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Review Article

On an Algebraical Computation of the Tensor and the Curvature for 3-Web

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The algebraic methods are used in the web geometry, in particular in the 3-web. Along the line, we suggest a new, alternative algebraic method for computation of the quantities $\overset{1}{\nabla}_l\, a^i_{jk}$, $\overset{2}{\nabla}_l\, a^i_{jk}$, and d^i_{jklm} by means of the embedding of local loops into Lie groups.

1. Introduction

Web geometry is one of the fields of mathematics which springs from two different fields of mathematics, namely, projective differential geometry and nomography. It was derived mostly from projective differential geometry. Initially, projective differential geometry mainly consisted of the study of projective properties of curves and surfaces in $\mathbb{R}^3$, that is, of their differential properties that are invariant up to homographies. Web geometry studied the properties of (curves and) surfaces in ordinary euclidian space that are invariant up to isometric transformations. Gauss and other mathematicians have shown the usefulness of the first and second fundamental forms in the study of surfaces. They also brought to light the relevance of derived concepts, such as the principal, asymptotic, and conjugated directions. When considering the integral curves of these tangent direction fields, the mathematicians of the 19th century were considering what they called 2-nets of lines on surfaces, that is, the data of 2 families of curves, or in more modern terms, 2-webs. It is when they tried to generalize these constructions to the projective differential geometry that some 3-nets projectively attached to surfaces in $\mathbb{R}^3$ quite naturally made their appearance, Darboux introduced a 3-web named after him in [1]. These webs were useful at that time because they encoded properties of the surfaces under study. Thomsen in [2] shows that a surface area in $\mathbb{R}^3$ is isothermally asymptotic if and only if its Darboux 3-web is hexagonal. At that time, the study of 3-web on surfaces from the point of view of projective differential geometry was on the agenda. Thomsens result has this particular feature of characterizing the geometric-differential property of being isothermally asymptotic by a closedness property of more topological nature that is (or not) verified by a configuration traced on the surface itself. It is this feature which struck some mathematicians and led to the study of webs at the beginning of the 1930s. The development of geometry of fiber bundles and foliations stimulates the interest for new investigation of three webs [3–17]. In [18–27], the techniques were developed for webs using the intrinsic geometry structure. In this investigation, we propose to give another approach of computation of some classical relations, using the technique of the projective space. Our approach is based on the embedding of a smooth loop into a Lie group, by means of a closed subgroup. This transports the geometric problem into an abstract algebraic problem, where the 3-web is seen as a homogeneous space coset in a generic position. Using this technique the computation of the tensor structure of local loop is made easier. Therefore, we give an application of the computation of the well-known tensor. We use algebraic methods to compute the relations $\overset{1}{\nabla}_l a^i_{jk}$, $\overset{2}{\nabla}_l a^i_{jk}$, and d^i_{jklm}. The paper is organized as follows. In Section 2, we derive the analytic representation of the law of composition of local smooth loops, embedding in Lie groups. In Section 3, we evaluate tensor structure of a smooth analytic loop. In Section 4, we look at the tensor structure of a smooth local loop, embedding in Lie group. In Section 5 we applied our method to compute $\overset{2}{\nabla}_l a^i_{jk}$ and $\overset{1}{\nabla}_l a^i_{jk}$. In Section 6 we deal with the computation of the tensor $d^i_{jklm} = \overset{2}{\nabla}_m b^i_{jkl}$. The last section is devoted to the hexagonal loops.

2. Analytic Representation of Law of Composition of Local Smooth Loops, Embedding in Lie Groups

Let $\langle G, \cdot, e\rangle$ be a local Lie group and let H be its local closed subgroup. Denote by $\mathfrak{G}$ and $\mathfrak{h}$ their corresponding Lie algebra and Lie subalgebra, and let Q be a smooth space section of left coset $G \mod H$ passing through e the unit element of G $(e \in G)$.

The composition law

$$\begin{gathered} \underline{\times} : Q \times Q \longrightarrow Q, \\ (x, y) \longmapsto x \underline{\times} y = \prod_Q (x \cdot y), \end{gathered} \tag{2.1}$$

where $\prod_Q : G \to Q$ is the projection on Q parallel to the subgroup H, defines in Q a structure of a local loop, that is, $\langle Q, \underline{\times}, e\rangle$-loop [25, 28–36].

Let us map the tangent space T_eQ with the vector subspace $V \subset G$ such that $T_eQ = V$. Then $\mathfrak{G} = V \dotplus \mathfrak{h}$ since the submanifolds Q and H are transversal in the Lie group G.

Let us introduce the mapping ϕ

$$\begin{gathered} \phi : V \longrightarrow \mathfrak{h}, \\ \xi \longmapsto \phi(\xi), \end{gathered} \tag{2.2}$$

defined by the condition $\exp(\xi + \phi(\xi)) \in Q$ (for every vector $\xi \in V$, in the neighborhood of O, and the map ϕ is well defined).

Then $\phi(O) = O$ and

$$\phi(\xi) = R(\xi, \xi) + S(\xi, \xi, \xi) + o(3), \tag{2.3}$$

where

$$\begin{gathered} R : V \times V \longrightarrow \mathfrak{h}, \\ S : V \times V \times V \longrightarrow \mathfrak{h} \end{gathered} \tag{2.4}$$

are bilinear and trilinear symmetric maps. A base $\langle e_1, e_2, \ldots, e_N \rangle$ is fixed in $\mathfrak{G}$ such that $\langle e_1, e_2, \ldots, e_n \rangle$ generates V, that is, $V = \langle e_1, e_2, \ldots, e_n \rangle$ and $\langle e_{n+1}, e_{n+2}, \ldots, e_N \rangle$ generates $\mathfrak{h}$: $\mathfrak{h} = \langle e_{n+1}, e_{n+2}, \ldots, e_N \rangle$. Introduce in the local Lie group G the following normal coordinates: the coordinate on the submanifold Q which is the projection from $\exp V$, that is, for all $x \in Q$, $x = (x^i)_{i=\overline{1,n}}$, this means $\exp(x^i e_i + \phi(x^i e_i)) = x \in Q$.

Introduce the map

$$\begin{gathered} Q \longrightarrow V, \\ x \longmapsto \overline{x} = x^i e_i. \end{gathered} \tag{2.5}$$

Then the condition written before is equivalent to

$$\overline{x} + \phi(\overline{x}) = x \in Q. \tag{2.6}$$

In what follows, we will compute the constructed coordinates, fixed on the submanifold Q.

It is known that the law of composition in a Lie group $G(\cdot)$ has the following representation up to the fourth order in the normal coordinates:

$$\begin{aligned} a \cdot b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] + \frac{1}{12}[b, [b, a]] \\ -\frac{1}{48}[b, [a, [a, b]]] - \frac{1}{48}[a, [b, [a, b]]] + o(4). \end{aligned} \tag{2.4$'$}$$

Consider the coordinate representation of the law of composition $\underline{\times}$, for $y : x = (\overline{x})$ and $y = (\overline{y})$ in Q. We have

$$\begin{aligned} \overline{(x \underline{\times} y)} = \overline{x} + \overline{y} + K(\overline{x}, \overline{y}) + L(\overline{x}, \overline{x}, \overline{y}) + M(\overline{x}, \overline{y}, \overline{y}) \\ + P(\overline{x}, \overline{x}, \overline{x}, \overline{y}) + Q(\overline{x}, \overline{x}, \overline{y}, \overline{y}) + U(\overline{x}, \overline{y}, \overline{y}, \overline{y}) + o(4). \end{aligned} \tag{2.7}$$

(Our notations are similar to the notations of the work [24]).

Denote the right side in (2.7) by $z = (\overline{z})$. Then, for its computation, we obtain the following:

$$\exp(\overline{z} + \phi(\overline{z})) = \exp(\overline{x} + \phi(\overline{x})) \cdot \exp(\overline{y} + \phi(\overline{y}))h, \tag{2.8}$$

where h is an element from $\mathfrak{h}$, and indeed we have $h = h(\overline{x}, \overline{y})$.

The following proposition holds.

