International Scholarly Research Network
ISRN High Energy Physics
Volume 2012, Article ID 503621, 12 pages
doi:10.5402/2012/503621

Research Article

On the Algebra of q-Deformed Pseudodifferential Operators

Abderrahman EL Boukili and Moulay Brahim Sedra

Département de Physique, Laboratoire des Hautes Energies, Sciences de l'Ingénierie et Réacteurs (LHESIR), Faculté des Sciences, Université Ibn Tofail, Kénitra, Morocco

Correspondence should be addressed to Abderrahman EL Boukili, aelboukili@gmail.com

Received 14 December 2011; Accepted 4 January 2012

Academic Editor: A. Belhaj

While basing on the study that we we achieved on pseudodifferential operators in the works [arXiv:0708.4046 and hep-th/0610056], we interest in this paper to the construction of the algebra of q-deformed pseudodifferential operators. We use this algebraic structure to study in particular q-Burgers and q-KdV differential operators by the Lax generating technique. We give q-deformed Lax equations as well as the report between these equations through the q-deformed Burgers-KdV mapping.

1. Basic Notions

1.1. q-Pseudodifferential Operators

We start this part with defining the q-derivation. For it, we are going to introduce the general case to know the α-derivation that is defined by

$$\partial_\alpha(fg) = \alpha(f)d_\alpha(g) + d_\alpha(f)g, \tag{1.1}$$

where the two functions f and g are polynomials in an indeterminant x and its inverse x^{-1}.

In (1.1), α is a linear mapping. An example of the α-derivation is given by Jackson's q-differential operator ∂_q, such as [1]

$$\partial_q(f) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{1.2}$$

which gives the following form for (1.1):

$$\partial_q(fg) = \eta_q(f)\cdot\partial_q(g) + \partial_q(f)\cdot g. \tag{1.3}$$

The q-shift operator η_q is given by

$$\eta_q(f(x)) = f(qx). \tag{1.4}$$

One can define the commutation relation as follows:

$$[f,g] = f\circ g - g\circ f, \tag{1.5}$$

where the multiplication law "$\circ$" is

$$\begin{gathered}\partial_q\circ f = \eta_q(f)\partial_q + \partial_q f,\\ \partial_q^{-1}\circ f = \sum_{k\geq 0}(-1)^k q^{-k(k+1)/2}\eta_q^{-k-1}\left(\partial_q^k f\right)\partial_q^{-k-1}.\end{gathered} \tag{1.6}$$

The last equation are obtained by using the following relation:

$$\partial_q^{-1}\circ\partial_q\circ f = \partial_q\circ\partial_q^{-1}\circ f = f, \tag{1.7}$$

where ∂_q^{-1} is the formal inverse of ∂_q.

We should note that η_q does not commute with ∂_q,

$$\partial_q\left(\eta_q^k(f)\right) = q^k\eta_q^k(\partial_q f), \qquad k\in\mathbb{Z} \tag{1.8}$$

or in the following general case:

$$\partial_q^m\left(\eta_q^k(f)\right) = q^{k+m}\eta_q^k\left(\partial_q^m f\right), \qquad k,m\in\mathbb{Z}. \tag{1.9}$$

Note that (1.6) can be unified as follows:

$$\partial_q^n\circ f = \sum_{k\geq 0}\binom{n}{k}_q \eta_q^{n-k}\left(\partial_q^k f\right)\partial_q^{n-k}, \tag{1.10}$$

for all n. In the last equation, the q-binomials take the form

$$\binom{n}{k}_q = \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}, \tag{1.11}$$

and the q-numbers are given by

$$(n)_q = \frac{q^n - 1}{q - 1}, \tag{1.12}$$

where the convention

$$\binom{n}{0}_q = 1, \tag{1.13}$$

is taken.

