characterizations using conjugacy invariants of the isometries. Another approach which has been used recently to characterize the isometries algebraically is in terms of the centralizers, up to conjugacy. We call two elements x, y in a group G to be in the same z-class if their centralizers are conjugate in G. Kulkarni has proposed that the notion of z-class may be used to make precise the intuitive idea of "dynamical types" in any "geometry" whose automorphism group contains a copy of G, see [12]. Motivated by Kulkarni's proposal, the z-classes have been used in the classification problem for isometries in [8, 13]. The characterization by z-classes is based purely in terms of the internal structure of the group alone, and this does not involve any conjugacy invariant. This approach is indeed useful in certain contexts, for example, see Remark 3.3 in this paper. The problem of classifying the z-classes in a group is a problem of independent interest as well, for example, see [14]. Using the linear or hyperboloid model, the z-classes in the full isometry group of $\mathbf{H}^n$ have been classified and counted in [2]. It would be interesting to classify the z-classes using the Clifford algebraic representation of the isometry group.

In this paper, we classify the z-classes of isometries of $\mathbf{H}^4$ using the representation of the isometries by quaternionic matrices. We describe the centralizers up to conjugacy in Section 4. The dynamical types of isometries are precisely classified by the isomorphism types of the centralizers, see Theorem 4.1. This demonstrates the usefulness of the z-classes in the classification problem of the isometries. Apart from the z-classes, we obtained another characterization of the isometries in terms of conjugacy invariants. One key idea used in [8] was to consider the quaternions as a subring of the 2×2 complex matrices $M_2(\mathbb{C})$ and then embed the quaternionic matrices into complex matrices. We use this approach for the isometry group of $\mathbf{H^4}$. This approach is different from that of Cao et al. or Kido. Using a geometric and simple approach, the conjugacy classes of isometries of $\mathbf{H^4}$ are obtained in Section 3. After the conjugacy classes are known, the characterization by conjugacy invariants is obtained essentially as an appendix to the author's earlier work [8, Theorem-1.1].

2. Preliminaries

2.1. Classification of Isometries

Before proceeding further, we briefly recall the finer classification of isometries from [2]. The basic idea of the classification is the following.

To each isometry T of $\mathbf{H^n}$, one associates an orthogonal transformation T_o in $SO(n)$. For each pair of complex conjugate eigenvalues $\{e^{i\theta}, e^{-i\theta}\}$, $0 < \theta \leq \pi$, one associates a *rotation angle* θ to T. An isometry is called *k-rotatory elliptic*, respectively, *k-rotatory parabolic*, respectively, *k-rotatory hyperbolic* if it is elliptic, respectively, parabolic, respectively, hyperbolic and has k-rotation angles. A 0-rotatory hyperbolic is called a *stretch*, and a 0-rotatory parabolic is called a *translation*.

To obtain their characterization, Cao et al. [6] also offered a finer classification of the dynamical types of the isometries. The classification of Cao et al. matches with the above classification when restricted to dimension four. However, the terminologies used by these authors are not the same. For the future reader's convenience, we set up a dictionary between the terminologies of [2] at dimension four and that of Cao et al. in the following.

(For $n = 4$) Comparison with the classification of Cao et al. What Cao et al. [6] called *simple elliptics* are the 1-*rotatory elliptics* in this paper. The *simple parabolics*, respectively, *simple hyperbolics* are the *translations*, respectively, *stretches* in this paper. What we call a 2-*rotatory elliptic* is the *compound elliptic* in [6]. The *compound parabolics*, respectively, *compound hyperbolics* in [6] are the 1-*rotatory parabolics*, respectively, *1-rotatory hyperbolics* here.

2.2. The Quaternions

The space of all quaternions $\mathbb{H}$ is the four-dimensional real division algebra with basis $\{1, i, j, k\}$ and multiplication rules $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The multiplicative group of nonzero quaternions is denoted by $\mathbb{H}^*$. For a quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$, we define $\Re x = x_0$ and $\Im x = x_1 i + x_2 j + x_3 k$. The *norm* of x is defined as $|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. The conjugate of x is defined by $\overline{x} = x_0 - x_1 i - x_2 j - x_3 k$.

