Review Article

Phenomenology of Neutrino Mixing in Vacuum and Matter

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The current status and some perspectives of the phenomenology of massive neutrinos is reviewed. We start with the phenomenology of neutrino oscillations in vacuum and in matter. We summarize the results of neutrino experiments using solar, atmospheric. The fundamental theory of flavor changing neutrinos that has confirmed the neutrino oscillations and the various parameters affecting these oscillations have been discussed in detail. Specifically we will take the solar and atmospheric neutrino case. The oscillation plots will be discussed in detail, based on their behavior in vacuum and matter. Both normal and inverted mass hierarchy hypotheses are tested and both are consistent with observation. Finally the sensitivity of theta 13 over these probability oscillations has been analyzed and commented.

1. Introduction

Neutrinos are the most elusive in nature. They interact much less through coupling with W^+, W^- and, Z^0 boson. Several past and currently going neutrino experiments have not only resolved the mystery of solar and atmospheric neutrinos but also proved the massive nature of neutrinos. The flavor oscillation phenomenon leads to nonzero mass in the theories beyond a standard model in contradiction to the well-established standard model. The history of neutrino oscillation traces back to Gribov and Pontecorvo [1] but the credit for providing the evidence of neutrino oscillation goes to Cleveland et al. [2] for observing deficit in ν_e's neutrino flux about 1/3rd as given by the standard solar model [3] through the chlorine Homestake experiment [2]. Several independent experiments like GALLEX [4] and SAGE [5] agreed with Cleveland et al. [2] by observing the neutrino flux more than half as expected from the solar model predictions. The mystery of solar neutrinos finally got resolved with the experiment Sudbury Neutrino Observatory [6] which proved that the deficit was due to flavor transitions. The experiment was different in the sense that it used a heavy water detector to detect ^{8}B neutrinos via charged current interaction with electron only and neutral current interaction with total neutrino flux. Atmospheric neutrino anomaly is related to the electron-to-muon-neutrino ratio which reduces to half of the

expected value and a possible explanation is the change of flavor from $\nu_\mu \Leftrightarrow \nu_\tau$, $\nu_\mu \Leftrightarrow \nu_e$. Solar and atmospheric neutrino oscillations, neutrino masses, and mixing parameters have been proved by various detectors with different sensitivity and accuracy. The recent journey of detectors in neutrino physics starts from Super-Kamiokande [7], KamLAND [8], T2K [9], MINOS, [10] and finally end with detectors like SOUDAN [11] and OPERA [12]. The KamioKande [13] measured solar neutrinos to be half as per the SM whereas two experiments Gallex and Sage [4, 5] measured 56–60% of neutrino capture rate as predicted by the standard model. Then Super-Kamiokande I, II, and III [14–16] provided evidence for nonzero mass and also produced observation consistent with μ-neutrinos changing into τ-neutrinos. The latest searches at these experiments are more concerned with oscillation parameters, CP violation, and mass hierarchies. KAMLAND experiment was also able to investigate geographically produced antineutrinos [15, 17] and the best fitted values of $\Delta m_{21}^2 = 7.58 + 0.14\text{(stat)}^{+0.15}_{-0.15}\text{(syst)} \times 10^{-5}$ eV^2 and $\tan\theta_{12} = 0.56 + ^{0.10}_{-0.07}\text{(stat)}^{+0.10}_{-0.06}\text{(syst)}$ for $\tan\theta_{12} < 1$ as calculated using KAMLAND experiment. K2K [16, 18] was another long baseline experiment to study oscillation from ν_μ to ν_e in the atmospheric region and confirmed the deficit of muon as observed in Super-Kamiokande. MINOS [17] at the Fermi National Laboratory studied muon oscillations

produced from pion and kaon decay in the energy range of 1–10 GeV and focused primarily on the measurement of Δm^2_{23} with the precision better than 10%.

The most recent parameter of our interest is θ_{13} because it has opened some of the most fundamental questions like CP violation. The nonzero value of θ_{13} may change the old picture of neutrino oscillation completely. Our main focus is the study of the flavor oscillations in the solar and atmospheric neutrinos. The detailed theory is presented with the aim that oscillation shows variations in vacuum and matter for the solar and atmospheric cases. The parameters affecting the neutrino oscillations are all included in our studies. Few experiments are analyzed in detail and their significance is mentioned. Special attention is given to the probability oscillation curve and its sensitivity to the recent bound on θ_{13}.

2. Theory

Neutrino oscillations are periodic transitions between different flavor neutrinos in neutrino beams. The theory of neutrino oscillations has been studied by various pioneer scientists via different approaches including classical treatment by moving them to quantum mechanical treatment of wave packets [18] and then finally with quantum field theoretic treatment [19]. In this paper we present a detailed formalism for neutrino using the classical treatment only. The dependence of states on the time is given by the Schrodinger equation:

$$i\frac{\partial |\Psi(t)\rangle}{\partial t} = H |\Psi(t)\rangle, \tag{1}$$

where H is the total Hamiltonian in flavor space and the general solution for (1) is

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle, \tag{2}$$

where $|\Psi(0)\rangle$ is the state at the initial time ($t = 0$).

Flavor neutrino states and mass eigen states can be related through leptonic mixing matrix U as

$$|\nu_l\rangle = \sum_{i=0}^{3} U^*_{li} |\nu_i\rangle. \tag{3}$$

At time ($t = 0$),

$$|\Psi(0)\rangle = |\nu_i\rangle. \tag{4}$$

Application of the Hamiltonian operator will give the following:

$$H |\nu_i\rangle = E_i |\nu_i\rangle, \tag{5}$$

where

$$E_i = \sqrt{p_i^2 + m_i^2}. \tag{6}$$

The left-handed neutrino at the time $t \geq 0$ is written as

$$|\nu_l\rangle_t = e^{-iHt} |\nu_l\rangle = \sum_{i=1}^{3} e^{-iE_i t} U^*_{li} |\nu_i\rangle. \tag{7}$$

Similarly,

$$|\overline{\nu_l}\rangle_t = e^{-iHt} |\overline{\nu_l}\rangle = \sum_{i=1}^{3} e^{-iE_i t} U_{li} |\overline{\nu_i}\rangle. \tag{8}$$

Neutrino energies E_i ($i = 1, 2, 3$) are different; hence the flavor oscillation in neutrinos gives the amplitude of the transition $\nu_l \rightarrow \nu_{l'}$ during the time t and can be written as

$$A(\nu_l \longrightarrow \nu_{l'}) = \sum_{i=1}^{3} U_{il'} e^{-iE_i t} U^*_{li}. \tag{9}$$

Analogously, the amplitude of the transition $\overline{\nu_l} \rightarrow \overline{\nu_{l'}}$ during the time t is given by

$$A(\overline{\nu_l} \longrightarrow \overline{\nu_{l'}}) = \sum_{i=1}^{3} U^*_{il'} e^{-iE_i t} U_{li}. \tag{10}$$

Probability for the transition of neutrinos and antineutrinos will be

$$P(\nu_l \longrightarrow \nu_{l'}) = \left| \sum_{i=1}^{3} U_{il'} e^{-iE_i t} U^*_{li} \right|^2,$$

$$P(\overline{\nu_l} \longrightarrow \overline{\nu_{l'}}) = \left| \sum_{i=1}^{3} U^*_{il'} e^{-iE_i t} U_{li} \right|^2. \tag{11}$$

From (11), we can find possible relations between probabilities:

$$\sum_l P(\nu_l \longrightarrow \nu_{l'}) = 1, \qquad \sum_{l'} P(\overline{\nu_l} \longrightarrow \overline{\nu_{l'}}) = 1,$$

$$\sum_{l'} P(\nu_l \longrightarrow \nu_{l'}) = 1, \qquad \sum_l P(\overline{\nu_l} \longrightarrow \overline{\nu_{l'}}) = 1. \tag{12}$$

A mixed neutrino state is characterized by their momentum p with $p_i = p$ and mass m_i with $m_i^2/p^2 \ll 1$. And

$$E_i \cong p + \frac{m_i^2}{2p}. \tag{13}$$

Hence the energy difference between two such states will be

$$E_i - E_j = \frac{\Delta m^2_{ji}}{2p}, \tag{14}$$

where $\Delta m^2_{ji} = m_i^2 - m_j^2$.

As $E \approx p$, therefore

$$E_i - E_j = \frac{\Delta m^2_{ji}}{2E}. \tag{15}$$

Here the assumption is made that neutrinos are having different energy in different mass eigenstates. Let us suppose the t denotes the difference between production and detection

time for the ultrarelativistic neutrinos and L is the distance between source and the detector:

$$t \cong L,$$

$$\left(E_i - E_j\right) t = \frac{\Delta m_{ji}^2}{2E} L, \tag{16}$$

$$P\left(\nu_l \longrightarrow \nu_{l'}\right) = \left| \sum_{i=1}^{3} U_{l'i} e^{-i(\Delta m_{ji}^2 L/2E)} U_{li}^* \right|^2.$$

The unitary condition suggests that

$$\sum_i U_{l'i} U_{li}^* = \delta_{ll'}. \tag{17}$$

Hence

$$P\left(\nu_l \longrightarrow \nu_{l'}\right) = \left| \delta_{ll'} + \sum_{i \neq j} U_{il'} (e^{-i(\Delta m_{ji}^2 L/2E)} - 1) U_{li}^* \right|^2. \tag{18}$$

Analogously, for the case of antineutrino,

$$P\left(\overline{\nu_l} \longrightarrow \overline{\nu_{l'}}\right) = \left| \delta_{ll'} + \sum_{i \neq j} U_{il'}^* \left(e^{-i(\Delta m_{ji}^2 L/2E)} - 1 \right) U_{li} \right|^2. \tag{19}$$