We can write out several explicit forms of (1.10) for q-derivative ∂_q^n and $\partial_q^{-n} (n \geq 0)$ as

$$\begin{aligned}
\partial_q \circ f &= (\partial_q f) + \eta_q(f)\partial_q, \\
\partial_q^2 \circ f &= \left(\partial_q^2 f\right) + (q+1)\eta_q(\partial_q f)\partial_q + \eta_q^2(f)\partial_q^2, \\
\partial_q^3 \circ f &= \left(\partial_q^3 f\right) + \left(q^2+q+1\right)\eta_q\left(\partial_q^2 f\right)\partial_q + \left(q^2+q+1\right)\eta_q^2(\partial_q f)\partial_q^2 + \eta_q^3(f)\partial_q^3, \\
\partial_q^{-1} \circ f &= \eta_q^{-1}(f)\partial_q^{-1} - q^{-1}\eta_q^{-2}(\partial_q f)\partial_q^{-2} + q^{-3}\eta_q^{-3}\left(\partial_q^2 f\right)\partial_q^{-3} - q^{-6}\eta_q^{-4}\left(\partial_q^3 f\right)\partial_q^{-4} \\
&\quad + \frac{1}{q^{10}}\eta_q^{-5}\left(\partial_q^4 f\right)\partial_q^{-5} + \cdots + (-1)^k q^{-(1+2+3+\cdots+k)}\eta_q^{-k-1}\left(\partial_q^k f\right)\partial_q^{-k-1} + \cdots, \\
\partial_q^{-2} \circ f &= \eta_q^{-2}(f)\partial_q^{-2} - \frac{1}{q^2}(2)_q\eta_q^{-3}(\partial_q f)\partial_q^{-3} + \frac{1}{q^{(2+3)}}(3)_q\eta_q^{-4}\left(\partial_q^2 f\right)\partial_q^{-4} \\
&\quad - \frac{1}{q^{(2+3+4)}}(4)_q\eta_q^{-5}\left(\partial_q^3 f\right)\partial_q^{-5} + \cdots \\
&\quad + \frac{(-1)^k}{q^{(2+3+\cdots+k+1)}}(k+1)_q\eta_q^{-2-k}\left(\partial_q^k f\right)\partial_q^{-2-k} + \cdots.
\end{aligned} \tag{1.14}$$

We also add that the residue of the symbol $\mathcal{L}(x, \partial_q)$ can be written as

$$\mathbf{Res}\left(\sum_{i=-\infty}^{N} u_i(x)\partial_q^i\right) = u_{-1}(x), \tag{1.15}$$

and its **Tr**-functional is

$$\mathbf{Tr}\left(\sum_{i=-\infty}^{N} u_i(x)\partial_q^i\right) = \int_{S^1} u_{-1}(x)dx. \tag{1.16}$$

1.2. Algebraic Structure of q-PDO

Now let us introduce the q-pseudodifferential operators algebra q-PDO. The latter is characterized by the relation [1]:

$$q\text{-PDO} = \left\{ \mathcal{L}(x, \partial_q) = \sum_{i=-\infty}^{N} u_i(x)\partial_q^i \right\}. \tag{1.17}$$

We can noted this space in the following way $\mathfrak{q}\mathcal{A} \equiv q-\Psi\mathrm{DO}$ is seen as being the algebra of all local and nonlocal q-differential operators of arbitrary conformal spins and arbitrary degrees, this spaces can be seen as being the q-deformation of pseudodifferential algebra $\mathcal{A}$ that we saw in [2–11]. One may expand $\mathfrak{q}\mathcal{A}$ as

$$\mathfrak{q}\mathcal{A} = \bigoplus_{m\leq n} \mathfrak{q}\mathcal{A}^{(m,n)} = \bigoplus_{m\leq n} \bigoplus_{s\in\mathbb{Z}} \mathfrak{q}\mathcal{A}_s^{(m,n)}, \qquad m, n, s, \in \mathbb{Z}, \tag{1.18}$$

where we have denoted by (m, n) the lowest and the highest degrees, respectively, and by s the conformal spin. To be explicit, consider the space $\mathfrak{q}\mathcal{A}_s^{(m,n)}$ of q-differential operators:

$$\mathcal{L}_s^{(m,n)} = \sum_{i=m}^{n} u_{s-i}(z)\partial_q^i. \tag{1.19}$$

The vector space $\mathfrak{q}\mathcal{A}^{(m,n)}$ of q-differential operators with given degrees (m, n) but undefined spin

$$\mathfrak{q}\mathcal{A}^{(m,n)} = \bigoplus_{s\in\mathbb{Z}} \mathfrak{q}\mathcal{A}_s^{(m,n)} \tag{1.20}$$

exhibits a Lie algebra's structure with respect to the Lie bracket for $m \leq n \leq 1$.

