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Research Article

Local Convexity Shape-Preserving Data Visualization by Spline Function

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The main purpose of this paper is the visualization of convex data that results in a smooth, pleasant, and interactive convexity-preserving curve. The rational cubic function with three free parameters is constructed to preserve the shape of convex data. The free parameters are arranged in a way that two of them are left free for user choice to refine the convex curve as desired, and the remaining one free parameter is constrained to preserve the convexity everywhere. Simple data-dependent constraints are derived on one free parameter, which guarantee to preserve the convexity of curve. Moreover, the scheme under discussion is, C^1 flexible, simple, local, and economical as compared to existing schemes. The error bound for the rational cubic function is $O(h^3)$.

1. Introduction

Spline interpolation is a significant tool in computer graphics, computer-aided geometric design and engineering as well. Convexity is prevalent shape feature of data. Therefore, the need for convexity preserving interpolating curves and surfaces according to the given data becomes inevitable. The aspiration of this paper is to preserve the hereditary attribute that is the convexity of data. There are many applications of convexity preserving of data, for instance, in the design of telecommunication systems, nonlinear programming arising in engineering, approximation of functions, optimal control, and parameter estimation.

The problem of convexity-preserving interpolation has been considered by a number of authors [1–21] and references therein. Bao et al. [1] used function values and first derivatives of function to introduce a rational cubic spline (cubic/cubic). A method for value control, inflection-point control and convexity control of the interpolation at a point was developed to be used in practical curve design. Asaturyan et al. [3] constructed a six-degree piecewise polynomial interpolant for the space curves to satisfy the shape-preserving properties for collinear and coplanar data.

Brodlie and Butt [4] developed a piecewise rational cubic function to preserve the shape of convex data. In [4], the authors inserted extra knots in the interval where the interpolation loses the convexity of convex data which is the drawback of this scheme. Carnicer et al. [5] analyzed the convexity-preserving properties of rational Bézier and non-uniform rational B-spline curves from a geometric point of view and also characterize totally positive systems of functions in terms of geometric convexity-preserving properties of the rational curves.

Clements [6] developed a C^2 parametric rational cubic interpolant with tension parameter to preserve the convexity. Sufficient conditions were derived to preserve the convexity of the function on strictly left/right winding polygonal line segments. Costantini and Fontanella [8] preserved the convexity of data by semi-global method. The scheme has some research gaps like the degree of rectangular patches in the interpolant that was too large; the resulting surfaces were not visually pleasing and smooth.

Delbourgo and Gregory [9] developed an explicit representation of rational cubic function with one free parameter which can be used to preserve the convexity of convex data. Meng and Shi Long [11] also developed an explicit representation of rational cubic function with two free parameters which can be used to preserve the convexity of convex data. In the schemes [9, 11], there was no choice for user to refine the convexity curve as desired. The rational spline was represented in terms of first derivative values at the knots and provided an alternative to the spline under tension to preserve the shape of monotone and convex data by Gregory [10].

McAllister [12], Passow [13], and Roulier [14] considered the problem of interpolating monotonic and convex data in the sense of monotonicity and convexity preserving. They used a piecewise polynomial Bernstein-Bézier function and introduce additional knots into their schemes. Such a scheme for quadratic spline interpolation was described by McAllister [12] and was further developed by Schumaker [15] using piecewise quadratic polynomial which was very economical, but the method generally inserts an extra knot in each interval to interpolate.

Sarfraz and Hussain [17] used the rational cubic function with two shape parameters to solve the problem of convexity preserving of convex data. Data-dependent sufficient constraints were derived to preserve the shape of convex data. Sarfraz [18] developed a piecewise rational cubic function with two families of parameters. In [18], the authors derived the sufficient conditions on shape parameters to preserve the physical shape properties of data. Sarfraz [19–21] used piecewise rational cubic interpolant in parametric context for shape preserving of plane curves and scalar curves with planar data. The schemes [17–21] are local, but, unfortunately, they have no flexibility in the convexity-preserving curves.

In this paper, we construct a rational cubic function with three free parameters. One of the free parameter is used as a constrained to preserve the convexity of convex data while the other two are left free for the user to modify the convex curve. Sufficient data-dependent constraints are derived. Our scheme has a number of attributes over the existing schemes.

(i) In this paper, the shape-preserving of convex data is achieved by simply imposing the conditions subject to data on the shape parameters used in the description of rational cubic function. The proposed scheme works evenly good for both equally and unequally spaced data. In contrast [1] assumed certain function values and derivative values to control the shape of the data.

(ii) In [12, 15], the smoothness of interpolant is C^0 while in this work the degree of smoothness is C^1.

(iii) The developed scheme has been demonstrated through different numerical examples and observed that the scheme is not only local, computationally economical, and easy to compute, time saving but also visually pleasant as compared to existing schemes [17–21].

(iv) In [9–11, 17–21], the schemes do not allow to user to refine the convex curve as desired while for more pleasing curve (and still having the convex shape preserved) an additional modification is required, and this task is more easily done in this paper by simply adjustment of free parameters in the rational cubic function interpolation on user choice.

(v) In [17–21], the authors did not provide the error analysis of the interpolants while a very good $O(h^3)$ error bound is achieved in this paper.

(vi) In [4, 12–15], the authors developed the schemes to achieve the desired shape of data by inserting extra knots between any two knots in the interval while we preserve the shape of convex data by only imposing constraints on free parameters without any extra knots.

The remaining part of this paper is organized as follows. A rational cubic function is defined in Section 2. The error of the rational cubic interpolant is discussed in Section 3. The problem of shape preserving convexity curve is discussed in Section 4. Derivatives approximation method is given in Section 5. Some numerical results are given in Section 6. Finally, the conclusion of this work is discussed in Section 7.

