International Scholarly Research Network
ISRN Mathematical Analysis
Volume 2012, Article ID 107846, 10 pages
doi:10.5402/2012/107846

Research Article

Traveling Wave Solutions of a Generalized Zakharov-Kuznetsov Equation

Wenbin Zhang[1] and Jiangbo Zhou[2]

[1] *Taizhou Institute of Science and Technology, NanJing University of Science and Technology, Taizhou 225300, China*
[2] *Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University, Zhenjiang 212013, China*

Correspondence should be addressed to Wenbin Zhang, zhangwbshux@163.com

Received 19 December 2011; Accepted 4 January 2012

Academic Editors: G. Akrivis and M. Winter

We employ the bifurcation theory of planar dynamical system to investigate the traveling-wave solutions of the generalized Zakharov-Kuznetsov equation. Four important types of traveling wave solutions are obtained, which include the solitary wave solutions, periodic solutions, kink solutions, and antikink solutions.

1. Introduction

Consider the following generalized Zakharov-Kuznetsov (ZK) equation:

$$u_t + \alpha(u^n)_x + \left(\beta u_{xx} + \gamma u_{yy} + \delta u_{zz}\right)_x = 0, \tag{1.1}$$

where $n \geq 2$, α, β, γ, δ are real constants. The ZK equation was first derived for describing weakly nonlinear ion acoustic waves in a strongly magnetized lossless plasma composed of cold ions and hot isothermal electrons [1]. The ZK equation is also known as one of the two-dimensional generalizations of the KdV equation (see [2, 3]), and it is not integrable by the inverse scattering transform method [4].

When $n = 2$, $\alpha = (1/2)a$, $\beta = 1, \gamma = 1$, and $\delta = 1$, (1.1) reduced to the equation

$$u_t + auu_x + \left(u_{xx} + u_{yy} + u_{zz}\right)_x = 0. \tag{1.2}$$

Wazwaz [5] obtained periodic solutions and solitary-wave solutions of (1.2) by using the sine-cosine algorithm method.

When $\alpha = a,\ \beta = b,\ \ \gamma = b$, and $\delta = 0$, (1.1) reduced to the equation

$$u_t + a(u^n)_x + b(u_{xx} + u_{yy})_x = 0. \tag{1.3}$$

Wazwaz [6] obtained some solitary-wave solutions and periodic structures of (1.2) by using the extended tanh method.

In this paper, we will employ the dynamical system theory [7] to investigate the traveling-wave solutions of (1.1). Numbers of smooth solitary-wave solutions, periodic solutions, kink solutions, and antikink solutions are given for each parameter condition. Here we note that such a powerful method has been employed by many authors to solve many partial differential equations [8–12].

2. Plane Phase Analysis

Let $\xi = x + y + z - ct$, where c is the wave speed. By using the traveling wave transformation $u(x, y, z, t) = \phi(x+y+z-ct) = \phi(\xi)$, we can reduce (1.1) to the following ordinary differential equation:

$$-a\phi_\xi + b(\phi^n)_\xi + \phi_{\xi\xi\xi} = 0, \tag{2.1}$$

where $(\cdot)_\xi$ denotes the derivative of the function with respect to ξ, $a = c/(\beta + \gamma + \delta)$, and $b = \alpha/(\beta + \gamma + \delta)$.

Integrating (2.1) once and setting the integration constant as 0, we have

$$-a\phi + b\phi^n + \phi_{\xi\xi} = 0. \tag{2.2}$$

Let $\phi' = y$; then (2.2) can be transformed into the following planar dynamical system:

$$\begin{aligned} \frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= a\phi - b\phi^n. \end{aligned} \tag{2.3}$$

We call it the traveling-wave system of (1.1). It is a planar dynamical system with Hamiltonian function

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{a}{2}\phi^2 + \frac{b}{n+1}\phi^{(n+1)} = h, \tag{2.4}$$

where h is a constant.

According to the theory of dynamical systems [7], we can obtain the properties of singular points as follows.