In fact, It's straightforward to check that the commutator of two operators of $\mathfrak{q}\mathcal{A}_s^{(p,q)}$ is an operator of conformal spin $2s$ and degrees $(p, 2q - 1)$. Since the Lie bracket $[\cdot,\cdot]$ acts as

$$[\cdot,\cdot] : \mathfrak{q}\mathcal{A}_s^{(m,n)} \times \mathfrak{q}\mathcal{A}_s^{(m,n)} \longrightarrow \mathfrak{q}\mathcal{A}_{2s}^{(m,2n-1)}, \tag{1.21}$$

imposing the closure, one gets strong constraints on the spin s and the degrees parameters (m, n), namely,

$$s = 0, \qquad m \leq n \leq 1. \tag{1.22}$$

From these equations, we learn in particular that the spaces $\mathfrak{q}\mathcal{A}_0^{(m,n)}$, $m \leq n \leq 1$ admit a Lie algebra's structure with respect to the bracket (1.5) provided that the Jacobi identity is fulfilled. This can be ensured by showing that the Leibnitz product is associative.

The spaces $\mathfrak{q}\mathcal{A}_0^{(m,n)}$, $m \leq n \leq 1$ as well as the vector space $\mathfrak{q}\mathcal{A}_0^{(0,1)}$ are in fact subalgebra of the Lie algebra $\mathfrak{q}\mathcal{A}_0^{(-\infty,1)}$ which can be decomposed as

$$\mathfrak{q}\mathcal{A}_0^{(-\infty,1)} = \mathfrak{q}\mathcal{A}_0^{(-\infty,-1)} \oplus \mathfrak{q}\mathcal{A}_0^{(0,1)}. \tag{1.23}$$

$\mathfrak{q}A_0^{(-\infty,-1)}$ is nothing but the Lie algebra of Lorentz scalar pure q-pseudodifferential operators of higher degree $n = -1$ and $\mathfrak{q}\mathcal{A}_0^{(0,1)}$ is the central extension of the Lie algebra $\mathfrak{q}\mathcal{A}_0^{(1,1)}$ of vector fields $\mathrm{Diff}(S^1)$:

$$\mathfrak{q}\mathcal{A}_0^{(0,1)} = \mathfrak{q}\mathcal{A}_0^{(0,0)} \oplus \mathfrak{q}\mathcal{A}_0^{(1,1)}, \tag{1.24}$$

and where $\mathfrak{q}\mathcal{A}_0^{(0,0)} \approxeq \mathcal{A}_0^{(0,0)}$ is the one dimensional trivial ideal.

The infinite dimensional huge space $\mathfrak{q}\mathcal{A}$ is the algebra of q-differential operators of arbitrary spins and arbitrary degrees. It's obtained from the space $\mathfrak{q}\mathcal{A}^{(m,n)}$ by summing over all allowed degrees $\mathfrak{q}\mathcal{A}$:

$$\begin{aligned}
\mathfrak{q}\mathcal{A} &= \bigoplus_{m\leq n} \mathfrak{q}\mathcal{A}^{(m,n)} \\
&= \bigoplus_{m\in\mathbb{Z}} \left[\bigoplus_{k\in\mathbb{N}} \mathfrak{q}\mathcal{A}^{(m,m+k)}\right] \\
&= \bigoplus_{m\in\mathbb{Z}} \left[\bigoplus_{k\in\mathbb{N}} \left[\bigoplus_{s\in\mathbb{Z}} \mathfrak{q}\mathcal{A}_s^{(m,m+k)}\right]\right].
\end{aligned} \tag{1.25}$$

This infinite dimensional space which is the combined conformal spin and degrees tensor algebra is closed under the Lie bracket without any constraint.