We choose $\mathbb{C}$ to be the subspace of $\mathbb{H}$ spanned by $\{1, i\}$. With respect to this choice of $\mathbb{C}$, we can write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} j$; that is, every element a in $\mathbb{H}$ can be uniquely expressed as $a = c_0 + c_1 j$, where c_0, c_1 are complex numbers. Similarly we can also write $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$. For a non-zero quaternion λ, the centralizer of λ in $\mathbb{H}$ is $Z(\lambda) = \mathbb{R} + \mathbb{R}\lambda$. If $\lambda \in \mathbb{C}$, then $Z(\lambda) = \mathbb{C}$.

Definition 2.1. Two quaternions a and b are *similar* if there exists a non-zero quaternion v such that $a = vbv^{-1}$.

Proposition 2.2. (see [15]). *Two quaternions are similar if and only if* $\Re a = \Re b$ *and* $|a| = |b|$.

Corollary 2.3. *The similarity class of every quaternion α contains a pair of complex conjugates with absolute-value $|\alpha|$ and real part equal to $\Re\alpha$.*

2.3. The Isometry Group Using Quaternions

2.3.1. The Upper Half-Space Model

First we associate three involutions to the quaternions:

(i) $* : q = q_o + q_i + q_2 j + q_3 k \mapsto q^* = q_o + q_i + q_2 j - q_3 k$. It determines an antiautomorphism of $\mathbb{H}$: $(ab)^* = b^* a^*$, $(a+b)^* = a^* + b^*$,

(ii) $' : q = q_o + q_i + q_2 j + q_3 k \mapsto q' = q_o - q_i - q_2 j + q_3 k$. It determines an automorphism of $\mathbb{H}$: $(ab)' = a'b'$, $(a+b)' = a' + b'$,

(iii) the conjugation $q \mapsto \overline{q}$. This again gives an anti-automorphism of $\mathbb{H}$. Note that $\overline{a} = (a')^* = (a^*)'$.

Following Ahlfors [3] and Waterman [4], we identify $\mathbb{R}^3$ with the additive subspace of the quaternions spanned by $\{1, i, j\}$, that is,

$$\mathbb{R}^3 = \{q \in \mathbb{H} \mid q = q_0 + q_1 i + q_2 j\}. \tag{2.1}$$

We consider the upper half-space model of the hyperbolic space which is given by

$$\mathbf{H}^4 = \{q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H} \mid q_3 > 0\}, \tag{2.2}$$

equipped with the metric induced from the differential

$$ds = \frac{|dq|}{q_3}. \tag{2.3}$$

The boundary of $\mathbf{H^4}$ is identified with $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$.

Let $Sl(2,\mathbb{H})$ be the subgroup of $GL(2,\mathbb{H})$ given by

$$Sl(2,\mathbb{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{H}) \mid ab^*,\ cd^*,\ c^*a,\ d^*b \in \mathbb{R}^3,\ ad^* - bc^* = 1 \right\}. \tag{2.4}$$

The group $Sl(2,\mathbb{H})$ acts on $\mathbf{H^4}$ by the linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : q \longmapsto (aq+b)(cq+d)^{-1}. \tag{2.5}$$

Then, $PSl(2,\mathbb{H}) = Sl(2,\mathbb{H})/\{\pm I\}$ is the group of orientation-preserving isometries of $\mathbf{H^4}$.