Proposition 2.1. *We have*

$$K(\overline{x}, \overline{y}) = \frac{1}{2}\prod[\overline{x}, \overline{y}], \tag{2.9}$$

where $\prod[\overline{x}, \overline{y}]$ *is the projection of the commutator* $[\overline{x}, \overline{y}]$ *on* V *parallel to the subalgebra* $\mathfrak{h}$

$$h(x, y) = -\frac{1}{2}[\overline{x}, \overline{y}] + \frac{1}{2}\prod[\overline{x}, \overline{y}] + 2R(\overline{x}, \overline{y}) + o(2). \tag{2.10}$$

Proof. we use the formulae (2.8). Comparing the terms from V and $\mathfrak{h}$ and considering only the terms of the first order, we obtain that

$$\begin{gathered} \overline{z} = \overline{x} + \overline{y} \in V, \\ h = o \in \mathfrak{h}. \end{gathered} \tag{2.11}$$

For computing the term of the second order, we denote

$$\begin{gathered} \overline{z} = \overline{x} + \overline{y} + K(\overline{x}, \overline{y}) \in V, \\ h = N(\overline{x}, \overline{y}) \in \mathfrak{h}, \end{gathered} \tag{2.12}$$

from (2.8) and considering (2.4) and (2.4′), we have

$$\begin{aligned} &\overline{x} + \overline{y} + K(\overline{x}, \overline{y}) + R(\overline{x}, \overline{x}) + R(\overline{y}, \overline{y}) + 2R(\overline{x}, \overline{y}) \\ &\quad = \overline{x} + \overline{y} + N(\overline{x}, \overline{y}) + R(\overline{x}, \overline{x}) + R(\overline{y}, \overline{y}) + \frac{1}{2}[\overline{x}, \overline{y}], \end{aligned} \tag{2.13}$$

then by comparing term from V and $\mathfrak{h}$ and noting that

$$\frac{1}{2}[\overline{x}, \overline{y}] = \frac{1}{2}\prod[\overline{x}, \overline{y}] + \left(\frac{1}{2}[\overline{x}, \overline{y}] - \frac{1}{2}\prod[\overline{x}, \overline{y}]\right), \tag{2.14}$$

hence

$$K(\overline{x},\overline{y}) = \frac{1}{2}\prod[\overline{x},\overline{y}],$$
$$h(x,y) = -\frac{1}{2}[\overline{x},\overline{y}] + \frac{1}{2}\prod[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y}). \tag{2.15}$$

□

Corollary 2.2. *From Proposition 2.1, it follows that*

$$\overline{(x\underline{\times}y)} = \overline{x} + \overline{y} + \frac{1}{2}\prod[\overline{x},\overline{y}] + o(2). \tag{2.16}$$

Proposition 2.3. *One can show that*

$$L(\overline{x},\overline{x},\overline{y}) = -\frac{1}{6}\prod[\overline{x},[\overline{x},\overline{y}]] + \frac{1}{2}\prod[R(\overline{x},\overline{x}),\overline{y}] + \frac{1}{4}\prod\Big[\overline{x},\prod[\overline{x},\overline{y}]\Big] + \prod[\overline{x},R(\overline{x},\overline{y})],$$
$$M(\overline{x},\overline{y},\overline{y}) = \frac{1}{3}\prod[\overline{y},[\overline{y},\overline{x}]] + \frac{1}{2}\prod[\overline{x},R(\overline{y},\overline{y})] - \frac{1}{4}\prod\Big[\overline{y},\prod[\overline{y},\overline{x}]\Big] + \prod[\overline{y},R(\overline{x},\overline{y})],$$
$$\begin{aligned} h(\overline{x},\overline{y}) = &-\frac{1}{2}[\overline{x},\overline{y}] + \frac{1}{2}\prod[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y}) + R\Big(\overline{x},\prod[\overline{x},\overline{y}]\Big) + 3S(\overline{x},\overline{x},\overline{y}) \\ &+\frac{1}{6}\Lambda[\overline{x},[\overline{x},\overline{y}]] - \frac{1}{4}\Lambda\Big[\overline{x},\prod[\overline{x},\overline{y}]\Big] - \frac{1}{2}\Lambda[R(\overline{x},\overline{x}),\overline{y}] - \Lambda[\overline{x},R(\overline{x},\overline{y})] \\ &+ R\Big(\overline{y},\prod[\overline{x},\overline{y}]\Big) + 3S(\overline{x},\overline{y},\overline{y}) - \frac{1}{3}\Lambda[\overline{y},[\overline{y},\overline{x}]] \\ &+\frac{1}{4}\Lambda\Big[\overline{y},\prod[\overline{y},\overline{x}]\Big] - \frac{1}{2}\Lambda[\overline{x},R(\overline{y},\overline{y})] - \Lambda[\overline{y},R(\overline{x},\overline{y})] + 0(3), \end{aligned} \tag{2.17}$$

where $\Lambda : \mathfrak{G} \to \mathfrak{h}$ *is the projection on* $\mathfrak{h}$ *parallel to* V.

Proof. The proof is based on the direct computation. Denote that

$$\overline{z} = \overline{x} + \overline{y} + \frac{1}{2}[\overline{x},\overline{y}] + L(\overline{x},\overline{x},\overline{y}) + M(\overline{x},\overline{y},\overline{y}),$$
$$h(\overline{x},\overline{y}) = -\frac{1}{2}[\overline{x},\overline{y}] + \frac{1}{2}\prod[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y}) + E(\overline{x},\overline{x},\overline{y}) + F(\overline{x},\overline{y},\overline{y}). \tag{2.18}$$

From (2.8) with the consideration of (2.4) and (2.4′), we obtain the following:

$$\begin{aligned}
&L(\overline{x},\overline{x},\overline{y}) + M(\overline{x},\overline{y},\overline{y}) + R\Big(\overline{x},\prod[\overline{x},\overline{y}]\Big) + R\Big(\overline{y},\prod[\overline{x},\overline{y}]\Big) + S(\overline{x},\overline{x},\overline{x})\\
&\quad + 3S(\overline{x},\overline{y},\overline{y}) + 3S(\overline{x},\overline{x},\overline{y}) + S(\overline{y},\overline{y},\overline{y}) + \cdots\\
&\quad = \frac{1}{12}[\overline{x},[\overline{x},\overline{y}]] + \frac{1}{12}[\overline{y},[\overline{y},\overline{x}]] + E(\overline{x},\overline{x},\overline{y}) + F(\overline{x},\overline{y},\overline{y})\\
&\quad + S(\overline{x},\overline{x},\overline{x}) + S(\overline{y},\overline{y},\overline{y}) + \frac{1}{2}[R(\overline{x},\overline{x}),\overline{y}] + \frac{1}{2}[\overline{x},R(\overline{y},\overline{y})] + \frac{1}{4}\Big[\overline{x}+\overline{y},\prod[\overline{x},\overline{y}]\Big]\\
&\quad - \frac{1}{4}[\overline{x}+\overline{y},[\overline{x},\overline{y}]] + [\overline{x}+\overline{y},R(\overline{x},\overline{y})] + \cdots .
\end{aligned} \tag{2.19}$$

Then by comparing term from V and $\mathfrak{h}$ in the last identity, we obtain the requirement for $L(\overline{x},\overline{x},\overline{y})$, $M(\overline{x},\overline{y},\overline{y})$ and $h(\overline{x},\overline{y})$ in addition

$$\begin{aligned}
E(\overline{x},\overline{x},\overline{y}) &= R\Big(\overline{x},\prod[\overline{x},\overline{y}]\Big) + 3S(\overline{x},\overline{x},\overline{y}) + \frac{1}{6}\Lambda[\overline{x},[\overline{x},\overline{y}]] - \frac{1}{4}\Lambda\Big[\overline{x},\prod[\overline{x},\overline{y}]\Big]\\
&\quad - \frac{1}{2}\Lambda[R(\overline{x},\overline{x}),\overline{y}] - \Lambda[\overline{x},R(\overline{x},\overline{y})],\\
F(\overline{x},\overline{y},\overline{y}) &= R\Big(\overline{y},\prod[\overline{x},\overline{y}]\Big) + 3S(\overline{x},\overline{y},\overline{y}) - \frac{1}{3}\Lambda[\overline{y},[\overline{y},\overline{x}]] + \frac{1}{4}\Lambda\Big[\overline{y},\prod[\overline{y},\overline{x}]\Big]\\
&\quad - \frac{1}{2}\Lambda[\overline{x},R(\overline{y},\overline{y})] - \Lambda[\overline{y},R(\overline{x},\overline{y})].
\end{aligned} \tag{2.20}$$

□

Corollary 2.4. *One can obtain that*

$$\begin{aligned}
\overline{(x\underline{\times}y)} &= \overline{x} + \overline{y} + \frac{1}{2}\prod[\overline{x},\overline{y}] - \frac{1}{6}\prod[\overline{x},[\overline{x},\overline{y}]] + \frac{1}{2}\prod[R(\overline{x},\overline{x}),\overline{y}] + \frac{1}{4}\prod\Big[\overline{x},\prod[\overline{x},\overline{y}]\Big]\\
&\quad + \prod[\overline{x},R(\overline{x},\overline{y})] + \frac{1}{3}\prod[\overline{y},[\overline{y},\overline{x}]] + \frac{1}{2}\prod[\overline{x},R(\overline{y},\overline{y})]\\
&\quad - \frac{1}{4}\prod\Big[\overline{y},\prod[\overline{y},\overline{x}]\Big] + \prod[\overline{y},R(\overline{x},\overline{y})] + o(3).
\end{aligned} \tag{2.21}$$