A remarkable property of $\mathfrak{q}\mathcal{A}$ is that it can splits into six infinite subalgebras $\mathfrak{q}\mathcal{A}_{j+}$ and $\mathfrak{q}\mathcal{A}_{j-}$, $j = 0, \pm 1$ related to each others by conjugation of the spin and degrees. Indeed given two integers m and $n \geq m$, it is not difficult to see that the vector spaces $\mathfrak{q}\mathcal{A}^{(m,n)}$ and $\mathfrak{q}\mathcal{A}^{(-n-1,-m-1)}$ are dual with respect to the pairing product $(\cdot,\cdot)$ defined as

$$\left(\mathcal{L}^{(m,n)}, \mathcal{L}^{(\alpha,\beta)}\right) = \delta_{0,1+m+\beta}\delta_{0,1+n+\alpha}\mathbf{Res}\left[\mathcal{L}^{(m,n)} \circ \mathcal{L}^{(\alpha,\beta)}\right], \tag{1.26}$$

where $d^{(\alpha,\beta)}$ are q-differential operators with fixed degrees $(\alpha, \beta;\ \beta \geq \alpha)$ but arbitrary spin and where the residue operation res is defined as:

$$\mathbf{Res}\left(\partial_q^i\right) = \delta_{0,i+1}. \tag{1.27}$$

This equation shows that the operation res exhibits a conformal spin $\Delta = 1$. Using the properties of this operation and the pairing product (1.26), one can decompose $\mathfrak{q}\mathcal{A}$ as follows:

$$\mathfrak{q}\mathcal{A} = \mathfrak{q}\mathcal{A}_+ \oplus \mathfrak{q}\mathcal{A}_- \tag{1.28}$$

with

$$\mathfrak{q}\mathscr{A}_{+} = \underset{m\geq 0}{\oplus}\left[\underset{k\in\mathbb{N}}{\oplus}\mathfrak{q}\mathscr{A}^{(m,m+k)}\right], \tag{1.29}$$

$$\mathfrak{q}\mathscr{A}_{-} = \underset{m\geq 0}{\oplus}\left[\underset{k\in\mathbb{N}}{\oplus}\mathfrak{q}\mathscr{A}^{(-m-k-1,-m-1)}\right]. \tag{1.30}$$

The indices + and − carried by $\mathfrak{q}\mathscr{A}_{+}$ and $\mathfrak{q}\mathscr{A}_{-}$ refer to the positive (local) and negative (nonlocal) degrees respectively. On the other hand one can decomposes the space $\mathfrak{q}\mathscr{A}^{(m,m+k)},\ k \geq 0$ as

$$\mathfrak{q}\mathscr{A}^{(m,m+k)} = q\Sigma_{-}^{(m,m+k)} \oplus q\Sigma_{0}^{(m,m+k)} \oplus q\Sigma_{+}^{(m,m+k)}. \tag{1.31}$$

$q\Sigma_{-}^{(m,m+k)}$ and $q\Sigma_{+}^{(m,m+k)}$ denote the spaces of q-differential operators of negative and positive definite spin. They are read as

$$q\Sigma_{-}^{(m,m+k)} = \underset{s>0}{\oplus}\mathfrak{q}\mathscr{A}_{-s}^{(m,m+k)}, \tag{1.32}$$

$$q\Sigma_{0}^{(m,m+k)} = \mathfrak{q}\mathscr{A}_{0}^{(m,m+k)}, \tag{1.33}$$

$$q\Sigma_{+}^{(m,m+k)} = \underset{s>0}{\oplus}\mathfrak{q}\mathscr{A}_{s}^{(m,m+k)}. \tag{1.34}$$

$q\Sigma_{0}^{(m,m+k)}$ is just the vector space of Lorenz scalar q-differential operators. Combining (1.28)–(1.34), one sees that $\mathfrak{q}\mathscr{A}$ decomposes into $6 = 3 \times 2$ subalgebras

$$\mathfrak{q}\mathscr{A} = \underset{j=0,+,-}{\oplus}\left[\mathfrak{q}\mathscr{A}_{j+} \oplus \mathfrak{q}\mathscr{A}_{j-}\right] \tag{1.35}$$

with

$$\begin{aligned}\mathfrak{q}\mathscr{A}_{j+} &= \underset{m\geq 0}{\oplus}\left[\underset{k\in\mathbb{N}}{\oplus}q\Sigma_{j}^{(m,m+k)}\right],\\ \mathfrak{q}\mathscr{A}_{j-} &= \underset{m\geq 0}{\oplus}\left[\underset{k\in\mathbb{N}}{\oplus}q\Sigma_{j}^{(-m-k-1,-m-1)}\right].\end{aligned} \tag{1.36}$$