Proposition 2.1. *When $n = 2k$ is even, system* (2.3) *has two singular points $o(0,0)$ and $A(\phi_1, 0)$, where $\phi_1 = (a/b)^{1/(2k-1)}$.*

(i) *When $a > 0$, $o(0,0)$ is a saddle point and $A(\phi_1, 0)$ is a center point.*

(ii) *When $a = 0$, there is only one degenerate saddle point $o(0,0)$.*

(iii) *When $a < 0$, $o(0,0)$ is a center point and $A(\phi_1, 0)$ is a saddle point.*

Proposition 2.2. (1) *When $n = 2k + 1$ is odd and $ab > 0$, system* (2.3) *has three singular points $o(0,0)$ and $B(\pm\phi_2, 0)$, where $\phi_2 = (a/b)^{1/2k}$.*

(i) *When $a > 0$, $o(0,0)$ is a saddle point and $B(\pm\phi_2, 0)$ are center points.*

(ii) *When $a = 0$, there is only one degenerate saddle point $o(0,0)$.*

(iii) *When $a < 0$, $o(0,0)$ is a center point and $B(\pm\phi_2, 0)$ are saddle points.*

(2) *When $n = 2k + 1$ is odd and $ac \leq 0$, system* (2.3) *only has one singular point $o(0,0)$.*

(i) *When $a < 0$, $o(0,0)$ is a saddle point or a high-order saddle point for $a = 0$.*

(ii) *When $a > 0$, $o(0,0)$ is a center point or a high-order center point for $a = 0$.*

From the above analysis, we can obtain the bifurcations of phase portraits of system (2.3) in Figures 1 and 2.

3. Traveling Wave Solutions of (1.1)

Suppose that $\phi(\xi)$ is a continuous solution of (1.1) for $\xi \in (-\infty, +\infty)$ and $\lim_{\xi \to \infty} \phi(\xi) = A$, $\lim_{\xi \to -\infty} \phi(\xi) = B$. Recall that (i) $\phi(\xi)$ is called a solitary wave solution if $A = B$ and (ii) $\phi(\xi)$ is called a kink solutions, or antikink solution if $A \neq B$. Usually, a solitary wave solution of (1.1) corresponds to a homoclinic orbit of its traveling wave system (2.3), a kink (or antikink) wave solution of (1.1) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of system (2.3), and a periodic solution of (1.1) corresponds to a periodic orbit of system (2.3).

The case $n = 2$. As a example, we discuss the parameter region $a > 0$, $b > 0$ (see Figure 1(a)). In this case, system (2.4) has the form

$$H_2(\phi, y) = \frac{1}{2}y^2 - \frac{a}{2}\phi^2 + \frac{b}{3}\phi^3 = h. \tag{3.1}$$

From Figure 1(a) we can see that system (2.4) has a homoclinic orbit and a family of periodic orbits.

Corresponding to the homoclinic orbit defined by $H_2(\phi, y) = H_2(0,0) = 0$, we have

$$y^2 = a\phi^2 - \frac{2b}{3}\phi^3. \tag{3.2}$$

Substituting (3.2) into the first equation of system (2.3) and integrating along the corresponding homoclinic orbit, we obtain a smooth solitary wave solution:

$$u_1(\xi) = \frac{3a}{2b} \sec h^2\left(\frac{\sqrt{a}}{2}\xi\right). \tag{3.3}$$