2.3.2. The Ball Model

The ball model of the hyperbolic space is given by

$$\mathbf{D}^4 = \{z \in \mathbb{H} \mid |z| = 1\}, \tag{2.6}$$

equipped with the hyperbolic metric $ds = 2|dq|/(1 - |q|^2)$. The isometry group in the ball model is given by $U(1,1;\mathbb{H})$ which acts by the linear fractional transformations, and an isometry in $U(1,1;\mathbb{H})$ is of the form, see [6, Lemma 1.1],

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad |a| = |d|, \quad |b| = |c|, \quad |a|^2 - |c|^2 = 1, \quad \overline{a}b = \overline{c}d, \quad a\overline{c} = b\overline{d}. \tag{2.7}$$

The diffeomorphism which identifies the disk model $\mathbf{D}^4$ to the upper half-space model $\mathbf{H}^4$ is given by $f : z \mapsto (z + k)(kz + 1)^{-1}$. The matrix

$$f_k = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix} \tag{2.8}$$

acts as the quaternionic linear fractional transformation f on $\widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$. This implies that $PSl(2,\mathbb{H})$ and $PU(1,1;\mathbb{H})$ are conjugate in $PGL(2,\mathbb{H})$.

3. The Conjugacy Classes

3.1. The Conjugacy Classes

Lemma 3.1. *Let A be an element in $Sl(2,\mathbb{H})$.*

(i) *If A acts as a 1-rotatory elliptic, then A is conjugate to $D_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, $0 < \theta < \pi$. If $\theta = 0$ or π, A acts as the identity.*

(ii) *If A acts as a 1-rotatory parabolic, then A is conjugate to $T_{\theta,j} = \begin{pmatrix} e^{i\theta} & j \\ 0 & e^{-i\theta} \end{pmatrix}$, or $-T_{\theta,j}$, $0 < \theta < \pi$.*

(iii) *If A acts as a translation, then A is conjugate to $T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $-T_1$.*

(iv) *If A acts as a 1-rotatory hyperbolic, then A is conjugate to either $D_{r,\theta} = \begin{pmatrix} re^{i\theta} & 0 \\ 0 & r^{-1}e^{-i\theta} \end{pmatrix}$ or $D_{-r,\theta}$, $0 < \theta < \pi$, $r > 0$.*

(v) *If A acts as a stretch, then A is conjugate to $D_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$, or D_{-r}.*

(vi) *Finally, if A acts as a 2-rotatory elliptic, then A has a unique fixed point on $\boldsymbol{H}^4$ and A cannot conjugate to an upper triangular matrix in $Sl(2,\mathbb{H})$. However, in the ball model of $\boldsymbol{H}^4$, A is conjugate to*

$$D_{\theta,\phi} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \quad \theta \neq \pm\phi. \tag{3.1}$$

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The induced linear fractional transformation is given by

$$f_A : q \longmapsto (aq + b)(cq + d)^{-1}. \tag{3.2}$$

Now there are two cases.

Case 1. Suppose f_A has a fixed point on $\mathbb{S}^3$. This is the case precisely when A acts as a parabolic, hyperbolic, or 1-rotatory elliptic isometry. Up to conjugacy, we assume the fixed point to be ∞. So up to conjugacy we can assume $c = 0$, and hence $A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, $ad^* = 1$. Since every quaternion is conjugate to an element in $\mathbb{C}$, let $a = vle^{i\theta}v^{-1}$, where $le^{i\theta}$ is a nonzero complex number. We can further consider v to have unit norm, that is, $|v| = 1$. Using the relation $ad^* = 1$, we see that $d = v'l^{-1}e^{-i\theta}(v')^{-1}$. Let $C = \left(\begin{smallmatrix} v & 0 \\ 0 & v' \end{smallmatrix}\right)$. Since $v(v')^* = v\overline{v} = 1$, hence C is an element in $Sl(2, \mathbb{H})$, and $C^{-1}AC = \left(\begin{smallmatrix} le^{i\theta} & v^{-1}bv' \\ 0 & l^{-1}e^{-i\theta} \end{smallmatrix}\right)$. Since $e^{i\theta} = je^{-i\theta}j$, hence, for $-\pi < \theta < 0$, conjugating $C^{-1}AC$ by $J = \left(\begin{smallmatrix} j & 0 \\ 0 & -j \end{smallmatrix}\right)$, we can further take θ to be in the interval $[0, \pi]$. Hence, up to conjugacy, every element in $Sl(2, \mathbb{H})$ which has a fixed point on $\mathbb{S}^3$ is of the form $T_{l,\theta,q} = \left(\begin{smallmatrix} le^{i\theta} & q \\ 0 & l^{-1}e^{-i\theta} \end{smallmatrix}\right)$, where $l > 0$, $0 \le \theta \le \pi$, $q \in \mathbb{H}$. When $q = 0$, $l = 1$, we denote it by D_θ, and when $l \neq 1$, $\theta = 0$, we denote it by $D_{l,\theta}$.