The duality of these $6 = 3 \times 2$ subalgebras is described by the combined scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ built out of the product equation(1.26) and conformal spin pairing:

$$\langle u_k, u_l\rangle = \int dz\, u_k(z) u_{1-k}(z)\delta_{k+l,1}, \tag{1.37}$$

as follows [2, 3]:

$$\left\langle\left\langle \mathcal{L}_s^{(\alpha,\beta)}, \mathcal{L}_r^{(m,n)}\right\rangle\right\rangle = \delta_{0,r+s}\delta_{0,1+n+\alpha}\delta_{0,1+m+\beta}\int dz \operatorname{res}\left[\mathcal{L}_s^{(\alpha,\beta)} \circ \mathcal{L}_{-s}^{(-\beta-1,-\alpha-1)}\right] \tag{1.38}$$

with respect to this new product, $\mathfrak{q}\mathcal{A}_{++}$, $\mathfrak{q}\mathcal{A}_{0+}$, and $\mathfrak{q}\mathcal{A}_{-+}$ behave as the dual algebras of $\mathfrak{q}\mathcal{A}_{--}$, $\mathfrak{q}\mathcal{A}_{0-}$, and $\mathfrak{q}\mathcal{A}_{+-}$, respectively, while $\mathfrak{q}\mathcal{A}_{0-}$ is just the algebra of Lorenz scalar pure q-pseudo operators. This algebra and its dual $\mathfrak{q}\mathcal{A}_{0+}$, the space of Lorenz scalar local q-differential operators, are very special subalgebras as they are systematically used to construct new realizations of the w_i-symmetry, $i \geq 2$ by using scalar q-differential operators type

$$\mathcal{L}^{(k)}(a) = a_{-k}(a)\partial_q^k. \tag{1.39}$$

We note that the space $\mathfrak{q}\mathcal{A}_{++}$ is the algebra of local q-differential operators of positive definite spins and positive degrees. $\mathfrak{q}\mathcal{A}_{--}$, however, is the Lie algebra of pure q-pseudodifferential operators of negative degrees and spins.

2. q-Deformed Lax Generating Technique

The aim of this section is to present some results related to the Lax representation in its q-deformed version. Using the convention notations and the analysis presented previously, we perform consistent algebraic computations, based on the Pseudodifferential analysis, to derive explicit Lax pair operators of some integrable systems in the q-deformation framework.

We underline that the present formulation is based on the (q-pseudo) operators ∂_q^n and ∂_q^{-n} instead of the (pseudo) operators ∂^n and ∂^{-n} used in several works. We note also that the obtained results are shown to be compatible with the ones already established in literature [12–16] in the case of $q = 1$.

The basic idea of the Lax formulation consists first in considering a noncommutative integrable system which possesses the Lax representation:

$$[\mathcal{L}, \partial_t - B]_q = 0, \tag{2.1}$$

with $\partial_t \equiv \partial/\partial t$ et $[f, g]_q = f \circ g - f \circ g$.

Equation (2.1) and the associated pair of operators $(\mathcal{L}, B)$ are called the Lax q-differential equation and the Lax pair, respectively. The q-differential operator $\mathcal{L}$ defines the integrable system which we should fix from the beginning.

Note that the sl_n-KdV hierarchy in the q-deformed version is defined as:

$$\frac{\partial \mathcal{L}}{\partial t_k} = [(\mathcal{L}^{k/2})_+, \mathcal{L}]_q, \tag{2.2}$$

and the way with which ones to writes the Lax q-differential equation as in (2.1) is equivalent to the following equation:

$$[\mathcal{L}, \partial_t - B]_q \equiv [\mathcal{L}, \partial_{t_k} - (\mathcal{L}^{k/2})_+]_q = 0, \tag{2.3}$$

where the operator B is the analogue of $(\mathcal{L}^{k/2})_+$ describing then an q-differential operator of conformal spin k.

Now, let us apply the q-deformation Lax-pair generating technique. We need to find an appropriate operator B which satisfies (2.1), for this we have to make some constraints on the operator B, namely,

Ansatz for the operator B:

$$B = \partial_q^n \circ \mathcal{L}^m + \widetilde{B}, \tag{2.4}$$

with ∂_q^n is the q-differential operator which acts on $\mathcal{L}^m$ according to (1.10) and $\widetilde{B}$ is another operator of same conformal weight than B. Then, with this ansatz, the problem reduces to find the operator $\widetilde{B}$.

To understand the situation, we will study two interesting examples to know q-KdV and q-Burgers equations.