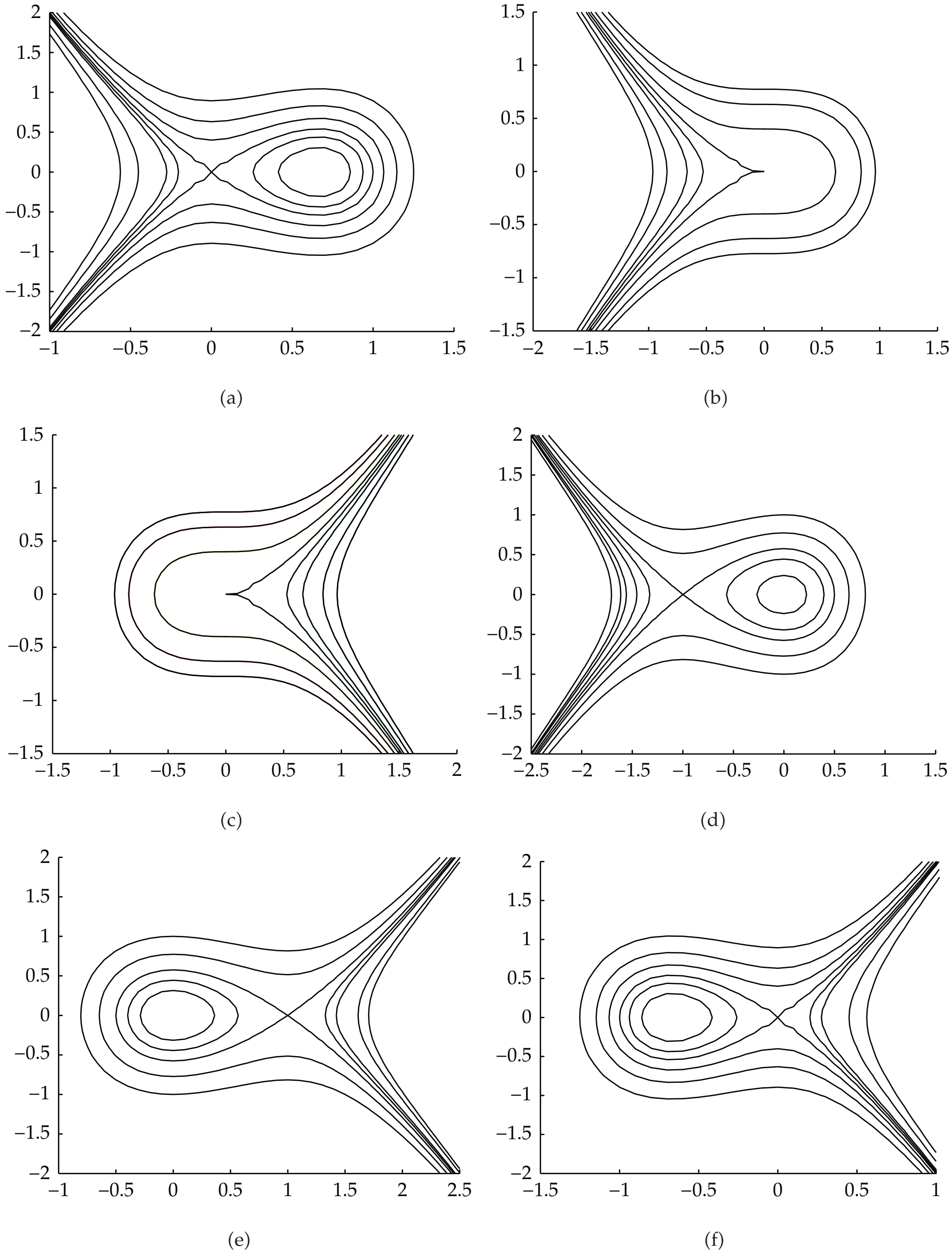

Figure 1: The phase portraits of system (2.3) for $n = 2k$. (a) $a > 0$, $b > 0$; (b) $a = 0$, $b > 0$; (c) $a = 0$, $b < 0$; (d) $a < 0$, $b > 0$; (e) $a < 0$, $b < 0$; (f) $a > 0$, $b < 0$.

Corresponding to the family of periodic orbits defined by $H_2(\phi, y) = h$, $h(h_1, 0)$, we have