For the computation of terms of the fourth order, denote that

$$\overline{z} = (2.21) + P(\overline{x},\overline{x},\overline{x},\overline{y}) + Q(\overline{x},\overline{x},\overline{y},\overline{y}) + U(\overline{x},\overline{y},\overline{y},\overline{y}), \tag{2.22}$$

and for $\mathfrak{h}$ to take terms of the third order

$$\begin{aligned} &P(\overline{x},\overline{x},\overline{x},\overline{y}) + Q(\overline{x},\overline{x},\overline{y},\overline{y}) + U(\overline{x},\overline{y},\overline{y},\overline{y}) \\ &\quad = [\overline{x} + R(\overline{x},\overline{x}) + S(\overline{x},\overline{x},\overline{x})] \cdot [\overline{y} + R(\overline{y},\overline{y}) + S(\overline{y},\overline{y},\overline{y})] \\ &\qquad \cdot \left(-\frac{1}{2}\Lambda[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y}) + E(\overline{x},\overline{x},\overline{y}) + F(\overline{x},\overline{y},\overline{y}) + \cdots\right), \end{aligned} \tag{2.23}$$

in the fourth order one needs to compute only the term in V. Conducting the reasoning as in the previous cases one obtains that

$$\begin{aligned} &P(\overline{x},\overline{x},\overline{x},\overline{y}) + Q(\overline{x},\overline{x},\overline{y},\overline{y}) + U(\overline{x},\overline{y},\overline{y},\overline{y}) \underset{=}{\operatorname{mod}\mathfrak{h}} \\ &\quad = \bigg\{\overline{x} + R(\overline{x},\overline{x}) + S(\overline{x},\overline{x},\overline{x}) + \overline{y} + R(\overline{y},\overline{y}) + S(\overline{y},\overline{y},\overline{y}) + \frac{1}{2}[\overline{x},\overline{y}] \\ &\qquad + \frac{1}{2}[\overline{x},R(\overline{y},\overline{y})] + \frac{1}{2}[R(\overline{x},\overline{x}),R(\overline{y},\overline{y})] + \frac{1}{2}[\overline{x},S(\overline{y},\overline{y},\overline{y})] + \frac{1}{2}[S(\overline{x},\overline{x},\overline{x}),\overline{y}] \\ &\qquad + \frac{1}{12}[\overline{x},[\overline{x},\overline{y}]] + \frac{1}{12}[\overline{x},[\overline{x},R(\overline{y},\overline{y})]] + \frac{1}{12}[\overline{y},[\overline{y},\overline{x}]] + \frac{1}{12}[\overline{y},[\overline{y},R(\overline{x},\overline{x})]] \\ &\qquad - \frac{1}{48}[\overline{y},[\overline{x},[\overline{x},\overline{y}]]] - \frac{1}{48}[\overline{x},[\overline{y},[\overline{x},\overline{y}]]] + \cdots\bigg\} \\ &\quad\ \cdot \left(-\frac{1}{2}\Lambda[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y}) + E(\overline{x},\overline{x},\overline{y}) + F(\overline{x},\overline{y},\overline{y}) + \cdots\right) \\ &\quad \underset{=}{\operatorname{mod}\mathfrak{h}} \frac{1}{2}\prod[\overline{x},E(\overline{x},\overline{x},\overline{y})] + \frac{1}{2}\prod[\overline{x},F(\overline{x},\overline{y},\overline{y})] + \frac{1}{2}\prod[\overline{y},E(\overline{x},\overline{x},\overline{y})] \\ &\qquad + \frac{1}{2}\prod[\overline{y},F(\overline{x},\overline{y},\overline{y})] - \frac{1}{8}\prod\left[\prod[\overline{x},\overline{y}],[\overline{x},\overline{y}]\right] + \frac{1}{2}\prod\left[\prod[\overline{x},\overline{y}],R(\overline{x},\overline{y})\right] \\ &\qquad + \frac{1}{12}\prod\left[\overline{x},\left[\overline{x},-\frac{1}{2}\Lambda[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y})\right]\right] + \frac{1}{12}\prod\left[\overline{y},\left[\overline{y},-\frac{1}{2}\Lambda[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y})\right]\right] \\ &\qquad + \frac{1}{12}\prod\left[\overline{x},\left[\overline{y},-\frac{1}{2}\Lambda[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y})\right]\right] + \frac{1}{12}\prod\left[\overline{y},\left[\overline{x},-\frac{1}{2}\Lambda[\overline{x},\overline{y}] + 2R(\overline{x},\overline{y})\right]\right] \\ &\qquad + \frac{1}{2}\prod[\overline{x},S(\overline{y},\overline{y},\overline{y})] + \frac{1}{2}\prod[S(\overline{x},\overline{x},\overline{x}),\overline{y}] + \frac{1}{12}\prod[\overline{x},[\overline{x},R(\overline{y},\overline{y})]] \\ &\qquad + \frac{1}{12}\prod[\overline{y},[\overline{y},R(\overline{x},\overline{x})]] - \frac{1}{48}\prod[\overline{y},[\overline{x},[\overline{x},\overline{y}]]] - \frac{1}{48}\prod[\overline{x},[\overline{y},[\overline{x},\overline{y}]]]. \end{aligned} \tag{2.24}$$

All the equalities in the above expression are modulo $\mathfrak{h}$.

Then the following proposition holds.

Proposition 2.5. *We have the following:*

$$
\begin{aligned}
P(\overline{x},\overline{x},\overline{x},\overline{y}) &= -\frac{1}{2}\prod[\overline{y},S(\overline{x},\overline{x},\overline{x})] + \frac{1}{12}\prod\left[\overline{x},\left[\overline{x},-\frac{1}{12}\Lambda[\overline{x},\overline{y}]+2R(\overline{x},\overline{y})\right]\right] \\
&\quad+\frac{1}{2}\prod[\overline{x},E(\overline{x},\overline{x},\overline{y})], \\
U(\overline{x},\overline{y},\overline{y},\overline{y}) &= \frac{1}{2}\prod[\overline{x},S(\overline{y},\overline{y},\overline{y})] + \frac{1}{12}\prod\left[\overline{y},\left[\overline{y},-\frac{1}{12}\Lambda[\overline{x},\overline{y}]+2R(\overline{x},\overline{y})\right]\right] \\
&\quad+\frac{1}{2}\prod[\overline{y},F(\overline{x},\overline{y},\overline{y})], \\
Q(\overline{x},\overline{x},\overline{y},\overline{y}) &= \frac{1}{2}\prod[\overline{y},E(\overline{x},\overline{x},\overline{y})] + \frac{1}{2}\prod[\overline{x},F(\overline{x},\overline{y},\overline{y})] - \frac{1}{8}\prod\left[\prod[\overline{x},\overline{y}],[\overline{x},\overline{y}]\right] \\
&\quad+\frac{1}{2}\prod\left[\prod[\overline{x},\overline{y}],R(\overline{x},\overline{y})\right] + \frac{1}{12}\prod\left[\overline{x},\left[\overline{y},-\frac{1}{2}\Lambda[\overline{x},\overline{y}]+2R(\overline{x},\overline{y})\right]\right] \\
&\quad+\frac{1}{12}\prod\left[\overline{y},\left[\overline{x},-\frac{1}{2}\Lambda[\overline{x},\overline{y}]+2R(\overline{x},\overline{y})\right]\right] + \frac{1}{12}\prod[\overline{x},[\overline{x},R(\overline{y},\overline{y})]] \\
&\quad+\frac{1}{12}\prod[\overline{y},[\overline{y},R(\overline{x},\overline{x})]] - \frac{1}{48}\prod[\overline{y},[\overline{x},[\overline{x},\overline{y}]]] - \frac{1}{48}\prod[\overline{x},[\overline{y},[\overline{x},\overline{y}]]].
\end{aligned}
\tag{2.25}
$$

Corollary 2.6. *We have the following:*

$$
\begin{aligned}
(\overline{x\underline{\times}y}) &= \overline{x}+\overline{y}+\frac{1}{2}\prod[\overline{x},\overline{y}] - \frac{1}{6}\prod[\overline{x},[\overline{x},\overline{y}]] + \frac{1}{2}\prod[R(\overline{x},\overline{x}),\overline{y}] + \frac{1}{4}\prod\left[\overline{x},\prod[\overline{x},\overline{y}]\right] \\
&\quad+\prod[\overline{x},R(\overline{x},\overline{y})] + \frac{1}{3}\prod[\overline{y},[\overline{y},\overline{x}]] + \frac{1}{2}\prod[\overline{x},R(\overline{y},\overline{y})] - \frac{1}{4}\prod\left[\overline{y},\prod[\overline{y},\overline{x}]\right] \\
&\quad+\prod[\overline{y},R(\overline{x},\overline{y})] + P(\overline{x},\overline{x},\overline{x},\overline{y}) + Q(\overline{x},\overline{x},\overline{y},\overline{y}) + U(\overline{x},\overline{y},\overline{y},\overline{y}) + 0(4),
\end{aligned}
\tag{2.26}
$$

where $P(\overline{x},\overline{x},\overline{x},\overline{y})$, $Q(\overline{x},\overline{x},\overline{y},\overline{y})$, *and* $U(\overline{x},\overline{y},\overline{y},\overline{y})$ *are from* (2.25).

3. Tensor Structure of a Smooth Analytic Loop

Let $\langle Q,\times,e\rangle$ be a smooth analytic loop with the neutral element e. In a standard way, see [26] on the Cartesian product $Q\times Q$, we introduce the structure of a three-web W such that the submanifold in the view of $\{a\}\times Q$ is a vertical foliations ($a\in Q$), $Q\times\{b\}$ is a horizontal foliations ($b\in Q$) and the set $\{(a,b): a\times b=c=\text{conts}\}$ foliations of the third family ($c\in Q$).