Now consider an element $T_{l,\theta,q}$ as above, where $q = q_0 + q_1 j$ and $l = 1$. Then, $T_{1,\theta,q}$ acts on $\mathbf{H}^4$ as

$$f_q : z \longmapsto e^{i\theta} z e^{i\theta} + q e^{i\theta}. \tag{3.3}$$

Let $q_0' = q_0 e^{i\theta}$, $q_1' = q_1 e^{-i\theta}$, $y = q_0'(1 - e^{2i\theta})^{-1}$. Conjugating f_q by the map

$$\tau_{-y} : z \longmapsto z - y = x = x_0 + x_1 j, \tag{3.4}$$

we have

$$\begin{aligned} \tau_{-y} f_q \tau_{-y}^{-1}(x) &= \tau_{-y} f(x + y) = \tau_{-y}\left(\left(e^{i2\theta}(x_0 + y) + q_0'\right) + (x_1 + q_1')j\right) \\ &= \left(e^{i2\theta} x_0 + c_0\right) + (x_1 + q_1')j \\ &= e^{i\theta} x e^{i\theta} + q_1 j e^{i\theta} \text{ since } c_0 = \left\{e^{i2\theta} y + q_0' - y\right\} = 0. \end{aligned} \tag{3.5}$$

Thus, $T_{1,\theta,q}$ is conjugate to $T_{1,\theta,q_1 j}$, where $q_1 \in \mathbb{C}$. If A acts as an elliptic, $q_1 = 0$. In this case we denote it by D_θ. If $q_1 \neq 0$, then A acts as a 1-rotatory parabolic when $\theta \neq 0, \pi$. Conjugating it further by a transformation of the form $z \mapsto q_1^{-1} z$, we can further assume, $q_1 = 1$; that is, A is conjugate to $T_{\theta,j}$. If $\theta = 0$ or π, A acts as a translation. In this case, we denote it by T_{q_1}, and further conjugating it by $R_j : z \mapsto -jz$ we get $R_j T_{q_1} R_j^{-1} : z \mapsto z + 1$. Thus, up to conjugation, a translation can be taken as $z \mapsto z + 1$.

Let $l \neq 1$ in the expression of $T_{l,\theta,q}$. Then, it acts as

$$g_q : z \longmapsto l^2 e^{i\theta} z e^{i\theta} + l q e^{i\theta}. \tag{3.6}$$

Let $q_0' = lq_0e^{i\theta}$, $q_1' = lq_1e^{-i\theta}$. Let $w_0 = q_0'(1 - l^2e^{2i\theta})^{-1}$, $w_1 = q_1'(1 - l^2)^{-1}$, $w = w_0 + w_1 j$. Conjugating g_q by the map

$$\tau_{-w} : z \longmapsto z - w = x = x_0 + x_1 j \tag{3.7}$$

we have

$$\begin{aligned}\tau_{-w} g_q \tau_{-w}^{-1}(x) = \tau_{-w} g_q(x + w) &= \tau_{-w}\left(\left(l^2e^{i2\theta}(x_0 + w_0) + q_0'\right) + \left(l^2(x_1 + w_1) + q_1'\right)j\right) \\ &= \left(l^2e^{i2\theta}x_0 + c_0\right) + \left(l^2x_1 + c_1\right)j.\end{aligned} \tag{3.8}$$

Thus, $T_{l,\theta,q}$ is conjugate to $D_{l,\theta}$. When $\theta \neq 0,\ \pi$, it acts as a 1-rotatory hyperbolic. Otherwise, it acts as a stretch.