2.1. *q-Deformed Burgers Equations*

The $\mathcal{L}$-operator for the q-deformed Burgers equation is given by

$$\mathcal{L}_{q-\text{burgers}} = \partial_q + u_1 \tag{2.5}$$

with

$$\mathcal{L}_{q-\text{burgers}} \in \mathfrak{q}\mathcal{A}_1^{(0,1)}. \tag{2.6}$$

Let's consider the constraint $n = 1 = m$, for the q-deformed Burgers operator B can be written, from the ansatz (2.4), as follows:

$$\begin{aligned} B &= \partial_q \circ \mathcal{L} + \widetilde{B} \\ &= \partial_q^2 + \eta_q(u_1)\partial_q + \partial_q(u_1) + \widetilde{B}. \end{aligned} \tag{2.7}$$

Simply algebraic computations give

$$\begin{aligned} \left[\mathcal{L}, \widetilde{B}\right] = {} & \left(\eta_q(u_1) - u_1\right)\partial_q^2 \\ & + \left[q\eta_q\left(\partial_q(u_1)\right) + \left(\eta_q(u_1)\right)^2 + \partial_q(u_1) - \partial_q\left(\eta_q(u_1)\right) - u_1\eta_q(u_1)\right]\partial_q \\ & + \eta_q(u_1)\partial_q(u_1) - u_1, \end{aligned} \tag{2.8}$$

where $u_1 = \partial u_1/\partial t$.

Now, our goal is to extract, from (2.1) and (2.8), the Lax equation called q-deformed Burgers or just q-Burgers equation. For this we will follow the following procedure:

Ansatz for the operator $\widetilde{B}$:

$$\widetilde{B} = \alpha\partial_q + \beta, \tag{2.9}$$

where α and β are arbitrary functions on u and its derivatives. one finds

$$\begin{aligned}\left[\mathcal{L},\widetilde{B}\right] = {} & (\eta_q(\alpha)-\alpha)\partial_q^2 \\ & + \left[\partial_q(\alpha)+\eta_q(\beta)+u_1\alpha-\alpha\eta_q(u_1)-\beta\right]\partial_q \\ & + \partial_q(\beta)-\alpha\partial_q(u_1).\end{aligned} \tag{2.10}$$

While identifying the two equations (2.8) and (2.10) we finds

$$a\partial_q^2(u_1)+(b-1)\left[\eta_q(u_1)\partial_q(u_1)+u_1\partial_q(u_1)\right]+u_1=0, \tag{2.11}$$

with a and b are arbitrary real constant.

Equation (2.11) is called *q-deformed Burgers equation* or *q-Burgers equation.* the characteristic of this equation is that it is linear for $b=1$ and that for $q=1$. (i.e., $\eta_q(u_1)=u_1$) we recover the same equation gotten in works [4, 5, 9]

$$au_1''+2(b-1)u_1u_1'+u_1=0. \tag{2.12}$$

2.2. *q-Deformed KdV Equations*

In this second example, we go worked on an q-differentials operator of conformal weight 2, this operator is given by the KdV Lax operator

$$\mathcal{L}_{q\text{-KdV}}=\partial_q^2+u_2. \tag{2.13}$$

We are going to follow the same method of the previous example, therefore the *Ansatz for the operator* B is

$$\begin{aligned}B &= \partial_q\circ\mathcal{L}+\widetilde{B} \\ &= \partial_q^3+\eta_q(u_2)\partial_q+\partial_q(u_2)+\widetilde{B}\end{aligned} \tag{2.14}$$

and the associated Lax equation:

$$[\partial_t-B,\mathcal{L}]_q=0 \tag{2.15}$$

after a calculation, one finds

$$[\mathcal{L},B]_q=-u_1, \tag{2.16}$$

by the same way of the case of Burgers, we finds the following q-KdV equation:

$$u_2=\left[u_2+\eta_q(u_2)\right]\partial_q(u_2)+\partial_q^2\left[\partial_q(u_2)+\eta_q(\partial_q u_2)\right] \tag{2.17}$$

as for $q = 1$, we finds the standard KdV

$$u_2 = u_2 u_2' + u_2'''. \tag{2.18}$$

2.3. q-Deformed Burgers-KdV Mapping

In this section, we present an approach to define the correspondence between integrables systems q-deformed-type Burgers and integrables systems q-deformed-type KdV. such correspondence named *q-deformed* Burgers-KdV *mapping* that is considered like a generalization of the Burgers-KdV mapping studied in works [7, 8, 11, 17].