In the coordinate $(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n)$, the indicated foliations are described by the system of differential 1-form [18, 19, 21, 23, 28, 37–40]

$$\omega_1^i = o, \qquad \omega_2^i = o, \qquad \omega_3^i = \omega_1^i + \omega_2^i = o, \tag{3.1}$$

where

$$\begin{gathered}
\omega_1^i = P_\alpha^i dx^\alpha, \qquad \omega_2^i = Q_\beta^i dy^\beta, \\
P_\alpha^i(x, y) = \frac{\partial \mu^i}{\partial x^\alpha}, \\
Q_\beta^i(x, y) = \frac{\partial \mu^i}{\partial y^\beta}, \\
\mu^i(x, y) = (x \times y)^i.
\end{gathered} \tag{3.2}$$

In the space of a 3-Web W, introduce the so-called Chern canonical connection $\nabla = (\overset{1}{\nabla}, \overset{2}{\nabla})$ [24, 38].

The indicated connection is described by

$$\begin{gathered}
\omega_j^k = \Gamma_{ij}^k \omega_1^i + \Gamma_{jl}^k \omega_2^j, \\
\Gamma_{ij}^k = -\widetilde{P}_i^\alpha \widetilde{Q}_j^\beta \frac{\partial^2 \mu^k}{\partial x^\alpha \partial y^\beta},
\end{gathered} \tag{3.3}$$

where $\widetilde{P}_i^\alpha$ and $\widetilde{Q}_j^\beta$ are inverse matrices for P_i^α and Q_j^β, respectively, in terms of the following structural equations:

$$\begin{aligned}
d\omega_1^k &= \omega_1^l \wedge \omega_l^k + a_{ij}^k \omega_1^i \wedge \omega_l^j, \\
d\omega_2^k &= \omega_2^l \wedge \omega_l^k - a_{ij}^k \omega_2^i \wedge \omega_2^j, \\
d\omega_j^k &= \omega_j^i \wedge \omega_i^k + b_{jlm}^k \omega_1^l \wedge \omega_2^m,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
a_{ij}^k &= -\frac{1}{2} \frac{\partial^2 \mu^k}{\partial x^\alpha \partial y^\beta} \left(\widetilde{P}_i^\alpha \widetilde{Q}_j^\beta - \widetilde{P}_j^\alpha \widetilde{Q}_i^\beta \right), \\
b_{jlm}^k &= \left(-\frac{\partial^3 \mu^k}{\partial x^\alpha \partial x^\beta \partial y^\gamma} \widetilde{P}_j^\beta + \frac{\partial^3 \mu^k}{\partial x^\alpha \partial y^\beta \partial y^\gamma} \widetilde{Q}_j^\beta \right) \widetilde{P}_l^\alpha \widetilde{Q}_m^\gamma - \Gamma_{pm}^k \frac{\partial^2 \mu^p}{\partial x^\alpha \partial x^\beta} \widetilde{P}_l^\alpha \widetilde{P}_j^\beta \\
&\quad + \Gamma_{lp}^k \frac{\partial^2 \mu^p}{\partial y^\alpha \partial y^\beta} \widetilde{P}_j^\alpha \widetilde{Q}_m^\beta - \Gamma_{pm}^k \Gamma_{lj}^p + \Gamma_{lp}^k \Gamma_{jm}^p.
\end{aligned} \tag{3.5}$$

The Chern connection in the 3-Web associated to the loop $\langle Q, \times, e\rangle$ admits an alternative description in terms of antiproduct of the loop Q by itself [31, 33]. In the set $Q\times Q$, introduce the covering loopuscular structure, by denoting for any pair $X = (x, x')$, $Y = (y, y')$, $A(u, v)$

$$L(X, A, Y) = \left(\left(x(u \setminus yv)\right)/v, u \setminus \left((uy'/v)x'\right)\right). \tag{3.6}$$

Then the Chern connection coincides with the connection tangent to the covering loopuscular structure [33].

In particular, for any tensor field $\Omega(u, v)$, in the space of 3-web $W = Q \times Q$

$$\overset{1}{\nabla}_i \Omega(u = e, v = e) = \frac{\partial}{\partial u^i}\left[\left\{\left[L^{(e,e)}_{(u,e)}\right]_{*,(e,e)}\right\}^{-1}\Omega(u, e)\right]\Bigg|_{u=e}, \tag{3.7}$$

$$\overset{2}{\nabla}_i \Omega(u = e, v = e) = \frac{\partial}{\partial v^i}\left[\left\{\left[L^{(e,e)}_{(e,v)}\right]_{*,(e,e)}\right\}^{-1}\Omega(e, v)\right]\Bigg|_{v=e}. \tag{3.8}$$

The value in the point (e, e) of the 3-Web $W = Q \times Q$ to the loop $\langle Q, \times, e\rangle$ fundamental tensor field a^i_{jk}, b^i_{jkl} and their corresponding derivations $\overset{1}{\nabla}_i$, $\overset{2}{\nabla}_i$ are called the tensors structure of the loop. The structure tensor of the smooth loop $\langle Q, \times, e\rangle$ is defined uniquely by its construction up to isomorphism [24, 28, 29, 38].

Proposition 3.1 (see [17, 38]). *The following relations hold*

$$\begin{aligned}\overset{1}{\nabla}_l a^i_{jk} &= b^i_{[j|l|k]},\\ \overset{2}{\nabla}_l a^i_{jk} &= b^i_{[jk]l}.\end{aligned} \tag{3.9}$$

For the proof of the proposition, it is sufficient to consider the first differential expression of the system (3.4).

Introduce the notation

$$\begin{aligned}c^i_{jklm} &= \overset{1}{\nabla}_m b^i_{jkl}\Big|_{(e,e)},\\ d^i_{jklm} &= \overset{2}{\nabla}_m b^i_{jkl}\Big|_{(e,e)}.\end{aligned} \tag{3.10}$$

And consider Proposition 2.3. The law of composition $(\times)$ of the smooth local loop $\langle Q, \times, e\rangle$ in the coordinate $x = (\overline{x})$ centralized at the point e is given by

$$\begin{aligned}\overline{(x \times y)} = {}&\overline{x} + \overline{y} + K(\overline{x}, \overline{y}) + L(\overline{x}, \overline{x}, \overline{y}) + M(\overline{x}, \overline{y}, \overline{y}) + P(\overline{x}, \overline{x}, \overline{x}, \overline{y})\\ &+ Q(\overline{x}, \overline{x}, \overline{y}, \overline{y}) + U(\overline{x}, \overline{y}, \overline{y}, \overline{y}) + o(4).\end{aligned} \tag{3.11}$$

Consider $\langle Q, \times, e\rangle$ as a coordinate loop of the 3-Web W, defined in the neighborhood of the point (e,e) of the manifold $Q\times Q$. Then in conformity with [24, 37], the basic tensor of the web can be expressed in terms of coefficient of the decomposition of the loop in the following way:

$$\begin{aligned} a(\overline{x},\overline{y}) &= -K(\overline{x},\overline{y}),\\ b(\overline{x},\overline{y},\overline{z}) &= -B(\overline{y},\overline{x},\overline{z}), \end{aligned} \tag{3.12}$$

$$\begin{aligned} c\left(\overline{x},\overline{y},\overline{z},\bar{t}\right) &= (4Q-6P)\left(\overline{y},\bar{t},\mathrm{x},\overline{z}\right) + a\left(\bar{t},b(\overline{x},\overline{y},\overline{z})\right) + a\left(\overline{y},b\left(\overline{x},\bar{t},\overline{z}\right)\right)\\ &\quad - b\left(\overline{x},a\left(\bar{t},\overline{y}\right),\overline{z}\right) + a\left(2L\left(\overline{y},\bar{t},\overline{x}\right),\overline{z}\right) - 2L\left(a(\overline{x},\overline{y}),\bar{t},\overline{z}\right)\\ &\quad - 2L\left(\overline{y},a\left(\overline{x},\bar{t}\right),\overline{z}\right) - 2L\left(\overline{y},\bar{t},a(\overline{x},\overline{z})\right),\\ d\left(\overline{x},\overline{y},\overline{z},\bar{t}\right) &= (4Q-6P)\left(\overline{y},\overline{x},\overline{z},\bar{t}\right) - a\left(b(\overline{x},\overline{y},\overline{z}),\bar{t}\right) - a\left(b\left(\overline{x},\overline{y},\bar{t}\right),\overline{z}\right)\\ &\quad + b\left(\overline{x},\overline{y},a\left(\overline{z},\bar{t}\right)\right) + a\left(\overline{y},2M\left(\overline{x},\overline{z},\bar{t}\right)\right) - 2M\left(a(\overline{y},\overline{x}),\overline{z},\bar{t}\right)\\ &\quad - 2M\left(\overline{y},a(\overline{z},\overline{x}),\bar{t}\right) - 2M\left(\overline{y},\overline{z},a\left(\bar{t},\overline{x}\right)\right), \end{aligned} \tag{3.13}$$

where

$$B(\overline{x},\overline{y},\overline{z}) = 2L(\overline{x},\overline{y},\overline{z}) - 2M(\overline{x},\overline{y},\overline{z}) - K(\overline{x},K(\overline{y},\overline{z})) + K(K(\overline{x},\overline{y}),\overline{z}). \tag{3.14}$$

4. Tensor Structure of a Smooth Local Loop, Embedding in Lie Group

Let $\langle Q, \times, e\rangle$ be a local smooth loop, the embedding in the Lie group G as a section of left coset $G \bmod H$, where H is a closed subgroup in G. In what follows, we will consider that $\langle Q, \times, e\rangle$ is referred to the normal coordinates $X = (\overline{x})$.