The transition probability $\nu_l \rightarrow \nu_{l'}$ is given as

$$P\left(\nu_l \longrightarrow \nu_{l'}\right) = \sum_{i,k} U_{l'i} U_{l'k}^* U_{li}^* U_{lk} e^{-i(\Delta m_{ji}^2/2E)L}$$

$$= \sum_i |U_{l'i}|^2 |U_{li}|^2$$

$$+ 2\operatorname{Re} \sum_{i>k} \left(U_{l'i} U_{l'k}^* U_{li}^* U_{lk} e^{-i(\Delta m_{ji}^2/2E)L} \right),$$

$$\sum_i |U_{l'i}|^2 |U_{li}|^2 = \delta_{ll'} - 2\operatorname{Re} \sum_{i>k} \left(U_{l'i} U_{lk}^* U_{li}^* U_{lk} \right)$$

$$= \delta_{ll'} - 2\operatorname{Re} \sum_{i>k} \left(U_{l'i} U_{l'k}^* U_{li}^* U_{lk} \right)$$

$$+ 2\operatorname{Re} \sum_{i>k} \left(U_{l'i} U_{l'k}^* U_{li}^* U_{lk} e^{-i(\Delta m_{ji}^2/2E)L} \right)$$

$$= \delta_{ll'} - 2\operatorname{Re} \sum_{i>k} \left(U_{l'i} U_{l'k}^* U_{li}^* U_{lk} \right)$$

$$\times \left(1 - e^{-i(\Delta m_{ki}^2 L/2E)} \right). \tag{20}$$

Finally for any complex a and b, $\operatorname{Re}(ab) = \operatorname{Re}(a)\operatorname{Re}(b) - \operatorname{Im}(a)\operatorname{Im}(b)$,

$$P\left(\nu_l \longrightarrow \nu_{l'}\right) = \delta_{ll'} - 2\operatorname{Re} \sum_{i>k} \left(U_{l'i} U_{l'i}^* U_{li}^* U_{lk} \right)$$

$$\times \left(1 - \cos \frac{\Delta m_{ki}^2 L}{2E} \right) \tag{21}$$

$$+ 2 \sum_{i>k} \operatorname{Im} \left(U_{l'i} U_{l'k}^* U_{li}^* U_{lk} \right) \sin \frac{\Delta m_{ki}^2 L}{2E}.$$

Similarly, for the antineutrino oscillation, probability becomes

$$P\left(\nu_{\bar{l}} \longrightarrow \nu_{\overline{l'}}\right) = \delta_{ll'} - 2\operatorname{Re} \sum_{i>k} \left(U_{l'i} U_{l'i}^* U_{li}^* U_{lk} \right)$$

$$\times \left(1 - \cos \frac{\Delta m_{ki}^2 L}{2E} \right)$$

$$- 2 \sum_{i>k} \operatorname{Im} \left(U_{l'i} U_{l'k}^* U_{li}^* U_{lk} \right) \sin \frac{\Delta m_{ki}^2 L}{2E}. \tag{22}$$

3. Two-Flavor Oscillation Probability in Vacuum

For two-flavor neutrino oscillations, the mixing matrix can be defined in terms of some rotating angle θ in 2 dimensions where U is actually PMNS matrix; therefore

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},$$

$$|\nu_e (t = 0)\rangle = |\nu_e\rangle = \cos\theta |\nu_1\rangle + \sin\theta |\nu_2\rangle, \tag{23}$$

$$|\nu_\mu (t = 0)\rangle = |\nu_\mu\rangle = -\sin\theta |\nu_1\rangle + \cos\theta |\nu_2\rangle.$$

The weak eigenstates are rotated by an angle θ with respect to the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ to allow mixing between ν_μ and ν_e. After some time t

$$|\nu_\mu (t = t)\rangle = |\nu_\mu (t)\rangle = -\sin\theta |\nu_1\rangle e^{(-iE_1 t/h)}$$

$$+ \cos\theta |\nu_2\rangle e^{-iE_2 t/h}$$

$$= -\sin\theta |\nu_1\rangle e^{-i(p+(m_1^2/2p))t/h} \tag{24}$$

$$+ \cos\theta |\nu_2\rangle e^{-i(p+(m_2^2/2p))t/h},$$

where we have used

$$E_1 = \left(p^2 + m_1^2\right)^{1/2}, \qquad E_2 = \left(p^2 + m_2^2\right)^{1/2},$$

$$P\left(\nu_e \longrightarrow \nu_\mu\right) = \left| \langle \nu_\mu \mid \nu_e (t) \rangle \right|^2,$$

$$\langle \nu_\mu | = \cos\theta \langle \nu_1 | + \sin\theta \langle \nu_2 |,$$

$$P\left(\nu_e \longrightarrow \nu_\mu\right) = \left| \langle \nu_\mu \mid \nu_e (t) \rangle \right|^2 \tag{25}$$

$$= e^{iz-iz} \sin^2\theta \cos^2\theta \left(1 - e^{(i\Delta m^2/2p)x} \right)$$

$$\times \left(1 - e^{(-i\Delta m^2/2p)x} \right).$$

For neutrinos to be relativistic, the substitution $p = E_v$ and $x = L$ can be made:

$$P\left(v_e \longrightarrow v_\mu\right) = \sin^2\theta \cos^2\theta \left(1 - e^{(i\Delta m^2/2E_v)L}\right)$$
$$\times \left(1 - e^{(-i\Delta m^2/2E_v)L}\right). \tag{26}$$

$$P\left(v_e \longrightarrow v_\mu\right)(L, E) = \sin^2 2\theta \sin^2\left(1.27\Delta m^2 \frac{L}{E_v}\right). $$

Here θ represents the mixing between two mass eigen states, L is length of source from the detector also known as baseline, E is the energy of neutrinos produced from the source. The functional dependence on L/E is called a spectral dependence where $F(L/E) = L/E$. The mass squared difference values and the mixing angle are the significant quantities to be measured. Physicists just probe the different mass eigenvalues and predict the mixing at which it occurs. Mixing angle dependence of the transition probability is expressed by $(\sin 2\theta)^2$. If we change from θ to $\pi/2 - \theta$, the mixing angle dependence remains as such which confirms with degeneracy of oscillation the probability for θ and $\pi/2 - \theta$. Two possibilities here correspond to two physically different mixings for two mass eigen states: if $\theta < \pi/4$, the electron neutrino is composed more of v_1, and if $\theta > \pi/4$, then muon neutrino is composed more of v_2. Moreover, transition to a different flavor is not possible if $\Delta m^2 L/2E \ll 1$ which led us to a survival probability. Direct information about the mixing angle can be obtained from the average neutrino oscillation probability $\langle P(v_\alpha \rightarrow v_\beta)\rangle = (1/2)(\sin 2\theta)^2$.

4. Three-Flavor Probability Oscillations in Vacuum

In case of three-flavor neutrino oscillations, the standard parameterization of mixing matrix can be achieved by using three vectors, and performing the Euler rotations introduces three mixing angles and one complex phase factor. For three flavor and three mass eigen states, it can be written as

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = U \begin{pmatrix} v_e \\ v_\mu \\ v_\tau \end{pmatrix}. \tag{27}$$

Consider the unitary 3×3 mixing matrix for the Dirac neutrinos and introduce the standard parameters (three mixing angles and one phase) which characterize it:

$$|i\rangle \quad (i = 1, 2, 3),$$
$$\langle i \mid k\rangle = \delta_{ik}. \tag{28}$$

The first Euler rotation performed at the angle θ_{12} around the vector $|3\rangle$ produces new orthogonal and normalized vectors as

$$|1\rangle^{(1)} = c_{12}|1\rangle + s_{12}|2\rangle,$$
$$|2\rangle^{(1)} = -s_{12}|1\rangle + c_{12}|2\rangle, \tag{29}$$
$$|3\rangle^{(1)} = |3\rangle.$$

Here $c_{12} = \cos\theta_{12}$ and $s_{12} = \sin\theta_{12}$;

$$|v\rangle^{(1)} = U^{(1)}|v\rangle \quad \text{such that}$$

$$|v\rangle^{(1)} = \begin{pmatrix} |1^{(1)}\rangle \\ |2^{(1)}\rangle \\ |3^{(1)}\rangle \end{pmatrix}, \qquad |v\rangle = \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \end{pmatrix}, \tag{30}$$

$$U^{(1)} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Second rotation at the angle θ_{13} around vector the $|2\rangle^{(1)}$ introduces the CP phase δ:

$$|1\rangle^{(2)} = c_{13}|1\rangle^{(1)} | +s_{13}e^{-i\delta}|3\rangle^{(1)},$$
$$|2\rangle^{(2)} = |2\rangle^{(1)}, \tag{31}$$
$$|3\rangle^{(2)} = -s_{13}e^{-i\delta}|1\rangle^{(1)} + c_{13}|3\rangle^{(1)}.$$

In the matrix $|v\rangle^{(2)} = U^{(2)}|v\rangle^{(1)}$, where $U^{(2)} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix}$.