We illustrate this idea with the example of KdV and Burgers equation and then we are going to make a generalization for cameraman q-differentials-operators-type sl_n-KdV.

Let's consider the Burgers q-differential operator (2.5):

$$\mathcal{L}_{q\text{-burgers}} = \partial_q + u_1 \in \mathfrak{q}\mathcal{A}_1^{(0,1)} \tag{2.19}$$

and the KdV q-differential operator (2.13):

$$\mathcal{L}_{q\text{-KdV}} = \partial_q^2 + u_2 \in \frac{\mathfrak{q}\mathcal{A}_2^{(0,2)}}{\mathfrak{q}\mathcal{A}_2^{(1,1)}}. \tag{2.20}$$

Proposition 2.1 (q-deformed Miura transformation). *If one considers the two previous q-differential operators, one can make the following decomposition:*

$$\mathcal{L}_{q\text{-}KdV}(u_2) = \mathcal{L}_{q\text{-}burgers}(u_1) \circ \mathcal{L}_{q\text{-}burgers}(v_1), \tag{2.21}$$

with $\mathbf{v}_1 = -\boldsymbol{\eta}_q(u_1)$ *and* $\mathbf{u}_2 = \partial_q(-\eta_q(u_1)) - \mathbf{u}_1\boldsymbol{\eta}_q(u_1)$. *This decomposition is called q-deformed Miura transformation. one can see this mapping under the following form:*

$$\begin{aligned} \mathcal{L}_{q\text{-}burgers}(u_1) \hookrightarrow \mathcal{L}_{q\text{-}KdV}(u_2) &= \mathcal{L}_{q\text{-}burgers}(u_1) \circ \mathcal{L}_{q\text{-}burgers}(-\eta_q(u_1)) \\ \text{with} \quad u_2 &= \partial_q(-\eta_q(u_1)) - u_1\eta_q(u_1). \end{aligned} \tag{2.22}$$

Proposition 2.2. *As basing on the conforms weights of the operators derivatives:* $[\partial_{t_{q\text{-}KdV}}] = 3$ *and* $[\partial_{t_{q\text{-}Burgers}}] = 2$, *one can make the following correspondence:*

$$\partial_{t_{q\text{-}Burgers}} \hookrightarrow \partial_{t_{q\text{-}KdV}} = \alpha\partial_q \circ \partial_{t_{q\text{-}Burgers}} + \beta\partial_q^3, \tag{2.23}$$

where α *and* β *are arbitrary real constants.*

Proposition 2.3 (Généralisation). *Being given an q -deformed Burgers operator* $\mathcal{L}_{q\text{-}Burgers}$ *and an q-deformed* sl_n*-KdV operator of type:*

$$\mathcal{L}_{q\text{-}sl_n\text{-}KdV} = \partial_q^n + u_2\partial_q^{n-2} + u_3\partial_q^{n-3} + \cdots + u_n, \tag{2.24}$$

then we can make the following decomposition:

$$\mathcal{L}_{q\text{-}sl_n-KdV} = \mathcal{L}_{q\text{-}Burgers}(v_1) \circ \mathcal{L}_{q\text{-}Burgers}(v_2) \circ \cdots \circ \mathcal{L}_{q\text{-}Burgers}(v_n), \tag{2.25}$$

where v_i, $i = 1, \ldots, n$ are the fields of conformal weight 1 and which can be written in functions of the fields u_j, $j = 2, \ldots, n$ and their q-derivatives.

3. Conclusion

The importance of the theory of pseudodifferential operators in the study of nonlinear integrable systems is point out. Principally, the algebra of nonlinear (local and nonlocal) differential operators acte on the ring of analytic functions $u_s(x,t)$.