Proposition 4.1. *The following relations holds*

$$a(\overline{x},\overline{y}) = -\frac{1}{2}\prod[\overline{x},\overline{y}], \tag{4.1}$$

$$b(\overline{x},\overline{y},\overline{z}) = -\frac{1}{2}\prod[[\overline{x},\overline{y}],\overline{z}] + \frac{1}{2}\prod\left[\prod[\overline{x},\overline{y}],\overline{z}\right] - 2\prod[R(\overline{x},\overline{y}),\overline{z}]. \tag{4.2}$$

Proof. The first relation follows from Proposition 2.1 and the relation (3.12). In the relation (3.14), we have

$$B(\overline{x},\overline{y},\overline{z}) = 2L(\overline{x},\overline{y},\overline{z}) - 2M(\overline{x},\overline{y},\overline{z}) - K(\overline{x},K(\overline{y},\overline{z})) + K(K(\overline{x},\overline{y}),\overline{z}), \tag{4.3}$$

and from Proposition 2.3 we have

$$\begin{aligned}
2L(\overline{x},\overline{y},\overline{z}) &= -\frac{1}{6}\prod[\overline{x},[\overline{y},\overline{z}]] + \prod[R(\overline{x},\overline{y}),\overline{z}] + \frac{1}{4}\prod\Big[\overline{x},\prod[\overline{y},\overline{z}]\Big] - \frac{1}{6}\prod[\overline{y},[\overline{x},\overline{y}]] \\
&\quad + \prod[\overline{x},R(\overline{y},\overline{z})] + \prod[\overline{y},R(\overline{x},\overline{z})] + \frac{1}{4}\prod\Big[\overline{y},\prod[\overline{x},\overline{z}]\Big], \\
2M(\overline{x},\overline{y},\overline{z}) &= \frac{1}{3}\prod[\overline{y},[\overline{z},\overline{x}]] + \prod[\overline{x},R(\overline{y},\overline{z})] - \frac{1}{4}\prod\Big[\overline{y},\prod[\overline{z},\overline{x}]\Big] + \frac{1}{3}\prod[\overline{z},[\overline{y},\overline{x}]] \\
&\quad + \prod[\overline{y},R(\overline{x},\overline{z})] + \prod[\overline{z},R(\overline{x},\overline{y})] - \frac{1}{4}\prod\Big[\overline{z},\prod[\overline{y},\overline{x}]\Big],
\end{aligned} \tag{4.4}$$

Furthermore,

$$\begin{aligned}
K(\overline{x},K(\overline{y},\overline{z})) &= \frac{1}{4}\prod\Big[\overline{x},\prod[\overline{y},\overline{z}]\Big], \\
K(K(\overline{x},\overline{y}),\overline{z}) &= \frac{1}{4}\prod\Big[\prod[\overline{x},\overline{y}],\overline{z}\Big].
\end{aligned} \tag{4.5}$$

Substituting these expressions in $B(\overline{x},\overline{y},\overline{z})$, we obtain that

$$B(\overline{x},\overline{y},\overline{z}) = -\frac{1}{2}\prod[[\overline{x},\overline{y}],\overline{z}] + \frac{1}{2}\prod\Big[\prod[\overline{x},\overline{y}],\overline{z}\Big] + 2\prod[R(\overline{x},\overline{y}),\overline{z}], \tag{4.6}$$

but from (3.12) we have $b(\overline{x},\overline{y},\overline{z}) = -B(\overline{y},\overline{x},\overline{z})$. Hence,

$$b(\overline{x},\overline{y},\overline{z}) = -\frac{1}{2}\prod[[\overline{x},\overline{y}],\overline{z}] - \frac{1}{2}\prod\Big[\prod[\overline{x},\overline{y}],\overline{z}\Big] - 2\prod[R(\overline{x},\overline{y}),\overline{z}]. \tag{4.7}$$

Let Ω be one of the structural tensor of the loop Q, and consider the expression of the fundamental tensor field $\Omega(u,v)$ in the space of three-web $W = Q\times Q$. Then $\Omega = \Omega(u = e, v = e)$ and for $\overset{1}{\nabla}_i\Omega(u = e, v = e)$, $\overset{2}{\nabla}_i\Omega(u = e, v = e)$, the formulae obtained in (3.7) hold.

Consider the computation of $\overset{1}{\nabla}_i\Omega(u = e, v = e)$, the value of the tensor field $\Omega(u,v)$ for $v = e$ can be seen as the structure of the smooth local loop $\langle Q, \underset{u}{\times}, u\rangle$, where

$$x \underset{u}{\times} y = x\times(u\setminus y). \tag{4.8}$$

As a result, ∇ is transported from T_uQ in T_eQ by means of the inverse transformation R_u, which coincide with the structure of the tensor $\widetilde{\Omega_u}$ and the smooth local loop $\langle Q, \underset{u}{\cdot}, e\rangle$ with the operation

$$x \underset{u}{\cdot} y = u\setminus((u\times x)\times y). \tag{4.9}$$

So that

$$\overset{1}{\nabla}_i \Omega(u = e, v = e) = \left.\frac{\partial \widetilde{\Omega_u}}{\partial u^i}\right|_{u=e}, \tag{4.10}$$

in addition the law of composition (4.9) allows an intuitive algebraic interpretation in terms of the enveloping Lie group G.

Consider the section $Q'_u = Q \cdot u^{-1}$ of the coset space $G/\widetilde{H_u}$, where $\widetilde{H_u} = u \cdot H \cdot u^{-1}$, $u \in Q$ and the map

$$\begin{gathered} \Psi_u : Q \longrightarrow Q'_u \\ x \longmapsto (u \times x) \times u^{-1}. \end{gathered} \tag{4.11}$$

Denote by $(\underset{u}{*})$ the law of composition in Q'_u, so that

$$a \underset{u}{*} b = \prod\nolimits'_u (ab), \tag{4.12}$$

where $\prod'_u : G \rightarrow Q'_u$ is the projection on Q'_u parallel to $\widetilde{H_u}$. The following proposition hold. □

Proposition 4.2. *The map $\Psi_u : Q \rightarrow Q'_u$ is an isomorphism of the smooth loops $\langle Q, \underset{u}{\cdot}, e\rangle$ and $\langle Q'_u, \underset{u}{*}, e\rangle$.*

Proof. Let $a = \Psi_u x$, $b = \Psi_u y$, and $a \underset{u}{*} b = \Psi_u z$, where $x, y, z \in Q$.

Then

$$\begin{gathered} a \underset{u}{*} b = \prod\nolimits'_u (ab) = \prod\nolimits'_u \Big((u \times x) \cdot u^{-1} \cdot (u \times y) \cdot u^{-1}\Big), \\ \Big(a \underset{u}{*} b\Big) \times u \cdot h \cdot u^{-1} = (u \times x) u^{-1} \cdot (u \times y) \cdot u^{-1}. \end{gathered} \tag{4.13}$$

Multiplying by u we obtain that

$$\Big(a \underset{u}{*} b\Big) \times u \cdot h = (u \times x) \times y. \tag{4.14}$$

Applying the projection to the last equality, we obtain that

$$\Big(a \underset{u}{*} b\Big) \times u = (u \times x) \times y. \tag{4.15}$$

Furthermore,

$$\left(a \underset{u}{*} b\right) \times u = (\Psi_u z) \times u = (u \times z) \cdot u^{-1} \times u = (u \times x) \times y. \tag{4.16}$$

Then $z = u \setminus (u \times x) \times y$ and

$$\left(a \underset{u}{*} b\right) = (\Psi_u x) \underset{u}{*} (\Psi_u y) = \Psi_u z = \Psi_u \{u \setminus (u \times x) \times y\} = \Psi_u \left(x \underset{u}{\cdot} y\right). \tag{4.17}$$

Therefore $\Psi_u(x \underset{u}{\cdot} y) = (\Psi_u x) * (\Psi_u y)$. Hence, here is the result. Similarly we establish that

$$\overset{2}{\nabla}_i \Omega(u = e, v = e) = \left.\frac{\partial \widetilde{\widetilde{\Omega}}_v}{\partial v^i}\right|_{v=e}, \tag{4.18}$$

where $\widetilde{\widetilde{\Omega}}$ correspond to the structure tensor of the local loop $\langle Q, 1/v, e\rangle$ with the composition law

$$x \frac{1}{v} y = \left(x \times \left(y \times v\right)\right)/v. \tag{4.19}$$

The law of composition (4.19) allows us to find an algebraic interpretation in terms of the enveloping Lie group G. □

Let us introduce in consideration the subgroup $H''_v = vHv^{-1}$ where $v \in Q$. The following proposition holds.