Similarly rotation around vector $|1\rangle^{(2)}$ at the angle θ_2 is as follows

$$|1\rangle^{\text{mix}} = |1\rangle^{(2)},$$
$$|2\rangle^{\text{mix}} = c_{23}|2\rangle^{(2)} + s_{23}|3\rangle^{(2)},$$
$$|3\rangle^{\text{mix}} = -s_{23}|2\rangle^{(2)} + c_{23}|3\rangle^{(2)},$$
$$\left|v^{\text{mix}}\right\rangle = U^{(3)}|v\rangle^{(2)},$$
$$U^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix},$$
$$\left|v^{\text{mix}}\right\rangle = U|v\rangle, \quad \text{where } U = U^{(3)}U^{(2)}U^{(1)},$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - s_{13}c_{12}c_{23}e^{i\delta} & c_{12}s_{23} - s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{pmatrix}. \tag{32}$$

The phase δ is responsible for the effects of the CP violation which can take values from 0 to 2π. The mixing angles are parameters for three-neutrino oscillation in vacuum; all real parts of the quadratic products of elements of the mixing matrix entering in the three-neutrino oscillation probabilities are given as $\text{Re}\left(U_{l'i}U_{l'k}^* U_{li}^* U_{lk}\right)$. The individual probability expression for three neutrino flavors changing

into others can be achieved by solving for individual matrix elements. It is shown below that the probability for neutrino oscillation depend on three mixing angle and two-mass-squared difference;

$$\text{Re}\left(U_{l'i}U_{l'k}^*\,U_{li}^*U_{lk}\right)$$

$$= \left(U_{22}U_{21}^*U_{12}^*U_{11}\right), \quad \text{for } l' = 2, l = 1, i = 2, k = 1$$

$$= \left(c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta}\right)\left(-c_{23}s_{12} - s_{13}s_{23}c_{12}e^{-i\delta}\right)$$

$$\times \left(c_{13}s_{12}\right)\left(c_{13}c_{12}\right)$$

$$= -\frac{1}{4}c_{13}^2 \sin 2\theta_{12}$$

$$\times \left[\sin 2\theta_{12}(c_{23}^2 - s_{23}^2 s_{13}^2) + \cos 2\theta_{12}\,\sin 2\theta_{23}s_{13}\,\cos\delta_{13}\right],$$

$$\text{Re}\left(U_{l'i}U_{l'k}^*U_{li}^*U_{lk}\right)$$

$$= \left(U_{23}U_{22}^*U_{13}^*U_{12}\right)$$

$$= \left(c_{13}s_{23}\right)\left(c_{23}c_{12} - s_{13}s_{23}s_{12}e^{-i\delta}\right)\left(s_{13}e^{i\delta}\right)\left(c_{13}s_{12}\right)$$

$$= -c_{13}^2 s_{13}s_{12}s_{23}\left(s_{13}s_{23}s_{12} - c_{23}c_{12}\cos\delta_{13}\right),$$

$$\text{Re}\left(U_{l'i}U_{l'k}^*U_{li}^*U_{lk}\right)$$

$$= \left(U_{23}U_{21}^*U_{13}^*U_{11}\right)$$

$$= \left(c_{13}s_{23}\right)\left(-c_{23}s_{12} - s_{13}s_{23}c_{12}e^{-i\delta}\right)\left(s_{13}e^{i\delta}\right)\left(c_{13}c_{12}\right)$$

$$= -c_{13}^2 s_{13}c_{12}s_{23}\left(c_{23}s_{12}e^{i\delta} + s_{13}s_{23}c_{12}\right). \tag{33}$$

Similarly other expressions can also be derived shown in Table 1.

From the table, it is clear that out of three types of mass mixing, Δm_{21}^2, Δm_{31}^2, and Δm_{32}^2 only two are independent:

$$\Delta m_{32}^2 + \Delta m_{21}^2 - \Delta m_{31}^2 = 0. \tag{34}$$

Thus neutrino oscillations are only sensitive to the mass squared difference in spite of the actual mass. The solar experiments have inferred the sign of Δm^2 from the MSW effect but the sign of mass squared difference in atmospheric is not known and the condition observed at experiments, $\Delta m_{\text{sol}}^2 \ll \Delta m_{\text{atmos}}^2$, predicts that both kinds of neutrino mixing can occur; one is normal hierarchy (NH) having two light states and one heavier ($m_1 \ll m_2 < m_3$), and the other is inverted hierarchy (IH), m_3 is lightest state which assume masses in order ($m_3 \ll m_1 < m_2$) [19]. The two hierarchies are shown in Figure 1.

5. Atmospheric Neutrino Oscillation Probability

Atmospheric neutrinos are created by interactions of primary cosmic rays with nuclei in the atmosphere. The neutrinos generate upwardly-going and horizontal muons through decays. The phenomena of atmospheric neutrino oscillations arise from the deficit in upwardly and downward going muon neutrino. Let us calculate the probability for an electron neutrino changing into muon neutrino, as we already discussed that there are three possible cases for electron neutrino changing into muon neutrino, that is, from one mass eigenstate to another:

$$P\left(\nu_e \longrightarrow \nu_\mu\right)$$

$$= \delta_{12} - 4\left[-\frac{1}{4}c_{13}^2 \sin 2\theta_{12}\right.$$

$$\times \left[\sin 2\theta_{12}\left(c_{23}^2 - s_{23}^2 s_{13}^2\right)\right.$$

$$\left. + \cos 2\theta_{12}\sin 2\theta_{23}s_{13}\cos\delta_{13}\right]\right]$$

$$\sin\frac{\Delta m_{21}^2 L}{2E} + \delta_{32}$$

$$- 4\left[-c_{13}^2 s_{13}s_{12}s_{23}\left(s_{13}s_{23}s_{12} - c_{23}c_{12}\cos\delta_{13}\right)\right]$$

$$\times \sin\frac{\Delta m_{32}^2 L}{4E}$$

$$+ \delta_{31} - 4\left[-c_{13}^2 s_{13}c_{12}s_{23}\left(c_{23}s_{12}e^{i\delta} + s_{13}s_{23}c_{12}\right)\right]$$

$$\times \sin\frac{\Delta m_{31}^2 L}{4E}. \tag{35}$$

For atmospheric neutrinos,

$$\Delta m_{32}^2 \cong \Delta m_{31}^2 \cong \Delta m_{\text{atm}}^2,$$

$$\Delta m_{21}^2 \cong \Delta m_{\text{sol}}^2 \cong 0,$$

$$P\left(\nu_e \longrightarrow \nu_\mu\right) = -4\left(-s_{12}^2 s_{23}^2 s_{13}^2 c_{13}^2\right.$$

$$+ s_{12}s_{23}s_{13}c_{23}c_{12}c_{13}^2\cos\delta_{13}$$

$$- s_{13}^2 c_{13}^2 - s_{12}s_{23}s_{13}c_{23}c_{12}c_{13}^2\cos\delta_{13}\right)$$

$$\times \sin\frac{\Delta m_{\text{atm}}^2 L}{4E}$$

$$= \sin^2\left(2\theta_{13}\right)\sin^2\left(\theta_{23}\right)\sin\frac{\Delta m_{\text{atm}}^2 L}{4E},$$

$$P\left(\nu_e \longrightarrow \nu_\tau\right) = \sin^2\left(2\theta_{13}\right)\cos^2\left(\theta_{23}\right)\sin\frac{\Delta m_{\text{atm}}^2 L}{4E}. \tag{36}$$

The survival probability of electron neutrinos is

$$P\left(\nu_e \longrightarrow \nu_e\right) = 1 - \left[P\left(\nu_e \longrightarrow \nu_\mu\right) + P\left(\nu_e \longrightarrow \nu_\tau\right)\right]$$

$$= 1 - \sin^2\left(2\theta_{13}\right)\sin\frac{\Delta m_{\text{atm}}^2 L}{4E}. \tag{37}$$

TABLE 1

Probability	i, k	$\mathrm{Re}(U_{l'i} U^*_{l'k} U^*_{li} U_{lk})$
$P(\nu_e \rightarrow \nu_\mu)$	$i=2, k=1$	$(-1/4)c_{13}^2 \sin 2\theta_{12}[\sin 2\theta_{12}(c_{23}^2 - s_{23}^2 s_{13}^2) + \cos 2\theta_{12}\sin 2\theta_{23}s_{13}\cos\delta_{13}]$
	$i=3, k=2$	$c_{13}^2 s_{13}c_{12}s_{23}(s_{13}s_{23}s_{12} - c_{23}c_{12}\cos\delta_{13})$
	$i=3, k=1$	$c_{13}^2 s_{13}c_{12}s_{23}(c_{23}s_{12}e^{i\delta} + s_{13}s_{23}c_{12})$
$P(\nu_e \rightarrow \nu_\tau)$	$i=2, k=1$	$(1/4)c_{13}^2 \sin 2\theta_{12}[\sin 2\theta_{12}(c_{23}^2 s_{13}^2 - s_{23}^2) + \cos 2\theta_{12}\sin 2\theta_{23}s_{13}\cos\delta_{13}]$
	$i=3, k=2$	$-c_{13}^2 s_{12}c_{23}(s_{13}c_{23}s_{12} + s_{23}c_{12}\cos\delta_{13})$
	$i=3, k=1$	$-c_{13}^2 s_{12}c_{23}(s_{13}c_{23}c_{12} - s_{23}s_{12}\cos\delta_{13})$
$P(\nu_e \rightarrow \nu_e)$	$i=2, k=1$	$(1/4)c_{13}^4 \sin^2 2\theta_{12}$
	$i=3, k=2$	$(1/4)s_{12}^2\sin^2 2\theta_{13}$
	$i=3, k=1$	$1/4\, c_{12}^2 \sin^2 2\theta_{13}$
$P(\nu_\mu \rightarrow \nu_\tau)$	$i=2, k=1$	$(1/16)\sin^2 2\theta_{12}\sin^2 2\theta_{23}(1+s_{13}^2)^2 - (1/4)(\sin^2 2\theta_{12} + \sin^2 2\theta_{23})s_{13}^2 - (1/16)\sin 4\theta_{12}\sin 4\theta_{23}(1+s_{13}^2)s_{13}\cos\delta_{13}$ $+(1/4)\sin^2 2\theta_{12}\sin^2 2\theta_{23}s_{13}^2\cos^2\delta_{13}$
	$i=3, k=2$	$(-1/4)\sin 2\theta_{23}c_{13}^2[\sin 2\theta_{23}(c_{12}^2 - s_{12}^2 s_{13}^2) + \sin 2\theta_{12}\cos 2\theta_{23}s_{13}\cos\delta_{13}]$
	$i=3, k=1$	$(1/4)\sin 2\theta_{23}c_{13}^2[\sin 2\theta_{23}(c_{12}^2 s_{13}^2 - s_{12}^2) + \sin 2\theta_{12}\cos 2\theta_{23}s_{13}\cos\delta_{13}]$
$P(\nu_\mu \rightarrow \nu_\mu)$	$i=2, k=1$	$(1/4)\sin^2 2\theta_{12}(c_{23}^4 + s_{23}^4 s_{13}^4) + (1/4)(1 - (1/2)\sin^2 2\theta_{12})\sin^2 2\theta_{23}s_{13}^2 + (1/4)\sin 4\theta_{12}\sin 2\theta_{23}$ $(c_{23}^2 - s_{23}^2 s_{13}^2)s_{13}\cos\delta_{13} - (1/4)\sin^2 2\theta_{12}\sin^2 2\theta_{23}s_{13}^2\cos^2\delta_{13}$
	$i=3, k=2$	$s_{23}^2 c_{13}^2(c_{12}^2 c_{23}^2 + s_{12}^2 s_{23}^2 s_{13}^2 - (1/2)\sin 2\theta_{12}\sin 2\theta_{23}s_{13}\cos\delta_{13})$
	$i=3, k=1$	$s_{23}^2 c_{13}^2(s_{12}^2 c_{23}^2 + c_{12}^2 s_{23}^2 s_{13}^2 + (1/2)\sin 2\theta_{12}\sin 2\theta_{23}s_{13}\cos\delta_{13})$
$P(\nu_\tau \rightarrow \nu_\tau)$	$i=2, k=1$	$(1/4)\sin^2 2\theta_{12}(s_{23}^4 + c_{23}^4 s_{13}^4) + (1/4)(1 - (1/2)\sin^2 2\theta_{12})\sin^2 2\theta_{23}s_{13}^2 + (1/4)\sin 4\theta_{12}\sin 2\theta_{23}$ $(s_{23}^2 - c_{23}^2 s_{13}^2)s_{13}\cos\delta_{13} - (1/4)\sin^2 2\theta_{12}\sin^2 2\theta_{23}s_{13}^2\cos^2\delta_{13}$
	$i=3, k=2$	$c_{23}^2 c_{13}^2(c_{12}^2 s_{23}^2 + s_{12}^2 c_{23}^2 s_{13}^2 + (1/2)\sin 2\theta_{12}\sin 2\theta_{23}s_{13}\cos\delta_{13})$
	$i=3, k=1$	$c_{23}^2 c_{13}^2(s_{12}^2 s_{23}^2 + c_{12}^2 c_{23}^2 s_{13}^2 - (1/2)\sin 2\theta_{12}\sin 2\theta_{23}s_{13}\cos\delta_{13})$