Case 2. Suppose f_A has no fixed point on $\mathbb{S}^3$. We claim that the fixed point of f_A is unique. To see this, if possible suppose that f_A has at least two fixed points x and y on $\mathbf{H}^4$. Since between two points there is a unique geodesic, f_A must fixes the end points of the geodesic joining x and y; consequently, f_A pointwise fixes the geodesic. Thus, the associated orthogonal transformation must have an eigenvalue 1. Since f_A is orientation preserving, in the hyperboloid model, its representation must have an eigenvalue 1 of multiplicity at least 3, and hence the number of rotation angles can be at most one. Hence, f_A must have a fixed point on $\mathbb{S}^3$. This is a contradiction. Thus, f_A must be a 2-rotatory elliptic with a unique fixed point.

Let A be a 2-rotatory elliptic. Then, A cannot be conjugated to an upper triangular matrix in $Sl(2, \mathbb{H})$. In this case, for computational purpose, it is easier to use the ball model of the hyperbolic space. Up to conjugation, we assume that, in the ball model, A has the unique fixed point 0. Hence $b = 0$. This implies $c = 0$. It follows from [6, Proposition 3.2], that we must have $\Re(a) \neq \Re(d)$; see Section 3.1 of [6]. In particular, a is not similar to d. As in the proof of Lemma 3.1, we may assume by further conjugation that $a, d \in \mathbb{C}$, $|a| = |d| = 1$, and thus we assume

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \quad \theta \neq \pm\phi. \tag{3.9}$$

This completes the proof. □

3.2. Algebraic Characterization

Write $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$. Express $A = A_1 + jA_2$, where $A_1, A_2 \in M_4(\mathbb{C})$. This gives an embedding $A \mapsto A_{\mathbb{C}}$ of $Sl(2, \mathbb{H})$ into $SL(4, \mathbb{C})$, compare with [8, 15], where

$$A_{\mathbb{C}} = \begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{pmatrix}. \tag{3.10}$$

We will use this embedding to characterize the elements in $Sl(2, \mathbb{H})$.

Corollary 3.2. *Embed the group $SL(2, \mathbb{H})$ into $SL(4, \mathbb{C})$. Let f be an orientation-preserving isometry of $\boldsymbol{H}^4$. Let f be induced by A in $SL(2, \mathbb{H})$. Let $A_{\mathbb{C}}$ be the corresponding element in $SL(4, \mathbb{C})$. Let the characteristic polynomial of $A_{\mathbb{C}}$ be*

$$\chi_{A_{\mathbb{C}}}(x) = x^4 - 2a_3x^3 + a_2x^2 - 2a_1x + 1. \tag{3.11}$$

Define

$$c_1 = a_1^2, \qquad c_2 = a_2, \qquad c_3 = a_3^2. \tag{3.12}$$

Then, one has the following.

(i) *A acts as a 2-rotatory elliptic if and only if*

$$c_1 = c_3, \quad c_2 < c_1 + 2. \tag{3.13}$$

(ii) *A acts as a 1-rotatory hyperbolic if and only if*

$$c_1 = c_3, \quad c_2 > c_1 + 2. \tag{3.14}$$

(iii) *A acts as a stretch if and only if*

$$c_1 = c_3, \quad c_2 = c_1 + 2, \quad c_1 > 4. \tag{3.15}$$

(iv) *A acts as a translation if and only if*

$$c_1 = c_3, \quad c_2 = c_1 + 2, \quad c_1 = 4, \tag{3.16}$$

and A is not $\pm I$.

(v) *A acts as a 1-rotatory elliptic or a 1-rotatory parabolic if and only if*

$$c_1 = c_3, \quad c_2 = c_1 + 2, \quad c_1 < 4. \tag{3.17}$$

Moreover, if the characteristic polynomial of $A_{\mathbb{C}}$ is equal to its minimal polynomial, then A acts as a 1-rotatory parabolic. Otherwise, it acts as a 1-rotatory elliptic.