Proposition 4.3. *We have the following:*

$$x \frac{1}{v} y = \prod\nolimits_v'' (xy) \tag{4.20}$$

for all $x, y \in Q$, where

$$\prod\nolimits_v'' : G \to Q \text{ is the projection on } Q \text{ parallel to } H''_v. \tag{4.21}$$

Proof. In the Lie group G, we have $xy = (x \perp y) \times vhv^{-1}$ which is equivalent to $xy \cdot v = (x \perp y) \times vh$. Applying $\prod$ to the last formula, we get the following:

$$x \times \left(y \times v\right) = \left(x \perp y\right) \times v. \tag{4.22}$$

Therefore, $x \perp y = x \times (y \times v)/v$. □

5. Application: Computation of $\overset{2}{\nabla}_l a^i_{jk}$ and $\overset{1}{\nabla}_l a^i_{jk}$

(I) Computation of $\overset{2}{\nabla}_l a^i_{jk}$

For $u \in Q$, introduce the map

$$\begin{aligned} Ad_u : G &\longrightarrow G, \\ x &\longmapsto uxu^{-1}. \end{aligned} \tag{5.1}$$

Let $u = \exp\zeta$, where $\zeta \in Q$ and $g \in H$. Then

$$Ad_u(g) = ugu^{-1} = Ad(\exp\zeta)(g) = \exp(ad\zeta(g)) = g + [\zeta, g] + o(\zeta) \tag{5.2}$$

and $g + [\zeta, g] + o(\zeta) \in H''_u$, where $H''_u = uHu^{-1}$.

Let $\prod''_u : \mathfrak{G} \to T_eQ$ be the projection on T_eQ parallel to $\mathfrak{h}''_u$ and $\exp\mathfrak{h}''_u = H''_u$.

By fixing ξ, η from $\mathfrak{G}$, we find that

$$[\xi, \eta] = \prod[\xi, \eta] + h_1, \tag{5.3}$$

$$[\xi, \eta] = \prod''_u[\xi, \eta] + h_2, \tag{5.4}$$

where $h_1 \in \mathfrak{h}$ and $h_2 \in \mathfrak{h}''_u$. From (5.2) we obtain that h_2 has the form $h_2 = h_1 + \widehat{h}(\zeta) + [\zeta, h_1] + o(\zeta)$, where $\widehat{h}(\zeta) \in \mathfrak{h}''_u$. From (5.3) and (5.4), it follows that

$$\prod''_u[\xi, \eta] = [\xi, \eta] - h_2 = \prod[\xi, \eta] - \widehat{h}(\zeta) - [\zeta, h_1] + o(\zeta) = \prod[\xi, \eta] - \prod[\zeta, h_1] + o(\zeta). \tag{5.5}$$

But from (5.3), we have $h_1 = [\xi, \eta] - \prod[\xi, \eta]$. It follows that

$$\begin{aligned} \prod''_u[\xi, \eta] &= \prod[\xi, \eta] - \prod[\zeta, [\xi, \eta]] + \prod\left[\zeta, \prod[\xi, \eta]\right] + o(\zeta) \\ &= \prod[\xi, \eta] + \prod[[\xi, \eta], \zeta] - \prod\left[\prod[\xi, \eta], \zeta\right] + o(\zeta). \end{aligned} \tag{5.6}$$

Denote by $a''_u(\xi, \eta) = -(1/2)\prod''_u[\xi, \eta]$. Then

$$a''_u(\xi, \eta) = a(\xi, \eta) - \frac{1}{2}\prod[[\xi, \eta]] + \frac{1}{2}\prod\left[\prod[\xi, \eta], \zeta\right]. \tag{5.7}$$

Finally we have

$$\overset{2}{\nabla}_l a^i_{jk}\xi^j\eta^k\zeta^l = \frac{d}{dt}\left(a''_{\exp t\zeta}(\xi, \eta)\right)\Big|_{t=0} = -\frac{1}{2}\prod[[\xi, \eta]] + \frac{1}{2}\prod\left[\prod[\xi, \eta], \zeta\right]. \tag{5.8}$$

We obtain a result in conformity with Proposition 3.1 and the relation (4.2) indeed, from the relation (4.2)

$$b(\xi,\eta,\zeta) = -\frac{1}{2}[[\xi,\eta],\zeta] + \frac{1}{2}\prod\left[\prod[\xi,\eta],\zeta\right] - 2\prod[R(\xi,\eta),\zeta]. \tag{5.9}$$

From which we find that

$$\frac{1}{2}[b(\xi,\eta,\zeta) - b(\eta,\xi,\zeta)] = -\frac{1}{2}\prod[[\xi,\eta],\zeta] + \frac{1}{2}\prod\left[\prod[\xi,\eta],\zeta\right], \tag{5.10}$$

so that $\overset{2}{\nabla}_l a^i_{jk} = b^i_{[jk]l}$.

(II) Computation of $\overset{1}{\nabla}_l a^i_{jk}$

Let us introduce the map

$$\begin{aligned}\Psi_u : Q &\longrightarrow Q'_u,\\ x &\longmapsto (u\times x)u^{-1}.\end{aligned} \tag{5.11}$$

Then $d\Psi_u|_e : T_eQ \to T_eQ'_u$. Then the following proposition holds.

Proposition 5.1. *The map defined from the tangent space T_eQ to tangent space $T_eQ'_u$ is defined as follows:*

$$\begin{aligned}d\Psi_u|_e : T_eQ &\longrightarrow T_eQ'_u,\\ \xi &\longmapsto \xi + \frac{1}{2}[u,\xi] + \frac{1}{2}\prod[u,\xi] + 2R(u,\xi) + o(u).\end{aligned} \tag{5.12}$$

Proof. For the proof of this proposition, using the notion from Section 2 and the relation (2.8), we have $u\times\xi = (u\cdot\xi)\cdot h$ but from Proposition 1.4, we have

$$h(u,\xi) = -\frac{1}{2}[u,\xi] + \frac{1}{2}[u,\xi] + 2R(u,\xi) + o(u). \tag{5.13}$$

Thus,

$$\begin{aligned}u\times\xi = (u\cdot\xi)\cdot h &= u + \xi + \frac{1}{2}[u,\xi] + \frac{1}{2}\prod[u,\xi] + 2R(u,\xi) + o(u),\\ (u\times\xi)\times u^{-1} &= u + \xi + \frac{1}{2}\prod[u,\xi] + 2R(u,\xi) - u - \frac{1}{2}[\xi,u] + o(u)\\ &= \xi + \frac{1}{2}\prod[u,\xi] + \frac{1}{2}[u,\xi] + 2R(u,\xi) + o(u).\end{aligned} \tag{5.14}$$

Let $\widetilde{\prod_u} : \mathfrak{G} \to T_eQ'$ be the projection on T_eQ' parallel to $\widetilde{\mathfrak{h}_u}'$ where $\exp \widetilde{\mathfrak{h}_u}' = uHu^{-1}$. Then we obtain the following:

$$\omega + h_1 = \omega' + h_1' + [u, h_1'] \tag{5.15}$$

with $\omega \in T_eQ$, $h_1 \in \mathfrak{h}$, $\omega' \in T_eQ'$, $h_1' \in \mathfrak{h}$. For the computation of $\omega' = \omega'(u, \omega)$. From Proposition 5.1, we have

$$\omega + h_1 = \tilde{\omega} + \frac{1}{2}\prod[u, \tilde{\omega}] + \frac{1}{2}[u, \tilde{\omega}] + 2R(u, \tilde{\omega}) + h_1' + [u, h_1'] + o(u), \tag{5.16}$$

where $\tilde{\omega} \in T_eQ$, so that

$$\tilde{\omega} + \frac{1}{2}\prod[u, \tilde{\omega}] + \frac{1}{2}[u, \tilde{\omega}] + 2R(u, \tilde{\omega}) = \omega'. \tag{5.17}$$

It follows that

$$\begin{gathered} \omega = \tilde{\omega} + \prod[u, \tilde{\omega}] + [u, h_1'], \\ h_1 = h_1' + \text{terms with } u, \end{gathered} \tag{5.18}$$

from which

$$\begin{gathered} \tilde{\omega} = \omega - \prod[u, \omega] - [u, h_1'], \\ h_1' = h_1 + \text{term with } u. \end{gathered} \tag{5.19}$$

Then substituting in ω' the expression from $\tilde{\omega}$, we obtain that

$$\begin{aligned} \omega' &= \omega - \prod[u, \omega] - \prod[u, h_1] + \frac{1}{2}\prod[u, h_1] + \frac{1}{2}[u, \omega] + 2R(u, \omega) + o(u) \\ &= \omega + \frac{1}{2}[u, \omega] - \frac{1}{2}\prod[u, \omega] - \prod[u, h_1] + 2R(u, \omega) + o(u), \end{aligned} \tag{5.20}$$

from which we find that

$$\widetilde{\prod_u}(\omega + h_1) = \omega' = \omega + \frac{1}{2}[u, \omega] - \frac{1}{2}[u, \omega] + 2R(u, \omega) - \prod[u, h_1]. \tag{5.21}$$