In case of vacuum,

$$P\left(\nu_e \longrightarrow \nu_\mu\right) = P\left(\nu_\mu \longrightarrow \nu_e\right),$$

$$P\left(\nu_\mu \longrightarrow \nu_e\right) = \sin^2\left(2\theta_{13}\right)\sin^2\left(\theta_{23}\right)\sin\frac{\Delta m_{atm}^2 L}{4E}. \quad (38)$$

Similarly we can calculate the probability for muon neutrinos changing into tau neutrinos:

$$P\left(\nu_\mu \longrightarrow \nu_\tau\right) = \sin^2\left(2\theta_{23}\right)\cos^4\left(\theta_{13}\right)\sin\frac{\Delta m_{atm}^2 L}{4E}. \quad (39)$$

Now we will calculate the survival probability of muon neutrinos; that is,

$$\begin{aligned} P\left(\nu_\mu \longrightarrow \nu_\mu\right) &= 1 - \left[P\left(\nu_\mu \longrightarrow \nu_e\right) + P\left(\nu_\mu \longrightarrow \nu_\tau\right)\right] \\ &= 1 - \sin^2\left(2\theta_{13}\right)\sin^2\left(\theta_{23}\right) \\ &\quad + \sin^2\left(2\theta_{23}\right)\cos^4\left(\theta_{13}\right)\sin\frac{\Delta m_{atm}^2 L}{4E}. \end{aligned} \quad (40)$$

These are six probability terms from where we can find the probability for one neutrino changing into another depending upon their flavor.

6. Solar Neutrino Oscillation Probability

The difference in the number of solar neutrinos predicted from solar models and the number of neutrinos flowing through earth led to a solar neutrino problem; this created the solar neutrinos as the target for researchers as it can provide a more elaborated picture of stellar evolution and energy resources. The various experiments are focused to measure the solar neutrino flux. Solar neutrino events can also be analyzed by a Monte Carlo simulation study of uncertainties which made use of fluxes from 100 standard solar models [20]. Solar neutrino experimental data constrains that mass squared difference Δm_{21}^2 is only taken where other mass differences are neglected:

$$P\left(\nu_e \longrightarrow \nu_\mu\right) = \delta_{12} - 4\left[-\frac{1}{4}c_{13}^2 \sin 2\theta_{12}\right.$$
$$\times \left[\sin 2\theta_{12}\left(c_{23}^2 - s_{23}^2 s_{13}^2\right)\right.$$
$$\left.\left. + \cos 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta_{13}\right]\right]$$
$$\times \sin\frac{\Delta m_{21}^2 L}{2E},$$

$$P\left(\nu_e \longrightarrow \nu_\mu\right) = -4\left[-\frac{1}{4}c_{13}^2 \sin 2\theta_{12}\right.$$
$$\times \left[\sin 2\theta_{12}\left(c_{23}^2 - s_{23}^2 s_{13}^2\right)\right.$$
$$\left.\left. + \cos 2\theta_{12} \sin 2\theta_{23} s_{13} \cos\delta_{13}\right]\right]$$

$$\times \sin\frac{\Delta m_{21}^2 L}{2E}$$
$$= \left[\sin^2 2\theta_{12}\cos^2\left(\theta_{13}\right)\right.$$
$$\times \left(\cos^2\theta_{23} - \sin^2\theta_{23}\sin^2\theta_{13}\right)$$
$$\left. + \frac{1}{4}\left(\sin 4\theta_{12} \sin 2\theta_{13} \cos\theta_{13}\right)\right]\sin\frac{\Delta m_{21}^2 L}{2E}. \quad (41)$$

Similarly,

$$P\left(\nu_e \longrightarrow \nu_\tau\right) = \delta_{12} - 4\left[\frac{1}{4}c_{13}^2 \sin 2\theta_{12}\right.$$
$$\times \left\{\sin 2\theta_{12}\left(c_{23}^2 s_{13}^2 - s_{23}^2\right) + \cos 2\theta_{12}\right.$$
$$\left.\left. \times \sin 2\theta_{23} s_{13} \cos\delta_{13}\right\}\right]$$
$$\times \sin\frac{\Delta m_{21}^2 L}{4E}$$
$$= \left[-\sin^2 2\theta_{12}\cos^2\theta_{13}\right.$$
$$\times \left(\cos^2\theta_{23}\sin^2\theta_{13} - \sin^2\theta_{23}\right)$$
$$\left. - \frac{1}{4}\sin 4\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \cos\theta_{13}\right]$$
$$\times \sin\frac{\Delta m_{21}^2 L}{2E},$$

$$P\left(\nu_e \longrightarrow \nu_e\right) = 1 - \left[\sin^2 2\theta_{12}\cos^4\theta_{13}\right]\sin\frac{\Delta m_{21}^2 L}{2E}. \quad (42)$$

7. Neutrino Oscillations in Matter

Since neutrinos are weakly interacting, they might interact with matter either through charged-current (CC) or neutral-current (NC) interactions [21]. For the charged current interactions, only the electrons participate via $W^\pm$ exchange. Mikheev and Smirnov [22] noticed a resonance behavior for specific oscillation and matter density parameters. Therefore the probabilities for neutrino oscillation differ from their vacuum counterparts. Neutral-current flavor interactions can occur for any type of neutrino flavor. Moreover, a neutral-current interaction leads to the addition of an extra term in Hamiltonian for flavor oscillation; such a term produces a shift in the eigenvalues, but charged-current interactions give the contribution not only in the form of a change in eigen states but also adds to Hamiltonian an energy proportional to $V = \sqrt{2}\,G_F N_e$, where G_F = Fermi coupling constant and N_e = number of electrons per unit volume.
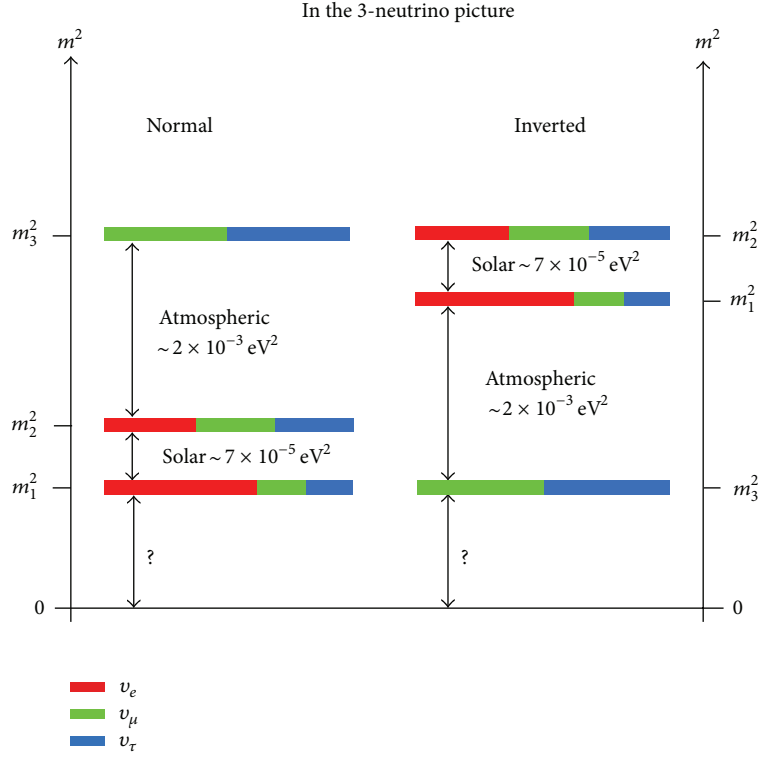
In the 3-neutrino picture

FIGURE 1: Normal and inverted hierarchy.