$$y^2 = \frac{2b}{3}(\phi - r_1)(\phi - r_2)(r_3 - \phi), \tag{3.4}$$

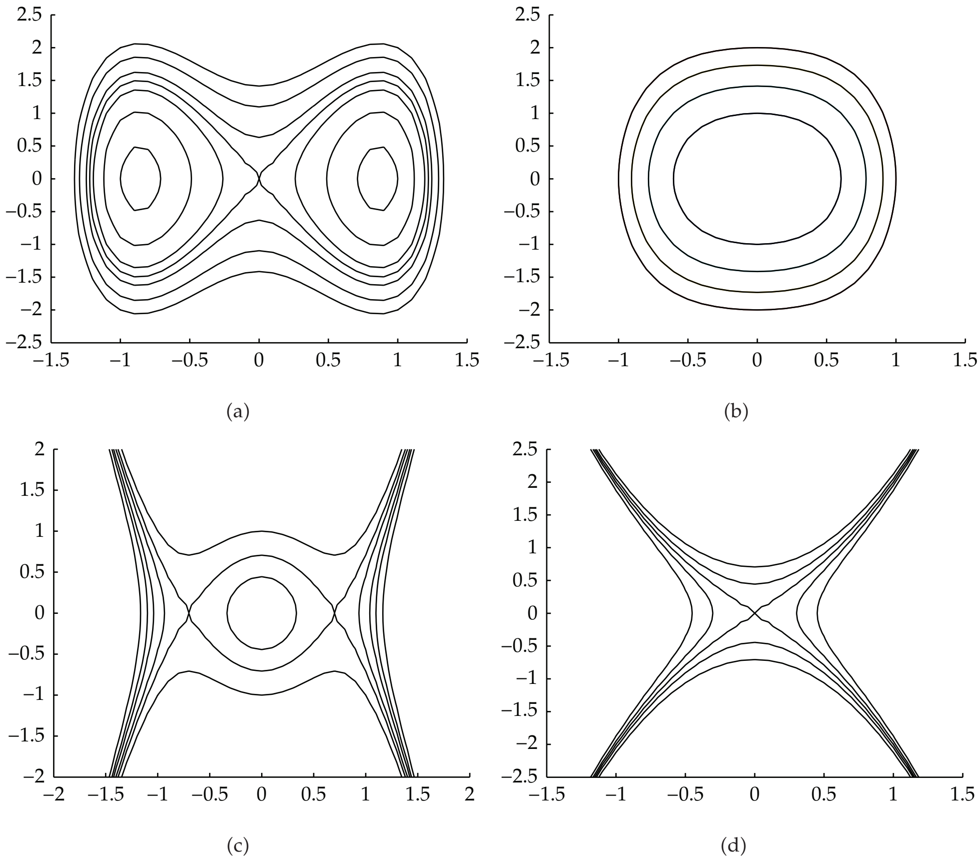

Figure 2: The phase portraits of system (2.3) for $n = 2k + 1$. (a) $a > 0, b > 0$; (b) $a \leq 0, b > 0$; (c) $a < 0, b < 0$; (d) $a \geq 0, b < 0$.

where r_1, r_2, r_3 are three real roots of the equation $a\phi^2 - (2b/3)\phi^3 + 2h = 0$ and $h_1 = H_2(\phi_1, 0) = -a^3/6b^2$. Thus, we obtain a periodic solution:

$$u_2(\xi) = r_3 - (r_3 - r_2) sn^2\left(\sqrt{\frac{b(r_3 - r_1)}{6}}\xi, \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}\right). \tag{3.5}$$

The case $n = 3$. In this case, system (2.4) has the form

$$H_3(\phi, y) = \frac{1}{2}y^2 - \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 = h. \tag{3.6}$$

(1) From Figure 2(a) we can see that system (2.4) has two homoclinic orbits and three families of periodic orbits.

Corresponding to the two homoclinic orbits defined by $H_3(\phi, y) = H_3(0,0) = 0$, we have

$$y^2 = a\phi^2 - \frac{b}{2}\phi^4. \tag{3.7}$$

Substituting (3.7) into the first equation of system (2.3), and integrating along the corresponding homoclinic orbits, we obtain two smooth solitary wave solutions:

$$u_{3,4}(\xi) = \pm\frac{2a}{b}\sec h\left(\sqrt{a}\xi\right). \tag{3.8}$$

Corresponding to the two families of periodic orbits defined by $H_3(\phi, y) = h$, $h(h_2, 0)$, we have

$$y^2 = a\phi^2 - \frac{b}{2}\phi^4 + 2h, \tag{3.9}$$

where $h_2 = H_3(\pm\phi_2, 0) = -a^2/4b$.

Substituting (3.9) into the first equation of system (2.3) and integrating along the corresponding periodic orbit, we obtain two periodic solutions:

$$u_{5,6}(\xi) = \pm\sqrt{\frac{a+k}{b}}dn\left(\sqrt{\frac{a+k}{2}}\xi, \sqrt{\frac{2k}{a+k}}\right), \tag{3.10}$$

where $k = \sqrt{a^2 + 4bh}$.