Now let us compute that

$$\widetilde{a_u}(\xi, \eta) = -\frac{1}{2}(d\Psi)^{-1}\widetilde{\prod_u}[d\Psi_\xi, d\Psi_\eta], \tag{5.22}$$

where $\xi, \eta \in T_e Q$,

$$
\begin{aligned}
&(d\Psi)^{-1}\widetilde{\prod_u}[d\Psi_\xi, d\Psi_\eta] \\
&\quad = (d\Psi)^{-1}\widetilde{\prod_u}\Big[\xi + \frac{1}{2}[u,\xi] + \frac{1}{2}\prod[u,\xi] + 2R(u,\xi), \eta + \frac{1}{2}[u,\eta] \\
&\qquad\qquad + \frac{1}{2}\prod[u,\eta] + 2R(u,\eta)\Big] \\
&\quad = (d\Psi)^{-1}\widetilde{\prod_u}\Big\{[\xi,\eta] + \frac{1}{2}[\xi,[u,\eta]] + \frac{1}{2}\Big[\xi,\prod[u,\eta]\Big] + 2[\xi, R(u,\eta)] \\
&\qquad\qquad - \frac{1}{2}[\eta,[u,\xi]] - \frac{1}{2}\Big[\eta,\prod[u,\xi]\Big] - 2[\eta, R(u,\xi)]\Big\} \\
&\quad = (d\Psi)^{-1}\Big\{\prod[\xi,\eta] + \frac{1}{2}\prod[\xi,[u,\eta]] + \frac{1}{2}\prod\Big[\xi,\prod[u,\eta]\Big] + 2\prod[\xi,R(u,\eta)] \\
&\qquad\qquad - \frac{1}{2}\prod[\eta,[u,\xi]] - \frac{1}{2}\prod\Big[\eta,\prod[u,\xi]\Big] - 2\prod[\eta,R(u,\xi)] + \frac{1}{2}\Big[u,\prod[\xi,\eta]\Big] \\
&\qquad\qquad - \frac{1}{2}\Big[u,\prod[\xi,\eta]\Big] + 2R\Big(u,\prod[\xi,\eta]\Big) - \prod[u,[\xi,\eta]] + \prod\Big[u,\prod[\xi,\eta]\Big]\Big\} \\
&\quad = \prod[\xi,\eta] + \frac{1}{2}[\xi,[u,\eta]] + \frac{1}{2}\prod\Big[\xi,\prod[u,\eta]\Big] + 2\prod[\xi,R(u,\eta)] - \frac{1}{2}\prod[\eta,[u,\xi]] \\
&\qquad - \frac{1}{2}\prod\Big[\eta,\prod[u,\xi]\Big] - 2\prod[\eta,R(u,\xi)] - \prod[u,[\xi,\eta]] \\
&\quad = \prod[\xi,\eta] + \frac{1}{2}[\xi,[\eta,u]] - \frac{1}{2}\prod\Big[\xi,\prod[\eta,u]\Big] + 2\prod[\xi,R(u,\eta)] \\
&\qquad - \frac{1}{2}\prod[\eta,[\xi,u]] + \frac{1}{2}\prod\Big[\eta,\prod[\xi,u]\Big] - 2\prod[\eta,R(u,\xi)],
\end{aligned}
\tag{5.23}
$$

where

$$
\begin{aligned}
\widetilde{a_u}(\xi,\eta) = {}& -\frac{1}{2}\prod[\xi,\eta] - \frac{1}{4}\prod[[\xi,u],\eta] + \frac{1}{4}\prod\Big[\prod[\xi,u],\eta\Big] - \prod[R(u,\xi),\eta] \\
& + \frac{1}{4}\prod[[\eta,u],\xi] - \frac{1}{4}\prod\Big[\prod[\eta,u],\xi\Big] + \prod[R(u,\eta),\xi].
\end{aligned}
\tag{5.24}
$$

From this last equation, it follows that

$$
\begin{aligned}
\overset{1}{\nabla}_l a^i_{jk}\xi^j\eta^k\zeta^l &= \frac{d}{dt}\widetilde{a_{\exp t\zeta}}(\xi,\eta)\Big|_{t=0} \\
&= -\frac{1}{4}\prod[[\xi,\zeta],\eta] + \frac{1}{4}\prod\Big[\prod[\xi,\zeta],\eta\Big] - \prod[R(\xi,\zeta),\eta] \\
&\quad + \frac{1}{4}\prod[[\eta,\zeta],\xi] - \frac{1}{4}\prod\Big[\prod[\eta,\zeta],\xi\Big] + \prod[R(\eta,\zeta),\xi].
\end{aligned}
\tag{5.25}
$$

We obtain a result in conformity with Proposition 3.1 and the relation (4.2) indeed from the formulae (4.2), it follows that

$$\begin{aligned}\frac{1}{2}[b(\xi,\zeta,\eta)-b(\eta,\zeta,\xi)] &= \frac{1}{2}\Big\{-\frac{1}{2}\prod[[\xi,\zeta],\eta]+\frac{1}{2}\prod\Big[\prod[\xi,\zeta],\eta\Big]-2\prod[R(\xi,\zeta),\eta]\\&\quad+\frac{1}{2}\prod[[\eta,\zeta],\xi]-\frac{1}{2}\prod\Big[\prod[\eta,\zeta],\xi\Big]+2\prod[R(\eta,\zeta),\xi]\Big\}\\&=-\frac{1}{4}\prod[[\xi,\zeta],\eta]+\frac{1}{4}\prod\Big[\prod[\xi,\zeta],\eta\Big]-\prod[R(\xi,\zeta),\eta]\\&\quad+\frac{1}{4}\prod[[\eta,\zeta],\xi]-\frac{1}{4}\prod\Big[\prod[\eta,\zeta],\xi\Big]+\prod[R(\eta,\zeta),\xi].\end{aligned}\tag{5.26}$$

Therefore,

$$\overset{1}{\nabla}_l a^i_{jk} = b^i_{[j|j|k]}.\tag{5.27}$$

□

6. Computation of the Tensor $d^i_{jklm} = \overset{2}{\nabla}_m b^i_{jkl}$

Denote that $u \cdot R(\eta,\eta) \cdot u^{-1}$ by $R''_u(\eta,\eta)$. For the computation of d^i_{jklm} let us firstly compute $R''_u(\eta,\eta)$.

The following proposition holds.

Proposition 6.1. *We have the following:*

$$R''_u(\eta,\eta) = R(\eta,\eta)+\prod[u,R(\eta,\eta)]+0\big(u,\eta^2\big).\tag{6.1}$$

The proof of this proposition is from Section 1. It is clear that $\xi+\phi(\xi)\in Q$, and from Section 4 $h''_u = h_1+[u,h_1]+0(u)$, where $h_1\in h$. Furthermore $\eta+R''_u(\eta,\eta)\in Q$ but $R''_u(\eta,\eta)\in h''_u$ that is why $R''_u(\eta,\eta)$ can be represented as $R''_u(\eta,\eta)=h_1+[u,h_1]+0(u)$, where $h_1 = R''_u(\eta,\eta)-[u,R''_u(\eta,\eta)]+0(u)$. Let us write $\eta+R''_u(\eta,\eta)$ as

$$\begin{aligned}\eta+R''_u(\eta,\eta) &= \Big\{\Big(\eta+\prod[u,R''_u(\eta,\eta)]\Big)+(R''_u(\eta,\eta)-[u,R''_u(\eta,\eta)])\\&\quad+\Big([u,R''_u(\eta,\eta)]-\prod[u,R''_u(\eta,\eta)]\Big)\Big\},\end{aligned}\tag{6.2}$$

put $\eta+\prod[u,R''_u(\eta,\eta)]=\xi$ then

$$\begin{aligned}\phi(\xi) &= R''_u(\eta,\eta)-[u,R''_u(\eta,\eta)]+[u,R''_u(\eta,\eta)]-\prod[u,R''_u(\eta,\eta)]\\&=R''_u(\eta,\eta)-\prod[u,R''_u(\eta,\eta)]+o(u),\end{aligned}\tag{6.3}$$

from the relation (2.4) we have $\phi(\xi)=R(\xi,\xi)+S(\xi,\xi,\xi)+0(3)$.

Therefore by comparing the term on the right hand sides of the last two relations, we obtain that

$$R''_u(\eta,\eta) = R(\eta,\eta) + \prod[u, R(\eta,\eta)] + 0\left(u,\eta^2\right). \tag{6.4}$$

Let $\prod''_u : \mathfrak{G} \to V = T_eQ$ be the projection of $\mathfrak{G}$ to V parallel to $\mathfrak{h}''_u$. Then we obtain the following:

$$\xi + \widetilde{h} = \widetilde{\xi} + h_1 + [u, h_1], \tag{6.5}$$

where $\xi, \widetilde{\xi} \in V$ and $\widetilde{h}, h_1 \in \mathfrak{h}$ for the search of $\widetilde{\xi} = \widetilde{\xi}(\xi, u)$, we have

$$\xi + \widetilde{h} = \widetilde{\xi} + h_1 + \prod[u, h_1] + \left([u, h_1] - \prod[u, h_1]\right), \tag{6.6}$$

where

$$\begin{gathered} \xi = \widetilde{\xi} + \prod[u, h_1], \\ \widetilde{h} = h_1 + [u, h_1] - \prod[u, h_1] = h_1 + \text{terms with } u. \end{gathered} \tag{6.7}$$

From these two equalities, we obtain that

$$\widetilde{\xi} = \xi - \prod\left[u, \widetilde{h}\right] + 0(u). \tag{6.8}$$

Hence

$$\prod\nolimits''_u\left(\xi + \widetilde{h}\right) = \xi - \prod\left[u, \widetilde{h}\right]. \tag{6.9}$$

We pass now to the computation of d^i_{jklm}.