Proof. Note that the coefficients c_1, c_2, and c_3 are conjugacy invariants for $Sl(2, \mathbb{H})$, in fact, for $PSl(2, \mathbb{H})$. Since $PSl(2, \mathbb{H})$ and $PU(1, 1; \mathbb{H})$ are conjugate in $PGL(2, \mathbb{H})$, $c_1, c_2,$ and c_3 serve as conjugacy invariants in the ball model also. Now the result follows from the above conjugacy classification applying [8, Theorem 1.1]. □

Remark 3.3. Observe that the conjugacy invariants c_1, c_2, and c_3 alone cannot distinguish between a 1-rotatory parabolic and a 1-rotatory elliptic with the same "rotation angle" θ; we are required to refer to the respective minimal polynomials to distinguish these classes further. However, without getting into the conjugacy invariants, one may also distinguish them algebraically by their centralizers. As we will see, their centralizers, up to conjugacy, are different and this gives another characterization of the isometries in terms of the z-classes.

4. The Centralizers, up to Conjugacy

Theorem 4.1. *There are seven z-classes of isometries of $\boldsymbol{H}^4$. The representative for each class and the isomorphism type of the centralizers in each class are given as follows:*

(i) *the trivial class: the identity map,*

(ii) *the 2-rotatory elliptics $D_{\theta,\phi} : Z(D_{\theta,\phi}) \approx \mathbb{S}^1 \times \mathbb{S}^1$,*

(iii) *the 1-rotatory elliptics $D_\theta : Z(D_\theta) \approx SL(2,\mathbb{C})$,*

(iv) *the 1-rotatory hyperbolics $D_{r,\theta} : Z(D_{r,\theta}) \approx \mathbb{C}^*$,*

(v) *the stretches $D_r : Z(D_r) \approx \mathbb{H}^*$,*

(vi) *the translations $T_1 : Z(T_1) \approx \mathbb{S}^1 \ltimes \mathbb{R}^3$,*

(vii) *the 1-rotatory translations $T_\theta :\ Z(T_\theta) \approx \mathbb{S}^1 \times \mathbb{C}$.*

Thus, the isometries are classified by the isomorphism classes of the centralizers.

Proof. First we note that the center of $Sl(2,\mathbb{H})$ is given by $\{I, -I\}$ and they form a single z-class. Now suppose $A \neq \pm I$ is given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ab^*, \quad cd^*, \quad c^*a, \quad d^*b \in \mathbb{R}^3, \quad ad^* - bc^* = 1. \tag{4.1}$$

Let $a = a_1 + a_2 j,\ b = b_1 + b_2 j$, and so forth, where for $i = 1, 2$, $a_i, b_i, c_i, d_i \in \mathbb{C}$.

(i) *Centralizer of* 1*-Rotatory Elliptics.* Note that $AD_\theta = D_\theta A$ implies

$$e^{i\theta} a = a e^{i\theta}, \qquad e^{i\theta} b = b e^{-i\theta}, \qquad e^{-i\theta} c = c e^{i\theta}, \qquad e^{-i\theta} d = d e^{-i\theta}. \tag{4.2}$$

This implies $a, d \in Z(e^{i\theta}) = \mathbb{C}$. Further

$$e^{i\theta}(b_1 + b_2 j) = (b_1 + b_2 j) e^{-i\theta} \Longrightarrow e^{i\theta} b_1 = b_1 e^{-i\theta}. \tag{4.3}$$

Since $\theta \neq 0,\ \pi$, this is possible only if $b_1 = 0$. Similarly, $c_1 = 0$. Hence,

$$Z(D_\theta) = \left\{ \begin{pmatrix} a & bj \\ c & dj \end{pmatrix} \mid a, b, c, d \in \mathbb{C},\ ad - b\overline{c} = -j \right\} \approx SL(2,\mathbb{C}). \tag{4.4}$$