Corresponding to the family of periodic orbits defined by $H_3(\phi, y) = h$, $h \in (0, \infty)$, we have

$$y^2 = a\phi^2 - \frac{b}{2}\phi^4 + 2h. \tag{3.11}$$

Substituting (3.11) into the first equation of system (2.3) and integrating along the corresponding periodic orbit, we obtain a periodic solution:

$$u_7(\xi) = \sqrt{\frac{a+k}{b}}dn\left(\sqrt{k}\xi, \sqrt{\frac{a+k}{2k}}\right). \tag{3.12}$$

(2) From Figure 2(c) we can see that system (2.4) has two heteroclinic orbits and a family of periodic orbits.

Corresponding to the two heteroclinic orbits defined by $H_3(\phi, y) = h_2$, we have

$$y^2 = a\phi^2 - \frac{b}{2}\phi^4 - \frac{a^2}{2b}. \tag{3.13}$$

Substituting (3.13) into the first equation of system (2.3) and integrating along the corresponding heteroclinic orbits, we obtain kink solutions, and antikink solutions:

$$u_{8,9}(\xi) = \pm\sqrt{\frac{a}{b}}\tan h\left(\sqrt{\frac{-a}{2}}\xi\right). \tag{3.14}$$

Corresponding to the family of periodic orbits defined by $H_3(\phi, y) = h$, $h \in (0, h_2)$, we have

$$y^2 = a\phi^2 - \frac{b}{2}\phi^4 + 2h. \tag{3.15}$$

Substituting (3.15) into the first equation of system (2.3) and integrating along the corresponding periodic orbits, we obtain a periodic solution:

$$u_{10}(\xi) = \sqrt{\frac{a+k}{b}}sn\left(\sqrt{-a}\xi, \sqrt{\frac{a-k}{2a}}\right). \tag{3.16}$$

(3) From Figure 2(b) we can see that system (2.4) has a family of periodic orbits.

Corresponding to the family of periodic orbits defined by $H_3(\phi, y) = h$, $h \in (0, \infty)$, we have the same periodic solution of $u(\xi)$ as (3.12).

Specifically, when $c = 0$, (3.12) has the form

$$y^2 = -\frac{b}{2}\phi^4 + 2h, \qquad h \in (0, \infty). \tag{3.17}$$

Substituting (3.17) into the first equation of system (2.3) and integrating along the corresponding periodic orbits, we obtain a periodic solution:

$$u_{11}(\xi) = 2\sqrt{\frac{h}{b}}cn\left(2\sqrt{h}\xi, \frac{\sqrt{2}}{2}\right). \tag{3.18}$$

The case $n > 3$.

(1) When $n = 2k$ is even, from Figure 1(a) we can see that system (2.4) has a homoclinic orbit.

Corresponding to the homoclinic orbit defined by $H(\phi, y) = H(0, 0) = 0$, we have

$$y^2 = a\phi^2 - \frac{2b}{(2k+1)}\phi^{2k+1}. \tag{3.19}$$

Substituting (3.19) into the first equation of system (2.3), we have

$$\int_0^{\phi} \frac{1}{s\sqrt{a-(2b/(2k+1))s^{(2k-1)}}} ds = \pm \int_0^{\xi} ds. \tag{3.20}$$

Let

$$\varphi = s^{2k-1}, \qquad \varphi_1 = \phi^{2k-1}. \tag{3.21}$$

Thus, (3.20) and (3.21) merge into

$$\int_0^{\varphi_1} \frac{1}{\varphi\sqrt{a-(2b/(2k+1))\varphi}} d\varphi = (2k-1)|\xi|. \tag{3.22}$$

Completing the integral in (3.22), we obtain

$$\varphi(\xi) = \frac{(2k+1)a}{2b} \sec h^2\left(\frac{(2k-1)\sqrt{a}}{2}\xi\right). \tag{3.23}$$

From (3.21) and (3.23), we have

$$u_{12}(\xi) = \left[\frac{(2k+1)a}{2b} \sec h^2\left(\frac{(2k-1)\sqrt{a}}{2}\xi\right)\right]^{1/(2k-1)}. \tag{3.24}$$

(2) When $n = 2k+1$ is odd, from Figure 2(a) we can see that system (2.4) has two homoclinic orbits. Corresponding to the homoclinic orbits defined by $H(\phi, y) = H(0,0) = 0$, we have

$$y^2 = a\phi^2 - \frac{b}{(k+1)}\phi^{2k+2}. \tag{3.25}$$

Substituting (3.25) into the first equation of system (2.3), we have

$$\int_0^{\phi} \frac{1}{s\sqrt{(A-s^k)(A+s^k)}} ds = \pm\sqrt{\frac{b}{(k+1)}} \int_0^{\xi} ds, \tag{3.26}$$

where $A = \sqrt{a(k+1)/b}$.