Neutrinos are produced in flavor eigen states, $|v_\alpha\rangle$, $(\alpha = e, \mu, \tau)$ created by the interaction of weak gauge bosons with the charged leptons between the source, production point of the neutrinos, and the detector. The state $|\psi(t)\rangle$ of neutrinos with momentum p satisfies the equation

$$i\frac{\partial}{\partial t} = H_0 |\psi(t)\rangle. \tag{43}$$

When H_0 is a free Hamiltonian, the state $|\psi(t)\rangle$ can be expanded over the total system of states of flavor neutrinos v_l, with momentum p,

$$|\psi(t)\rangle = \sum a_l(t) |v_l\rangle. \tag{44}$$

Here

$$|v_l\rangle = \sum_i U_{li}^* |v_i\rangle, \tag{45}$$

$H_0|v_i\rangle = E_i|v_i\rangle$, $H_0|v_i\rangle = E_i|v_i\rangle$, $E_i = \sqrt{p_i^2 + m_i^2} \simeq p + m_i^2 / 2E$, and $a_l(t) = \langle v_l | \psi(t)\rangle$ is the amplitude of probability to find v_l in state which is described by $|\psi(t)\rangle$.

Therefore,

$$i\frac{\partial a_l'(t)}{\partial t} = \sum_l \langle v_l' | H_0 | a_l(t)\rangle, \tag{46}$$

where

$$\langle v_l' | v_l\rangle = U_{l'l}, \qquad (47)$$
$$\langle v_l | v_{l'}\rangle = U_{l'l}^*. \qquad$$

Taking into account this relation, for the free Hamiltonian in the flavor representation we have the following expression:

$$\langle v_{l'} | H_0 | v_l\rangle = \sum_l U_{l'i}E_iU_{li}^* \sim p + \sum_l U_{l'i}\frac{m_i^2}{2E}U_{li}^*. \tag{48}$$

Therefore neutrino evolution equation in the flavor representation is as follows:

$$i\frac{\partial a(t)}{\partial t} = U\frac{m^2}{2E}U^\dagger a(t). \tag{49}$$

Let us introduce the function

$$a'(t) = U^\dagger a(t). \tag{50}$$

We find that the function $a'(t)$ satisfies the following equation, multiplying by $U^\dagger$ on both sides, then

$$i\frac{\partial}{\partial t}a'(t) = \frac{m^2}{2E}a'(t). \tag{51}$$

It is obvious that the solution of the equation has the form $a'(t) = e^{-i(m^2/2E)(t-t_0)}a'(t_0)$, where, $a'(t_0)$ is the wave function at the initial time t_0.

As the neutrinos propagate in matter, electron neutrino plays a very special role due to coherent forward scattering of neutrinos from electrons so that it leads to an additional contribution in oscillation probability. The forward scattering of electrons of matter with scattering of electrons of matter with neutrinos is called charge current interaction. Therefore

the probabilities for neutrino oscillation differ from their vacuum counterparts. Earth matter effects for the long baseline have been nicely studied earlier for both hierarchies [23]. Charged current interaction can give contribution only to the process of the elastic scattering of ν_e on electrons. Moreover, neutral current interactions occur for all flavors, leading to the addition of an extra term in Hamiltonian for the flavor oscillation probability. For the low energy, an effective Hamiltonian of the neutrino interaction was obtained from the diagonal matrix element $\langle p\mathrm{mat}|H_I^{CC}|p\mathrm{mat}\rangle$, where $H_I^{CC} = (G_F/\sqrt{2})2\overline{\nu_{eL}}(x)\gamma_\alpha\nu_{eL}\bar{e}(x)\gamma^\alpha(1 - \gamma_5)e(x)$ and the vector $|p\mathrm{mat}\rangle = |p\rangle|\mathrm{mat}\rangle$. Now substituting the Hamiltonian we obtain $\langle p\mathrm{mat}|(G_F/\sqrt{2})2\overline{\nu_{eL}}(x)\gamma_\alpha\nu_{eL}\bar{e}(x)\gamma^\alpha(1 - \gamma_5)e(x)|p\mathrm{mat}\rangle = \langle p|\langle\mathrm{mat}|(G_F/\sqrt{2})2\overline{\nu_{eL}}(x)\gamma_\alpha\nu_{eL}\bar{e}(x)\gamma^\alpha(1 - \gamma_5)e(x)|p\rangle|\mathrm{mat}\rangle$.

But $\langle p|\langle\mathrm{mat}|(G_F/\sqrt{2})2\overline{\nu_{eL}}(x)\gamma_\alpha\nu_{eL}\bar{e}(x)\gamma^\alpha\gamma_5 e(x)|p\rangle|\mathrm{mat}\rangle = 0$ as for unpolarized matter $\langle\mathrm{mat}|\bar{e}(x)\gamma^\alpha\gamma_5 e(x)|\mathrm{mat}\rangle = 0$.

Also $\langle\mathrm{mat}|\bar{e}(x)\gamma^\alpha e(x)|p\rangle|\mathrm{mat}\rangle = \langle\mathrm{mat}|\bar{e}(x)e(x)|p\rangle|\mathrm{mat}\rangle \delta_{\alpha0} = n_e(x)\delta_{\alpha0}$, where $n_e(x)$ is the number density at the point x. Also, $\langle p|\overline{\nu_{eL}}(x)\gamma_\alpha\nu_{eL}|p\rangle = 1$.

Substituting all these values, $H_I^{\mathrm{mat}}(t) = \sqrt{2}G_F n_e(t)\beta$. Using $\beta_{\nu_e,\nu_e} = 1$, all other elements of matrix β are equal to zero.

Let us now consider the NC interaction. Induced by the Z^0 exchange, the Hamiltonian of NC interactions of neutrinos with electrons and nucleons has the form

$$H_I^{CC}(x) = 2\frac{G_F}{\sqrt{2}}\sum_{l=e,\mu,\tau}\bar{\nu}_{lL}(x)\gamma^\alpha\nu_{lL}(x)j_\alpha^{NC}(x), \qquad (52)$$

where $j_\alpha^{NC}(x)$ is the sum of electron and nucleon (quark) neutral current. For the vector part of the effective hadron neutral current,

$$\nu_\alpha^{NC(N)}(x) = \frac{1}{2}\overline{N}(x)\gamma_\alpha\tau_3 N(x) - 2\sin^2\theta_w\overline{p}(x)\gamma_\alpha p(x),$$

$$\text{as } N = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (53)$$

where θ_w is the weak angle. Similarly, for the effective part of electron current,

$$\nu_\alpha^{NC(n)}(x) = \left(-\frac{1}{2}\right)\overline{n}(x)\gamma_\alpha n(x),$$

$$\nu_\alpha^{NC(e)}(x) = \left(-\frac{1}{2} + 2\sin^2\theta_w\right)\bar{e}(x)\gamma_\alpha e(x). \qquad (54)$$

For the corresponding matter matrix elements, we have

$$\left\langle\mathrm{mat}\mid\nu_\alpha^{NC(e)}(x)\mid\mathrm{mat}\right\rangle = \left(-\frac{1}{2} + 2\sin^2\theta_w\right)n_e(x)\delta_{\alpha0},$$

$$\left\langle\mathrm{mat}\mid\nu_\alpha^{NC(p)}(x)\mid\mathrm{mat}\right\rangle = \left(\frac{1}{2} - 2\sin^2\theta_w\right)\rho_p(x)\delta_{\alpha0},$$

$$\left\langle\mathrm{mat}\mid\nu_\alpha^{NC(n)}(x)\mid\mathrm{mat}\right\rangle = -\frac{1}{2}\rho_n(x)\delta_{\alpha0}. \qquad (55)$$

For the neutral matter, $n_e(x) = \rho_p(x)$ we conclude that the contributions of electron and proton NC to the effective Hamiltonian cancel each other. Thus

$$\left\langle\mathrm{mat}\mid j_\alpha^{NC}(x)\mid\mathrm{mat}\right\rangle = -\frac{1}{2}\rho_n(x)\delta_{\alpha0}. \qquad (56)$$

By taking into account the effective charged current interaction, only $\nu_e - e$ CC interaction gives a contribution to the effective Hamiltonian. Thus the evolution equation of neutrino has the form

$$i\frac{\partial a(t)}{\partial t} = \left(U\frac{m^2}{2E}U^\dagger + \sqrt{2}G_F n_e(t)\beta\right)a(t). \qquad (57)$$

Similarly, for the antineutrinos, effective Hamiltonian differs in sign from the neutrino-electron interactions; thus

$$\overline{H}_I^{\mathrm{mat}}(x) = -\sqrt{2}G_F n_e(t)\beta. \qquad (58)$$

Here we are more concerned with the matter effects of a constant density. The total Hamiltonian of neutrino in matter $H = U(m^2/2E)U^\dagger + \sqrt{2}G_F n_e\beta$, where $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ then the total effective Hamiltonian is $H = (1/2)\,\mathrm{Tr}\,H + H^m$. Here $(1/2)\,\mathrm{Tr}\,H = ((m_1^2 + m_2^2)/4E) + (1/2)\sqrt{2}G_F n_e$ and H^m is the traceless part of Hamiltonian:

$$H^m = \frac{1}{4E}\begin{pmatrix} -\Delta m^2\cos 2\theta + A & \Delta m^2\sin 2\theta \\ \Delta m^2\sin 2\theta & \Delta m^2\cos 2\theta - A \end{pmatrix},$$

$$\text{where } A = 2\sqrt{2}G_F n_e E$$

$$H^m = U^m E^m U^{m\dagger},$$

$$U^m = \begin{pmatrix} \cos\theta^m & \sin\theta^m \\ -\sin\theta^m & \cos\theta^m \end{pmatrix},$$

$$E^m = \begin{pmatrix} E_1^m & 0 \\ 0 & E_2^m \end{pmatrix}, \qquad (59)$$

where $E_{1,2}^m = \pm(1/4E)\sqrt{(\Delta m^2\cos 2\theta - A)^2 + (\Delta m^2\sin 2\theta)^2}$.