(ii) *Centralizer of Translations.* Let $\mathfrak{T}$ denote the group generated by all translations:

$$T_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}^3 - \{0\}, \tag{4.5}$$

then $\mathfrak{T} \approx \mathbb{R}^3$. Up to conjugacy, we consider T_1. It follows from $AT_1 = T_1 A$ that $a = d,\ c = 0$. Hence, A is of the form

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \quad a, b \in \mathbb{H},\ ab^* \in \mathbb{R}^3,\ aa^* = 1. \tag{4.6}$$

Now let $a = a_0 + a_1 i + a_2 j + a_3 k$. Then, $aa^* = 1$ implies $a^* = (1/|a|^2)\ \overline{a}$ which in turn implies that $|a| = 1$ and $a_1 = 0 = a_2$. Thus,

$$Z(T_1) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a = a_0 + a_1 k,\ a_0, a_1 \in \mathbb{R},\ |a| = 1,\ b \in \mathbb{H},\ \overline{a}b \in \mathbb{R}^3 \right\}. \tag{4.7}$$

Further more, we can write

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & \overline{a}b \\ 0 & 1 \end{pmatrix}, \tag{4.8}$$

this implies

$$Z(T_1) = \mathbb{S}^1 \ltimes \mathfrak{T} \approx \mathbb{S}^1 \ltimes \mathbb{R}^3. \tag{4.9}$$

(iii) *Centralizer of 1-Rotatory Parabolics.* Next consider the 1-rotatory parabolic $T_{\theta,j}$. Conjugating it further, we consider the 1-rotatory parabolic

$$T_{\theta, e^{i\theta} j} = \begin{pmatrix} e^{i\theta} & e^{i\theta} j \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{4.10}$$

Note that $T_{\theta, e^{i\theta} j}$ has the Jordan decomposition $T_{\theta, e^{i\theta} j} = D_\theta T_j = T_j D_\theta$, where $T_j = \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$. Hence, $Z(T_{\theta, e^{i\theta} j}) = Z(D_\theta) \cap Z(T_j)$. We see from the equation $AT_j = T_j A$ that $c = 0$, $aj = jd$. From (i), we have $a, d \in \mathbb{C}$, $b_1 = 0$. Now, combining the relations $ad = 1$ and $aj = jd$ implies $d = \overline{a},\ |a| = 1$. Hence, we have

$$Z\left(T_{\theta, e^{i\theta} j}\right) = \left\{ \begin{pmatrix} a & bj \\ 0 & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C},\ |a| = 1 \right\}. \tag{4.11}$$

Let $P = \left(\begin{smallmatrix} a & bj \\ 0 & \bar{a} \end{smallmatrix}\right)$. Then, $P = D_\theta T_{cj}$ for some $0 \le \theta \le 2\pi$, where $a = e^{i\theta}$ and $c \in \mathbb{C}$, in fact $c = a^{-1}bj$. Since matrices of the form D_θ commute with matrices of the form $T_{cj},\ c \in \mathbb{C}$, hence we have

$$Z\left(T_{\theta, e^{i\theta}j}\right) \approx \mathbb{S}^1 \times \mathbb{C}. \tag{4.12}$$

(iv) *Centralizer of Hyperbolics.* Up to conjugacy, we consider $D_{r,\theta} = \left(\begin{smallmatrix} re^{i\theta} & 0 \\ 0 & r^{-1}e^{-i\theta} \end{smallmatrix}\right)$, $0 \le \theta \le \pi$. Then,

$$\begin{gathered} AD_{r,\theta} = D_{r,\theta}A \Longrightarrow re^{i\theta}a = rae^{i\theta}, \qquad re^{i\theta}b = r^{-1}be^{-i\theta}, \\ r^{-1}e^{-i\theta}c = rce^{i\theta}, \qquad r^{-1}e^{-i\theta}d = dr^{-1}e^{-i\theta}. \end{gathered} \tag{4.13}$$