Let

$$\psi = s^k, \qquad \psi_1 = \phi^k. \tag{3.27}$$

Thus, (3.26) and (3.27) merge into

$$\int_0^{\psi_1} \frac{1}{\psi\sqrt{(A-\psi)(A+\psi)}}d\psi = k\sqrt{\frac{b}{(k+1)}}|\xi|. \tag{3.28}$$

Completing the integral in (3.28), we obtain

$$\psi(\xi) = \pm\frac{(k+1)a}{b}\sec h\left(\sqrt{a}k\xi\right). \tag{3.29}$$

From (3.27) and (3.29), we have

$$u_{13,14}(\xi) = \pm\left[\frac{(k+1)a}{b}\sec h\left(\sqrt{a}k\xi\right)\right]^{1/k}. \tag{3.30}$$

Acknowledgments

This work was supported by the Startup Fund for Advanced Talents of Jiangsu University (no. 09JDG013), the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (no. 09KJB110003), the Jiangsu Planned Projects for Postdoctoral Research Funds (no. 0902107C), the Jiangsu Government Scholarship for Overseas Studies, the Taizhou Social Development Project (no. 2011213), and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] V. E. Zakharov and E. A. Kuznetsov, "On three-dimensional solitons," *Soviet Physics*, vol. 39, pp. 285–288, 1974.

[2] Y. S. Kivshar and D. E. Pelinovsky, "Self-focusing and transverse instabilities of solitary waves," *Physics Reports*, vol. 331, no. 4, pp. 117–195, 2000.

[3] A.-M. Wazwaz, "Nonlinear dispersive special type of the Zakharov-Kuznetsov equation ZK(n,n) with compact and noncompact structures," *Applied Mathematics and Computation*, vol. 161, no. 2, pp. 577–590, 2005.

[4] B. K. Shivamoggi, "The Painlevé analysis of the Zakharov-Kuznetsov equation," *Physica Scripta*, vol. 42, no. 6, pp. 641–642, 1990.

[5] A.-M. Wazwaz, "Exact solutions with solitons and periodic structures for the Zakharov-Kuznetsov (ZK) equation and its modified form," *Communications in Nonlinear Science and Numerical Simulation*, vol. 10, no. 6, pp. 597–606, 2005.

[6] A.-M. Wazwaz, "The extended tanh method for the Zakharov-Kuznetsov (ZK) equation, the modified ZK equation, and its generalized forms," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 6, pp. 1039–1047, 2008.

[7] D. Luo, X. Wang, D. Zhu, and M. Han, *Bifurcation Theory and Methods of Dynamical Systems*, World Scientific, River Edge, NJ, USA, 1997.

[8] J.-B. Li, "Exact traveling wave solutions to 2D-generalized Benney-Luke equation," *Applied Mathematics and Mechanics. English Edition*, vol. 29, no. 11, pp. 1391–1398, 2008.

[9] J. Zhou and L. Tian, "Solitons, peakons and periodic cusp wave solutions for the Fornberg-Whitham equation," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 356–363, 2010.

[10] J. Zhou, L. Tian, and X. Fan, "Soliton, kink and antikink solutions of a 2-component of the Degasperis-Procesi equation," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2529–2536, 2010.

[11] C. Deng, "New exact solutions to the Zakharov-Kuznetsov equation and its generalized form," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 4, pp. 857–868, 2010.

[12] L.-H. Zhang, "Travelling wave solutions for the generalized Zakharov-Kuznetsov equation with higher-order nonlinear terms," *Applied Mathematics and Computation*, vol. 208, no. 1, pp. 144–155, 2009.