From equation, we find that the mixing angle θ^m is given as:

$$\cos 2\theta^m = \frac{\Delta m^2\cos 2\theta - A}{\sqrt{(\Delta m^2\cos 2\theta - A)^2 + (\Delta m^2\sin 2\theta)^2}},$$

$$\sin 2\theta^m = \frac{\Delta m^2\sin 2\theta}{\sqrt{(\Delta m^2\cos 2\theta - A)^2 + (\Delta m^2\sin 2\theta)^2}}. \qquad (60)$$

Three expressions for atmospheric neutrinos for the normal hierarchy are given as follows:

$$
\begin{aligned}
P_{\mu\tau\text{matter}} &= \left(\cos\left[\theta_{13}^m\right]\right)^2 * \left(\sin\left[2\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \left(\Delta m_{31} + A + \Delta m_{31}^m\right) * L\right)}{E}\right]\right)^2 \\
&\quad + \left(\sin\left[\theta_{13}{}^m\right]\right)^2 * \left(\sin\left[2\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \left(A + \Delta m_{31} - \Delta m_{31}^m\right) * L\right)}{E}\right]\right)^2 \\
&\quad - \left(\cos\left[2\theta_{23}\right]\right)^2 * \left(\sin\left[2\theta_{13}{}^m\right]\right)^2 \\
&\quad * \left(\sin\left[\theta_{23}\right]\right)^2 * \left(\sin\left[\frac{\left(1.27 * \Delta m_{31}^m * L\right)}{E}\right]\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
P_{\mu\mu\text{matter}} &= \left(1 - \cos\left[\theta_{13}^m\right]\right)^2 * \left(\sin\left[2\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \left(A + \Delta m_{31} - \Delta m_{31}^m\right) * L\right)}{E}\right]\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
P_{\mu e\text{matter}} &= \left(\sin\left[2\theta_{13}{}^m\right]\right)^2 * \left(\sin\left[\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \Delta m_{31}^m * L\right)}{E}\right]\right)^2.
\end{aligned}
\tag{61}
$$

And for the inverted hierarchy,

$$
\begin{aligned}
P_{\mu\tau\text{matter}} &= \left(\cos\left[\theta_{13}^m\right]\right)^2 * \left(\sin\left[2\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \left(A - \Delta m_{31} + \Delta m_{31}^m\right) * L\right)}{E}\right]\right)^2 \\
&\quad + \left(\sin\left[\theta_{13}{}^m\right]\right)^2 * \left(\sin\left[2\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \left(A + \Delta m_{31} - \Delta m_{31}^m\right) * L\right)}{E}\right]\right)^2 \\
&\quad - \left(\cos\left[2\theta_{23}\right]\right)^2 * \left(\sin\left[2\theta_{13}{}^m\right]\right)^2 \\
&\quad * \left(\sin\left[\theta_{23}\right]\right)^2 * \left(\sin\left[\frac{\left(1.27 * \Delta m_{31}^m * L\right)}{E}\right]\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
P_{\mu\mu\text{matter}} &= \left(1 - \cos\left[\theta_{13}^m\right]\right)^2 * \left(\sin\left[2\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \left(A - \Delta m_{31} - \Delta m_{31}^m\right) * L\right)}{E}\right]\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
P_{\mu e\text{matter}} &= \left(\sin\left[2\theta_{13}{}^m\right]\right)^2 * \left(\sin\left[\theta_{23}\right]\right)^2 \\
&\quad * \left(\sin\left[\frac{\left(1.27 * \Delta m_{31}^m * L\right)}{E}\right]\right)^2.
\end{aligned}
\tag{62}
$$

8. Oscillation Parameters and Masses

Neutrino oscillation is new physics beyond the standard model and requires the addition of a new field and new parameters to the standard model. Although atmospheric and solar neutrinos are having different origin, yet they exhibit the same phenomenon of oscillations among their flavors. The parameters involved in three-flavor oscillations can be estimated from the neutrinos coming from the sun and atmospheric and nuclear reactors and accelerators. Solar experiments detect neutrinos generated in the core of the sun due to thermonuclear reactions and must have the energy of the order of 0.2–15 MeV, whereas atmospheric experiments detect the neutrino produced in a cascade initiated by Cosmic rays collisions with nuclei in the Earth's atmosphere and has a source detector at a distance of several tons of meters with a range $L/E < 1$ m/MeV. Oscillation parameters historically fall into four categories in mixing angle-mass splitting parameter space: vacuum oscillations (VAC), "LOW", small mixing angle (SMA), and large mixing angle (LMA). Similarly based on the distance from source to detector, detectors can be SBL, LBL, VSBL, and VLBL. A global analysis of different experiments can provide best fit values of different parameters. The two large mixing angles θ_{12} and θ_{23} have been found to be of the order of ~34° and ~45°, respectively, but the third mixing angle θ_{13} is suppressed by different experiments. Recent searches for θ_{13}, however, oppose suppression of this mixing angle. Moreover sign of Δm_{31}^2 is still unknown; therefore two types of neutrino spectrum are possible, one is normal hierarchy ($m_1 \ll m_2 < m_3$) whereas the other is inverted hierarchy which assume masses in order ($m_3 \ll m_1 < m_2$). Three mixing angles and the mass differences from global analysis of data [24] are given below:

$$
\begin{aligned}
\Delta m_{21}^2 &= 7.59 \pm 0.20 \,{}^{+0.61}_{-0.69} \times 10^{-5}\ \text{eV}^2, \\
\Delta m_{31}^2 &= +2.46 \pm 0.12\ (\pm 0.37) \times 10^{-3}\ \text{eV}^2 \\
&= -2.36 \pm 0.11 \pm 0.37\ \text{eV}^2, \\
\theta_{23} &= 42.8^{+4.7}_{-2.9}\left({}^{+10.7}_{-7.3}\right), \\
\theta_{12} &= 34.4 \pm 1.0 \left({}^{+3.2}_{-2.9}\right), \\
\theta_{13} &= 5.6^{+3.0}_{-2.7}.
\end{aligned}
\tag{63}
$$

The parameter affecting the neutrino oscillation in matter is given by $A = 2\sqrt{2}G_F n_e E$ where G_F is Fermi constant, n_e is the electron number density, and E is the energy of neutrino beam. The parameter A is almost constant for the long baseline experiment with high precision as the beam does not penetrate deeply into the earth without affecting the different layers of the earth, but in sun, density decreases exponentially from center to core of earth which further leads to resonance enhancement of oscillations known as the MSW effect [22]. We here restrict ourselves to the matter effects with constant density for atmospheric neutrinos only. The graph for probability versus energy at three different lengths in vacuum as well as in matter (for normal and inverted
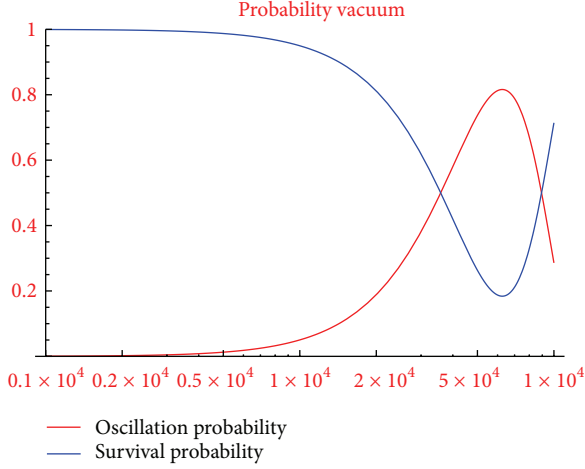
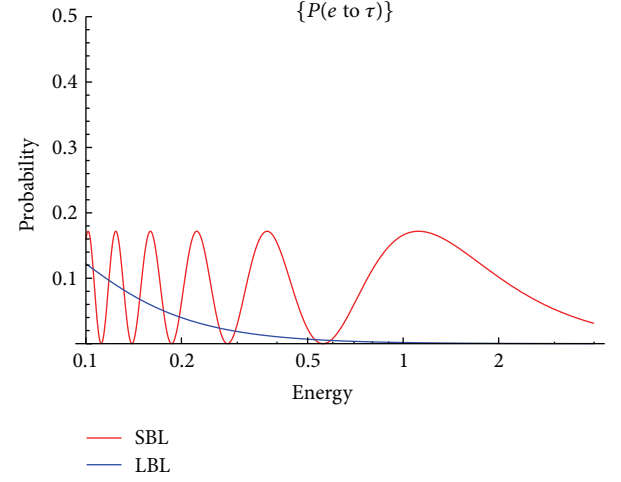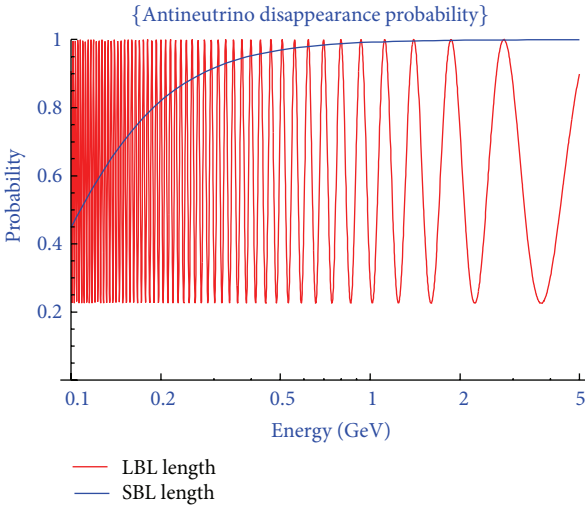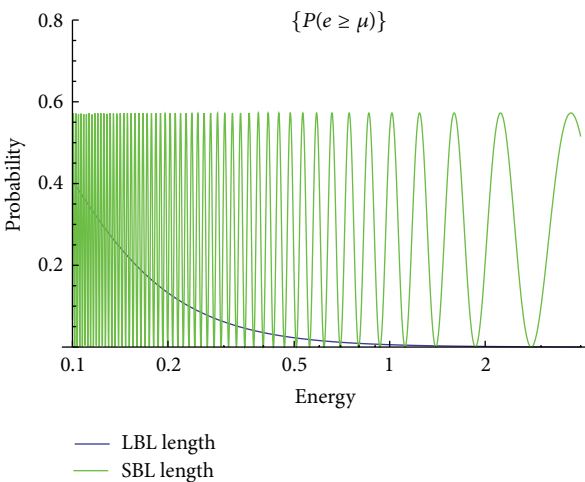FIGURE 2: Oscillations and survival probabilities.