In This paper, we have devoted to a brief account of the basic properties of the space of q-pseudo differential Lax operators in the bosonic case. Presently, we know that any q-pseudodifferential operator is completely specified by a conformal spin $s, s \in \mathbb{Z}$, two integers p, and $q = p + n$, $n \geq 0$ defining the lowest and the highest degrees, respectively, and finally $(1 + q - p) = n + 1$ analytic fields $u_j(z)$. We recall that the space $\mathfrak{q}\mathcal{A}$ of all local and nonlocal q-pseudodifferential operators admits a Lie algebra's structure with respect to the commutator buildout of the Leibnitz product. Moreover, we find that A splits into $3 \times 2 = 6$ subalgebr as $\mathfrak{q}\mathcal{A}_{j+}$ and $\mathfrak{q}\mathcal{A}_{j-}$, $j = 0$, ± 1 related to each others by two types of conjugations, namely, the spin.

Finally, we have focused in this work to present the basics steps towards constructing the q-deformed integrable systems and the associated Lax generating technique. Particular interest is devoted to the q-Burgers and the q-KdV systems and their underlying mapping.

References

[1] C. Kassel, "Cyclic homology of differential operators, the Virasoro algebra and a q-analogue," *Communications in Mathematical Physics*, vol. 146, no. 2, pp. 343–356, 1992.

[2] E. H. Saidi and M. B. Sedra, "The Feĭgin-Fuchs representation of the N = 4 SU(2) conformal current," *Classical and Quantum Gravity*, vol. 10, no. 10, pp. 1937–1946, 1993.

[3] E. H. Saidi, M. B. Sedra, and J. Zerouaoui, "On D = 2(1/3, 1/3) supersymmetric theories. II," *Classical and Quantum Gravity*, vol. 12, p. 2705, 1995.

[4] A. Boulahoual and M. B. Sedra, "The Moyal momentum algebra applied to (theta)-deformed 2d conformal models and KdV-hierarchies," *Chinese Journal of Physics*, vol. 43, no. 3, pp. 408–444, 2005.

[5] A. Boulahoual and M. B. Sedra, "The moyal momentum algebra," *Afr. J. Math. Phys.*, vol. 2, no. 1, pp. 111–113, 2005, http://arxiv.org/abs/hep-th/0207242.

[6] A. El Boukili, M. B. Sedra, and A. Zemate, "Super gelfand-dickey algebra and integrable models," *Chinise Journal of Physics*, vol. 47, no. 6, 2009.

[7] M. B. Sedra, A. El Boukili, H. Erguig, and J. Zerouaoui, "Some aspect of 2d integrability," *Advanced Studies in Theoretical Physics*, vol. 2, no. 2, pp. 87–97, 2008.

[8] O. Dafounansou, A. El Boukili, and M. B. Sedra, "Some aspects of moyal deformed integrable systems," *Chinese Journal of Physics*, vol. 44, no. 4, pp. 274–289, 2006.

[9] A. El Boukili and M. B. Sedra, "Some Physical Aspects of Moyal Noncommutativity," *Chinise Journal of Physics*, vol. 47, no. 3, 2009.

[10] K. Huitu and D. Nemeschansky, "Supersymmetric gelfand-dickey algebra," *Modern Physics Letters*, vol. 6, pp. 3179–31690, 1991.

[11] M. B. Sedra, "Moyal noncommutative integrability and the Burgers-KdV mapping," *Nuclear Physics B*, vol. 740, no. 3, pp. 243–270, 2006.

[12] J. Wess, "q-deformed Heisenberg-algebra," in *Corfu Summer Institute on Elementary Particle Physics (Kerkyra, 1998)*, Journal of High Energy Physics, 1999.

[13] M. Aganagic, H. Ooguri, N. Saulina, and C. Vafa, "Black holes, q-deformed 2d Yang-Mills, and non-perturbative topological strings," *Nuclear Physics B*, vol. 715, no. 1-2, pp. 304–348, 2005.

[14] M. Hamanaka and K. Toda, "Towards noncommutative integrable systems," *Physics Letters A*, vol. 316, no. 1-2, pp. 77–83, 2003.

[15] R. Herrmann, "Common aspects of q-deformed Lie algebras and fractional calculus," *Physica A*, vol. 389, no. 21, pp. 4613–4622, 2010.

[16] G. Fiore, "q-deformed su(2) instantons by q-quaternions," *Journal of High Energy Physics*, no. 2, article 010, 2007.

[17] A. EL Boukili and M. B. Sedra, "Integrability of q-deformed Lax equations," *Advanced Studies in Theoretical Physics*, vol. 4, no. 5–8, pp. 225–231, 2010.