Since $r \ne 1$, this implies $b = 0 = c$, $a, d \in Z(e^{i\theta})$. The equation $ad^* = 1$ implies $d = (a^*)^{-1}$. Note that if $\theta \ne 0$, $Z(e^{i\theta}) = \mathbb{C}$ and if $\theta = 0$, then $Z(1) = \mathbb{H}$.
Thus, *if A is 1-rotatory hyperbolic,* that is, $\theta \ne 0$, then

$$Z(D_{r,\theta}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^* \right\} \approx \mathbb{C}^*. \tag{4.14}$$

If A is stretch, then

$$Z(D_r) = \left\{ \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} \mid a \in \mathbb{H}^* \right\} \approx \mathbb{H}^*. \tag{4.15}$$

(v) *Centralizer of 2-Rotatory Elliptics.* We use the ball model. Up to conjugation, we consider

$$D_{\theta,\phi} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \quad \theta \ne \pm\phi. \tag{4.16}$$

Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ commute with A. The equation $D_{\theta,\phi}T = TD_{\theta,\phi}$ implies (since $\theta \ne \pm\phi$), $a, d \in \mathbb{C}$, $b = 0 = c$. Hence,

$$Z\left(D_{\theta,\phi}\right) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{C},\ |a| = |d| = 1 \right\} \approx \mathbb{S}^1 \times \mathbb{S}^1. \tag{4.17}$$

This completes the proof. □

Acknowledgment

The author gratefully acknowledges the support of SERC-DST FAST Grant SR/FTP/MS-004/2010.

References

[1] A. F. Beardon, *The Geometry of Discrete Groups*, vol. 91 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1983.
[2] K. Gongopadhyay and R. S. Kulkarni, "z-classes of isometries of the hyperbolic space," *Conformal Geometry and Dynamics*, vol. 13, pp. 91–109, 2009.
[3] L. V. Ahlfors, "Möbius transformations and Clifford numbers," in *Differential Geometry and Complex Analysis*, pp. 65–73, Springer, Berlin, Germany, 1985.
[4] P. L. Waterman, "Möbius transformations in several dimensions," *Advances in Mathematics*, vol. 101, no. 1, pp. 87–113, 1993.
[5] M. Wada, "Conjugacy invariants of Möbius transformations," *Complex Variables. Theory and Application*, vol. 15, no. 2, pp. 125–133, 1990.
[6] W. Cao, J. R. Parker, and X. Wang, "On the classification of quaternionic Möbius transformations," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 137, no. 2, pp. 349–361, 2004.
[7] J. R. Parker and I. Short, "Conjugacy classification of quaternionic Möbius transformations," *Computational Methods and Function Theory*, vol. 9, no. 1, pp. 13–25, 2009.
[8] K. Gongopadhyay, "Algebraic characterization of the isometries of the hyperbolic 5-space," *Geometriae Dedicata*, vol. 144, pp. 157–170, 2010.
[9] W. Cao, "On the classification of four-dimensional Möbius transformations," *Proceedings of the Edinburgh Mathematical Society. Series II*, vol. 50, no. 1, pp. 49–62, 2007.
[10] B. Foreman, "Conjugacy invariants of $SL(2,\mathbb{H})$," *Linear Algebra and its Applications*, vol. 381, pp. 25–35, 2004.
[11] T. Kido, "Möbius transformations on quaternions," *Tohoku Mathematical Journal*. In press.
[12] R. S. Kulkarni, "Dynamical types and conjugacy classes of centralizers in groups," *Journal of the Ramanujan Mathematical Society*, vol. 22, no. 1, pp. 35–56, 2007.
[13] W. Cao and K. Gongopadhyay, "Algebraic characterization ofisometries of the complex and the quaternionic hyperbolic planes," *Geometriae Dedicata*, pp. 1–17, 2011.
[14] R. S. Kulkarni, "Dynamics of linear and affine maps," *Asian Journal of Mathematics*, vol. 12, no. 3, pp. 321–343, 2008.
[15] F. Zhang, "Quaternions and matrices of quaternions," *Linear Algebra and its Applications*, vol. 251, pp. 21–57, 1997.