FIGURE 3

FIGURE 4

FIGURE 5

hierarchy) for ν_μ to ν_e, ν_μ to ν_τ, and ν_μ to ν_μ shows that the presence of matter affects the neutrino oscillation probability the most in case of the ν_μ to ν_e oscillations. This may be due to the reason that the charged current interaction with the electrons of the medium directly affects the oscillation probability in case of electrons only. This proves the fact that the matter effects are flavor dependent. Moreover, oscillations become more frequent at low energy and the increase in length results in increase of the energy range which is more sensitive to oscillations but affects to a very lesser extent of magnitude of transition probability. Oscillation probability is the more of magnitude in case of μ to τ rather than an electron which concludes that a mass eigen state is more composed of μ and τ rather than an electron for atmospheric neutrinos especially. The smallness of ν_e part of total mass eigen state is measured by the mixing angle θ_{13}. For solar neutrino oscillation probability, reactor-based neutrino experiments and detectors are designed as different experiments are designed in order to be sensitive to different values of Δm^2, by choosing the appropriate value of L/E, the neutrino beam consisting of ν_e when coming out of the source and having energy only ~1 MeV that is, in order to satisfy $\Delta m^2 L/2E \sim 1$ and L can be of order such that $L/E \ <= \ 1$ km/MeV. Solar neutrino experiments have sensitivity for Δm^2 of very small value. SNO has been extremely successful so far to prove neutrino flavor change because it was designed in such a way to detect the changed flavor of solar neutrinos, that is, the flavor in which ν_e changed. After analyzing the results from flux ratios, it was found that day-night asymmetry has more sensitivity to the value of Δm^2_{12}. The results favored large mixing angle solutions. The CHOOZ [25] experiment put forward that the electron antineutrino survival probability and its oscillation probability from electron to muon and tau are excluded for $\Delta m^2 \geq 8 \times 10^{-4}$ eV^2 and at maximum mixing $\sin^2 2\theta \geq 0.17$. The chain of experiment continued with the KamLAND [8] experiment which provided evidence of antineutrino disappearance probability for the very first time with a long baseline length of the order of ~175 km as shown in Figure 6. All reactor-based neutrinos consider only

$\{P(\mu \geq \tau)\}$

$L = 3000$ m

— Vacuum
— Matter NH
— Matter IH

(a)

$\{P(\mu \geq \tau)\}$

$L = 6000$ m

— Vacuum
— Matter NH
— Matter IH

(b)

$\{P(\mu \geq \tau)\}$
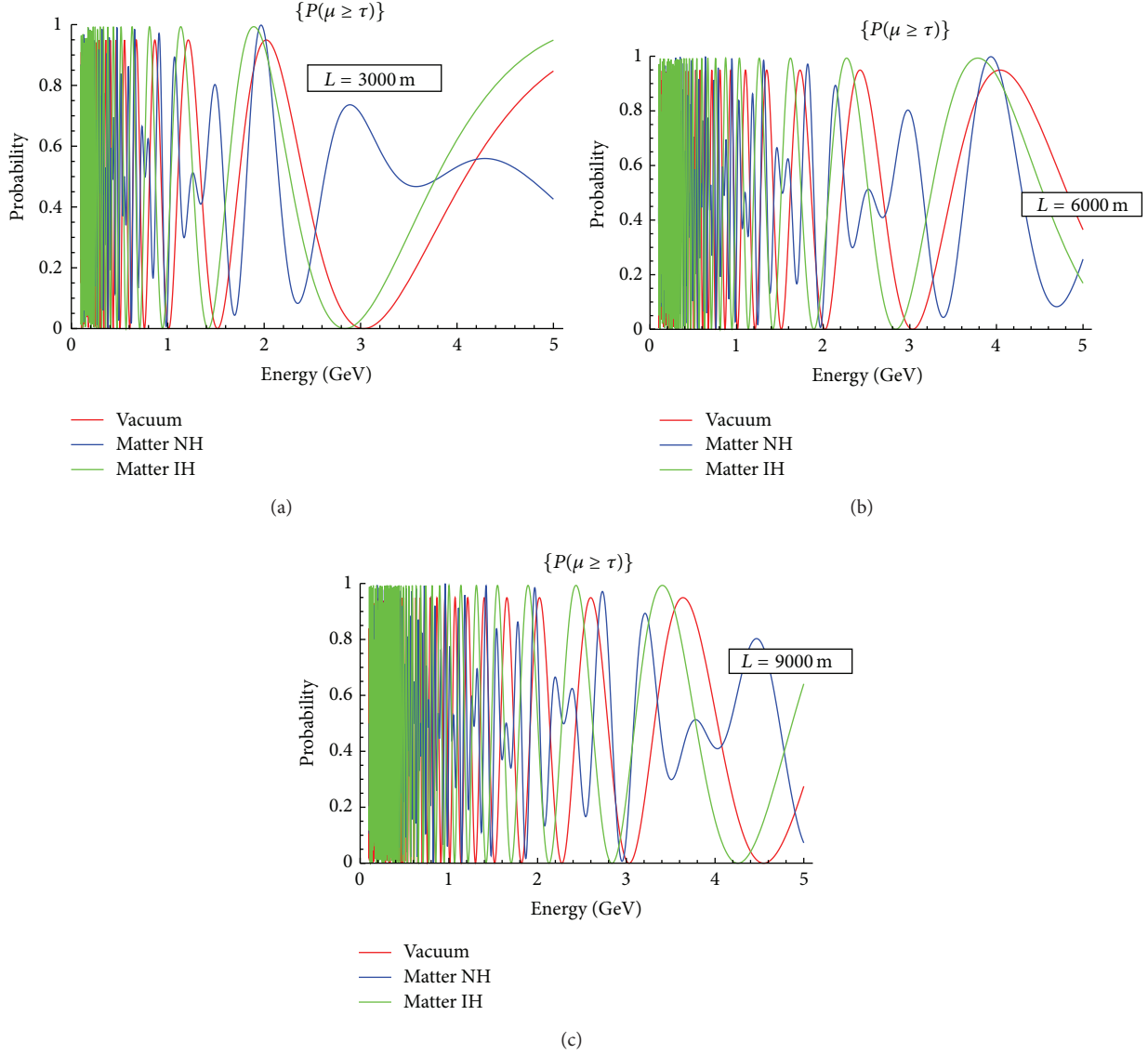
$L = 9000$ m

— Vacuum
— Matter NH
— Matter IH

(c)

Figure 6

the vacuum part of the oscillation probability as matter effects are negligible at low energy. Transitions from electron to μ's and τ's are shown in Figures 7 and 8. Two plots here show that the maximum oscillations are from e to μ for solar neutrinos.

9. Summary and Conclusion

In summary, two flavor oscillations are discussed here for the probability measurements and moreover generalized to three flavor despite in vacuum and matter. In the era of neutrino physics, vast amount of the information for oscillation parameters we have in hand, still the situation is not clear yet. Neutrino oscillations helped to solve the greatest mystery in solar world but actual mass of the neutrinos is still unknown. The recent searches for neutrinos are more focused on the estimation of parameter θ_{13} [25] and the

speed of neutrinos. Some more pieces of information on the mass hierarchy and CP violating phase can help us to get more detailed information on oscillation phenomenology. Future experiments are more concerned to find the accurate and precise measurement of θ_{13}. The accurate determination of θ_{13} can provide the more elaborated information on CP violating phase δ and mass hierarchy. It is proposed that a sign of Δm^2_{31} can also be well determined. Reactor experiments are unique in the sense that they can give a value of θ_{13} up to a higher precision value. That's why more recent experiments like Double CHOOZ, T2K, Nova, Daya Bay, and MINOS are designed in such a way that they are expected to measure this angle. Nuclear reactor experiments are more efficient in determination of θ_{13} as for the case of long baseline experiments sensitivity is reduced due to effect of transition parameters and matter effects. Nuclear reactors are
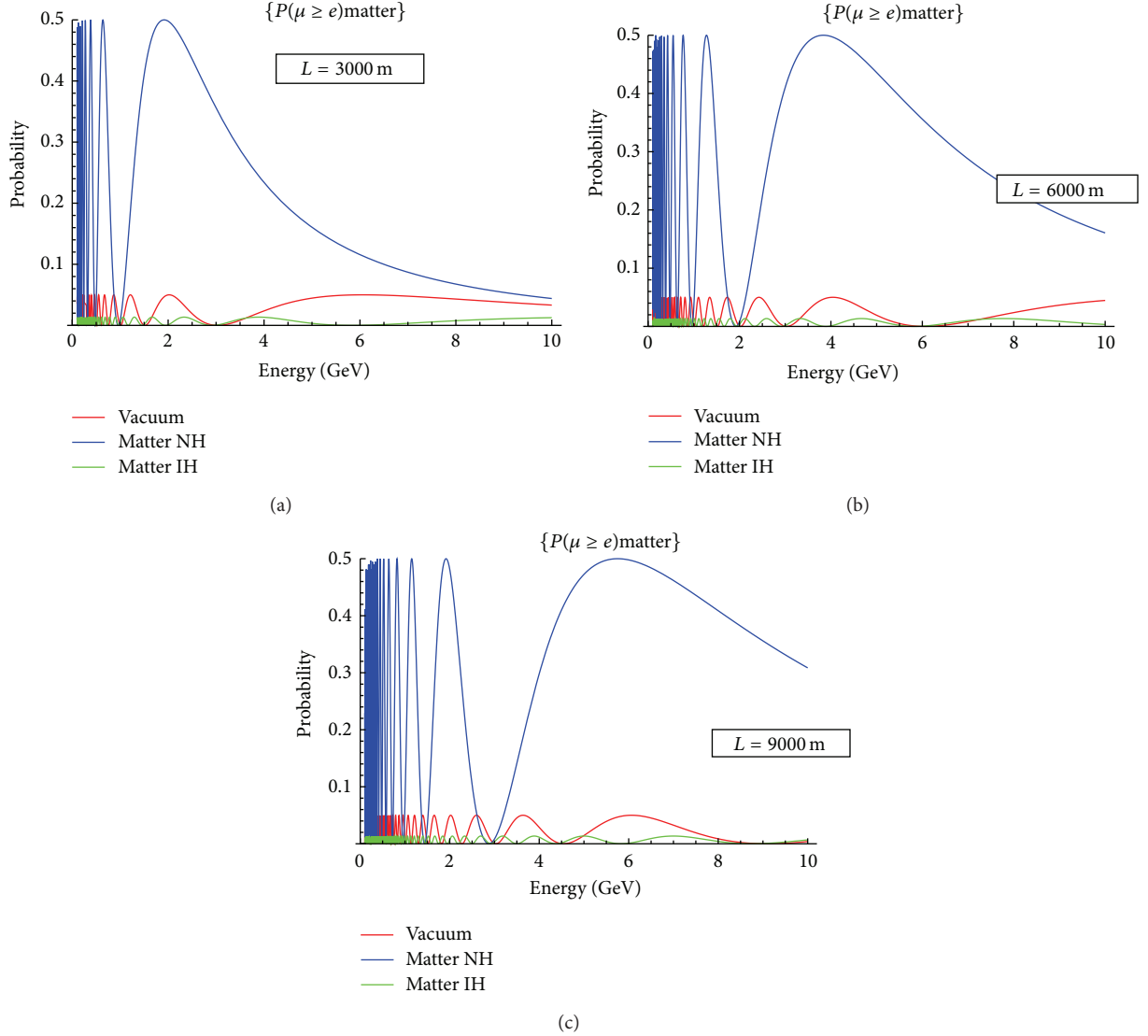
$\{P(\mu \geq e)\text{matter}\}$