From (4.2) it follows that

$$b(\xi,\eta,\zeta) = -\frac{1}{2}\prod[[\xi,\eta],\zeta] + \frac{1}{2}\prod\left[\prod[\xi,\eta],\zeta\right] - 2\prod[R(\xi,\eta),\zeta], \tag{6.10}$$

that is why

$$b''_u(\xi,\eta,\zeta) = -\frac{1}{2}\prod\nolimits''_u[[\xi,\eta],\zeta] + \frac{1}{2}\prod\nolimits''_u\left[\prod_u[\xi,\eta],\zeta\right] - 2\prod\nolimits''_u[R''_u(\xi,\eta),\zeta]. \tag{6.11}$$

From (6.9) it follows that

$$-\frac{1}{2}\prod\nolimits''_u[[\xi,\eta],\zeta] = -\frac{1}{2}\prod[[\xi,\eta],\zeta] + \frac{1}{2}\prod[u,[[\xi,\eta],\zeta]] - \frac{1}{2}\prod\left[u,\prod[[\xi,\eta],\zeta]\right]. \tag{6.12}$$

Furthermore,

$$\begin{aligned}
\frac{1}{2}\prod\nolimits_u''\left[\prod\nolimits_u''[\xi,\eta],\zeta\right] &= \frac{1}{2}\prod\nolimits_u''\left[\prod[\xi,\eta],\zeta\right] - \frac{1}{2}\prod\nolimits_u''\left[\prod[u,[\xi,\eta]],\zeta\right] \\
&\quad + \frac{1}{2}\prod\nolimits_u''\left[\prod\left[u,\prod[\xi,\eta]\right],\zeta\right] \\
&= \frac{1}{2}\prod\left[\prod[\xi,\eta],\zeta\right] - \frac{1}{2}\prod\left[u,\left[\prod[\xi,\eta],\zeta\right]\right] + \frac{1}{2}\prod\left[u,\prod\left[\prod[\xi,\eta],\zeta\right]\right] \\
&\quad + o(u).
\end{aligned} \tag{6.13}$$

Finally from (6.1) and (6.9), it follows that

$$\begin{aligned}
-2\prod\nolimits_u''[R_u''(\xi,\eta),\zeta] &= -2\prod\nolimits_u''[R_u(\xi,\eta),\zeta] - 2\prod\nolimits_u''\left[\prod[u,R(\xi,\eta)],\zeta\right] \\
&= -2\prod[R(\xi,\eta),\zeta] + 2\prod[u,[R(\xi,\eta),\zeta]] - 2\prod\left[u,\prod[R(\xi,\eta),\zeta]\right] \\
&\quad - 2\prod\left[\prod[u,R(\xi,\eta)],\zeta\right] + o(u)
\end{aligned} \tag{6.14}$$

from (6.12), (6.13), and (6.14), it follows that

$$\begin{aligned}
d(\xi,\eta,\zeta,\tau) &= \overset{2}{\nabla}_m b^i_{jkl}\Big|_{(e,e)}\xi^j\eta^k\zeta^l\tau^m = \frac{d}{dt}\left(b''_{\exp t\tau}(\xi,\eta,\zeta)\right)\Big|_{t=0} \\
&= \frac{1}{2}\prod[\tau,[[\xi,\eta],\zeta]] - \frac{1}{2}\prod\left[\tau,\prod[[\xi,\eta],\zeta]\right] - \frac{1}{2}\prod\left[\tau,\left[\prod[\xi,\eta],\zeta\right]\right] \\
&\quad + \frac{1}{2}\prod\left[\tau,\prod\left[\prod[\xi,\eta],\zeta\right]\right] - \frac{1}{2}\prod\left[\prod[\tau,[\xi,\eta]],\zeta\right] + \frac{1}{2}\prod\left[\prod\left[\tau,\prod[\xi,\eta]\right],\zeta\right] \\
&\quad + 2\prod[\tau,[R(\xi,\eta),\zeta]] - 2\prod\left[\tau,\prod[R(\xi,\eta),\zeta]\right] - 2\prod\left[\prod[\tau,R(\xi,\eta)],\zeta\right].
\end{aligned} \tag{6.15}$$

In the theory of 3-Web [17, 37, 39], the following relation is known:

$$d^i_{jk[lm]} = -b^i_{jkp}a^p_{lm}. \tag{6.16}$$

Let us verify that:

$$
\begin{aligned}
&\frac{1}{2}(d(\xi,\eta,\zeta,\tau)-d(\xi,\eta,\tau,\zeta))\\
&\quad=\frac{1}{4}\prod[\tau,[[\xi,\eta],\zeta]]-\frac{1}{4}\prod[\zeta,[[\xi,\eta],\tau]]-\frac{1}{4}\prod\Big[\tau,\prod[[\xi,\eta],\zeta]\Big]\\
&\qquad+\frac{1}{4}\prod\Big[\zeta,\prod[[\xi,\eta],\tau]\Big]-\frac{1}{4}\prod\Big[\tau,\Big[\prod[\xi,\eta],\zeta\Big]\Big]+\frac{1}{4}\prod\Big[\zeta,\Big[\prod[\xi,\eta],\tau\Big]\Big]\\
&\qquad+\frac{1}{4}\prod\Big[\tau,\prod\Big[\prod[\xi,\eta],\zeta\Big]\Big]-\frac{1}{4}\prod\Big[\zeta,\prod\Big[\prod[\xi,\eta],\tau\Big]\Big]-\frac{1}{4}\prod\Big[\prod[\tau,[\xi,\eta]],\zeta\Big]\\
&\qquad+\frac{1}{4}\prod\Big[\prod[\zeta,[\xi,\eta]],\tau\Big]+\frac{1}{4}\prod\Big[\prod\Big[\tau,\prod[\xi,\tau]\Big],\zeta\Big]-\frac{1}{4}\prod\Big[\prod\Big[\zeta,\prod[\xi,\eta]\Big],\tau\Big]\\
&\qquad+\prod[\tau,[R(\xi,\eta),\zeta]]-\prod[\zeta,[R(\xi,\eta),\tau]]-\prod\Big[\tau,\prod[R(\xi,\eta),\zeta]\Big]\\
&\qquad+\prod\Big[\zeta,\prod[R(\xi,\eta),\tau]\Big]-\prod\Big[\prod[\tau,R(\xi,\eta)],\zeta\Big]+\prod\Big[\prod[\zeta,R(\xi,\eta)],\tau\Big]\\
&\quad=-\frac{1}{4}\prod[[\xi,\eta],[\zeta,\tau]]+\frac{1}{4}\prod\Big[\prod[\xi,\eta],[\zeta,\tau]\Big]-\prod[R(\xi,\eta),[\zeta,\tau]].
\end{aligned}
\tag{6.17}
$$

In addition, considering that

$$
[\zeta,\tau]=\prod[\zeta,\tau]+\Big([\zeta,\tau]-\prod[\zeta,\tau]\Big).
\tag{6.18}
$$

One obtain that

$$
\begin{aligned}
\frac{1}{2}(d(\xi,\eta,\zeta,\tau)-d(\xi,\eta,\tau,\zeta))&=-\frac{1}{4}\prod\Big[[\xi,\eta],\prod[\zeta,\tau]\Big]+\frac{1}{4}\prod\Big[\prod[\xi,\eta],\prod[\zeta,\tau]\Big]\\
&\quad-\prod\Big[R(\xi,\eta),\prod[\zeta,\tau]\Big].
\end{aligned}
\tag{6.19}
$$

From relations (4.1) and (4.2), it follows that

$$
\begin{aligned}
b(\xi,\eta,a(\zeta,\tau))&=\frac{1}{2}b\Big(\xi,\eta,\prod[\zeta,\tau]\Big)\\
&=-\frac{1}{4}\prod\Big[[\xi,\eta],\prod[\zeta,\tau]\Big]\\
&\quad+\frac{1}{4}\prod\Big[\prod[\xi,\eta],\prod[\zeta,\tau]\Big]-\prod\Big[R(\xi,\eta),\prod[\zeta,\tau]\Big].
\end{aligned}
\tag{6.20}
$$

Hence $d^i_{jk[lm]}=-b^i_{jkp}a^p_{lm}$.

7. Hexagonal Loops

The analytic hexagonal 3-Web and their corresponding loops can be characterize by the following condition:

$$b^i_{(jkl)} = 0, \tag{7.1}$$

where $b(\xi, \eta, \zeta) = -(1/2)\prod[[\xi, \eta], \zeta] + (1/2)\prod[\prod[\xi, \eta], \zeta] - 2\prod[R(\xi, \eta), \zeta]$, that is, way, $b^i_{(jkl)} = 0$ is equivalent to the following condition:

$$\prod[R(\xi, \eta), \zeta] + \prod[R(\eta, \zeta), \xi] + \prod[R(\zeta, \xi), \eta] = 0, \tag{7.2}$$

which can be written as follows:

$$\underset{\xi\eta\zeta}{\sigma}\prod[R(\xi, \eta), \zeta] = 0, \tag{7.3}$$

where $\underset{\xi\eta\zeta}{\sigma}$ is the cyclic sum for ξ, η, ζ.

We have furthermore, for the hexagonal three webs the following relation:

$$d^i_{(jkl)m} = 0. \tag{7.4}$$

Considering (6.15) and (7.2), one obtain that

$$\underset{\xi\eta\zeta}{\sigma}\prod\left\{[\tau, [R(\xi, \eta), \zeta]] - \left[\prod[\tau, R(\xi, \eta)], \zeta\right]\right\} = 0, \tag{7.5}$$

where $\underset{\xi\eta\zeta}{\sigma}$ is the cyclic sum for ξ, η, ζ.

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