$L = 3000\,\text{m}$

Vacuum
Matter NH
Matter IH

(a)

$\{P(\mu \geq e)\text{matter}\}$

$L = 6000\,\text{m}$

Vacuum
Matter NH
Matter IH

(b)

$\{P(\mu \geq e)\text{matter}\}$

$L = 9000\,\text{m}$

Vacuum
Matter NH
Matter IH

(c)

FIGURE 7

very intense sources of $\overline{\nu}_e$'s coming from the β-decay of the neutron-rich fission fragments. If relaxation θ_{13} is avoided, that antineutrino disappearance probability becomes

$$P\left(\overline{\nu}_e \longrightarrow \overline{\nu}_e\right) = 1 - \sin^2_{2\theta_{13}} \sin^2 \Delta m^2_{31} \frac{L}{2E}$$
$$- \cos 4_{\theta_{13}} \sin^2_{\theta_{12}} \sin^2 \Delta m^2_{31} \frac{L}{2E}. \tag{64}$$

Thus sensitivity of θ_{13} can be seen more at small energy and short baseline rather than long baseline as shown in Figure 9. The present information about this angle can be determined through global analysis of data. Gonzalez-Garcia et al. predicted the following limits on θ_{13} from various experiments [26]:

$$\sin^2_{\theta_{13}} < \left\{ \begin{array}{l} 0.060\,(0.089)\,(\text{solar} + \text{KAMLAND}) \\ 0.027\,(0.058)\,(\text{CHOOZ}+\text{atm}+\text{K2K}+\text{MINOS}) \\ 0.035\,(0.056)\,(\text{Global data}) \end{array} \right\}. \tag{65}$$

But what would happen if this angle is large? Earlier experiments assumed θ_{13} as zero because of some discrete symmetries present in the mixing parameters. Now the question arises how these symmetries are changed if θ_{13} is large or the new parameter ranges are compatible with the PMNS matrix U and whether the more resolved values of two mass differences is expected or not? What are the changes expected in the neutrino beam energy to measure more accurately the mixing parameters?
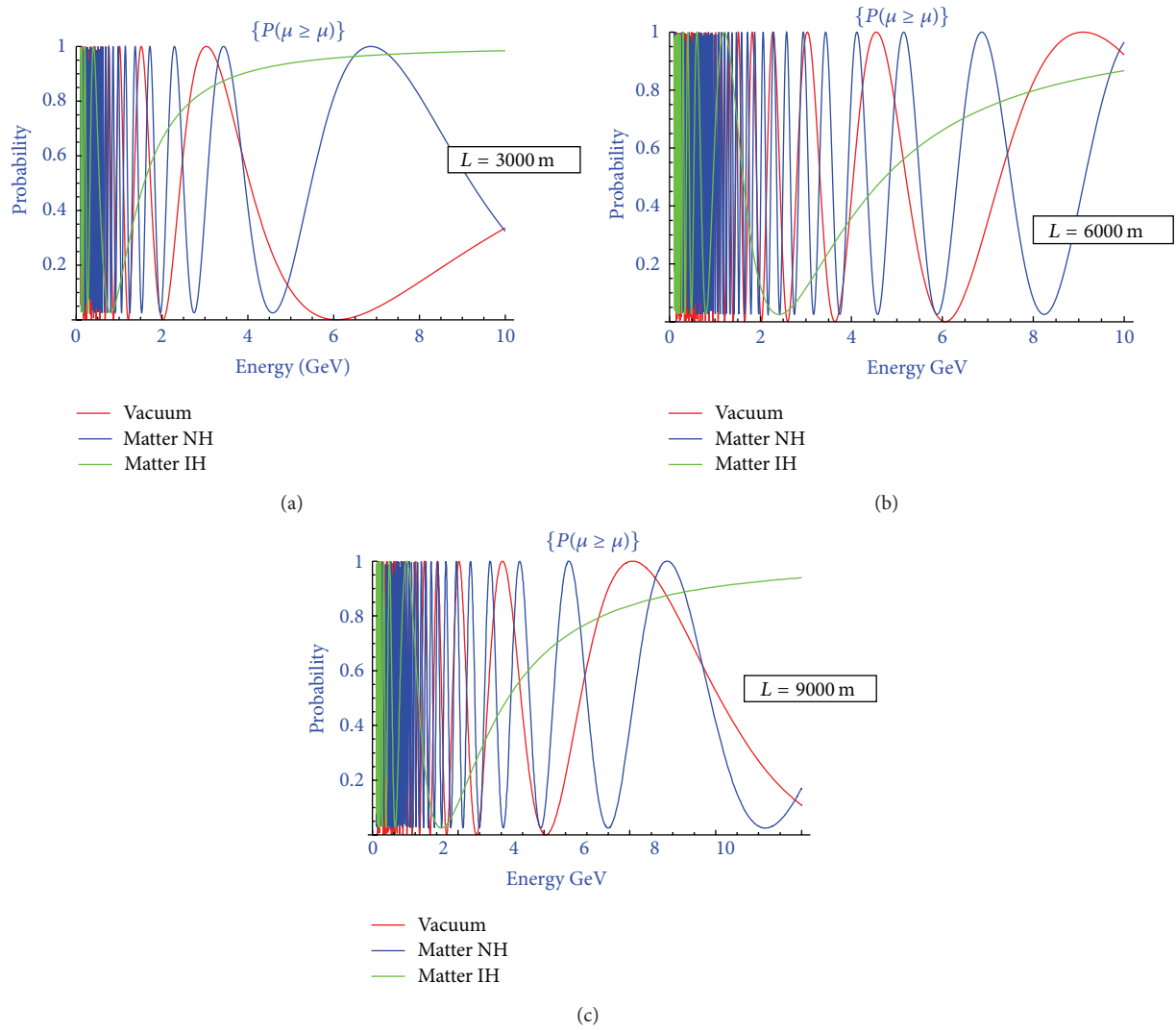
$\{P(\mu \geq \mu)\}$

$L = 3000\,\text{m}$

Probability

Energy (GeV)

— Vacuum
— Matter NH
— Matter IH

(a)

$\{P(\mu \geq \mu)\}$

$L = 6000\,\text{m}$

Probability

Energy GeV

— Vacuum
— Matter NH
— Matter IH

(b)

$\{P(\mu \geq \mu)\}$

$L = 9000\,\text{m}$

Probability

Energy GeV

— Vacuum
— Matter NH
— Matter IH

(c)

FIGURE 8

$\{\text{Sensitivity to } \theta_{13}\}$

Probability

Energy (MeV)

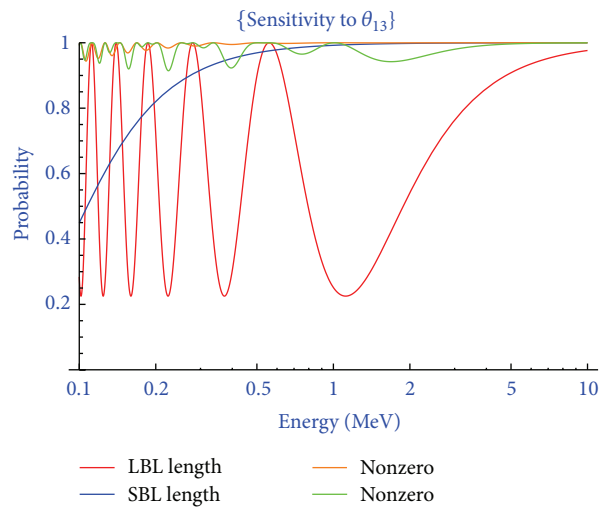— LBL length   — Nonzero
— SBL length   — Nonzero

FIGURE 9

Appendix

Oscillation Plots for Vacuum and Matter

For more details, see Figures 2, 3, 4, 5, 6, 7, 8, and 9 and Table 1.

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