2. Rational Cubic Spline Function

Let $\{(x_i, f_i),\ i = 0, 1, 2, \dots, n\}$ be the given set of data points such as $x_0 < x_1 < x_2 < \cdots < x_n$. The rational cubic function with three free parameters introduced by Abbas et al. [2], in each subinterval $I_i = [x_i, x_{i+1}],\ i = 0, 1, 2, \dots, n-1$, is defined as

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{2.1}$$

with

$$\begin{gathered} p_i(\theta) = u_i f_i (1-\theta)^3 + (w_i f_i + u_i h_i d_i)\theta(1-\theta)^2 + (w_i f_{i+1} - v_i h_i d_{i+1})\theta^2(1-\theta) + v_i f_{i+1}\theta^3, \\ q_i(\theta) = u_i(1-\theta)^3 + w_i\theta(1-\theta) + v_i\theta^3, \end{gathered} \tag{2.2}$$

where $\theta = x - x_i/h_i,\ h_i = x_{i+1} - x_i$, and u_i, v_i, w_i are the positive free parameters. It is worth noting that when we use the values of these free parameters as $u_i = 1,\ v_i = 1$ and $w_i = 3$, then the C^1 piecewise rational cubic function (2.1) reduces to standard cubic Hermite spline discussed in Schultz [16].

The piecewise rational cubic function has the following interpolatory conditions:

$$S_i(x_i) = f_i, \qquad S_i(x_{i+1}) = f_{i+1}, \qquad S_i'(x_i) = d_i, \qquad S_i'(x_{i+1}) = d_{i+1}, \tag{2.3}$$

where $S_i'(x)$ denotes the derivative with respect to "x," and d_i denotes the derivative values at knots.

3. Interpolation Error Analysis

The error analysis of piecewise rational cubic function (2.1) is estimated, without loss of generality, in the subinterval $I_i = [x_i, x_{i+1}]$. It is to mention that the scheme constructed in Section 2 is local. We suppose that $f(x) \in C^3[x_0, x_n]$, and $S_i(x)$ is the interpolation of function $f(x)$ over arbitrary subinterval $I_i = [x_i, x_{i+1}]$. The Peano Kernel Theorem, Schultz [16] is used to obtain the error analysis of piecewise rational cubic interpolation in each subinterval $I_i = [x_i, x_{i+1}]$, and it is defined as

$$R[f] = f(x) - S_i(x) = \frac{1}{2}\int_{x_i}^{x_{i+1}} f^{(3)}(\tau) R_x\left[(x-\tau)_+^2\right] d\tau. \tag{3.1}$$

In each subinterval, the absolute value of error is

$$\left|f(x) - S_i(x)\right| \leq \frac{1}{2}\left\|f^{(3)}(\tau)\right\| \int_{x_i}^{x_{i+1}} \left|R_x\left[(x-\tau)_+^2\right]\right| d\tau, \tag{3.2}$$

where

$$\begin{aligned} &R_x\left[(x-\tau)_+^2\right] \\ &= \begin{cases} (x-\tau)^2 - \dfrac{\left(w_i(x_{i+1}-\tau)^2 - 2h_i v_i(x_{i+1}-\tau)\right)\theta^2(1-\theta) + v_i(x_{i+1}-\tau)^2\theta^3}{q_i(\theta)} & x_i < \tau < x, \\ \dfrac{\left(w_i(x_{i+1}-\tau)^2 - 2h_i v_i(x_{i+1}-\tau)\right)\theta^2(1-\theta) + v_i(x_{i+1}-\tau)^2\theta^3}{q_i(\theta)} & x < \tau < x_{i+1}, \end{cases} \\ &= \begin{cases} a(\tau, x) & x_i < \tau < x, \\ b(\tau, x) & x < \tau < x_{i+1}, \end{cases} \end{aligned} \tag{3.3}$$

where $R_x[(x-\tau)_+^2]$ is called the Peano Kernel of integral. To derive the error analysis, first of all we need to examine the properties of the kernel functions $a(\tau, x)$ and $b(\tau, x)$, and then to find the values of following integrals:

$$\int_{x_i}^{x_{i+1}} \left|R_x\left[(x-\tau)_+^2\right]\right| d\tau = \int_{x_i}^{x} |a(\tau, x)| d\tau + \int_{x}^{x_{i+1}} |b(\tau, x)| d\tau. \tag{3.4}$$

So, we calculate these values in two parts. The proof of Theorem will be completed by combining these two parts.

3.1. Part 1

By simple computation, the roots of $a(x,x) = (\theta^2(1-\theta)^2((w_i - v_i)\theta + (2v_i - w_i))h_i^2)/q_i(\theta)$ in $[0,1]$ are $\theta = 0$, $\theta = 1$ and $\theta^* = 1 - v_i/(w_i - v_i)$. It is easy to show that when $\theta \le \theta^*$, $a(x,x) \le 0$ and $\theta \ge \theta^*$, $a(x,x) \ge 0$. The roots of quadratic function $a(\tau, x) = 0$ are

$$\tau_1^* = x - \frac{\theta h_i(\theta(w_i - v_i) + A)}{(1-\theta)u_i + \theta w_i}, \qquad \tau_1^{**} = x - \frac{\theta h_i(\theta(w_i - v_i) - A)}{(1-\theta)u_i + \theta w_i}, \tag{3.5}$$

where $A = \sqrt{v_i((w_i - 2v_i) + 3\theta) + w_i(w_i - 4v_i)\theta}$.

So, when $\theta > \theta^*$, $x_i < \tau_1^{**} < x$ and when $\theta < \theta^*, \tau_1^{**} > x$. Thus, $\theta < \theta^*$, $a(\tau, x) < 0$ for all $\tau \in [x_i, x]$,

$$\begin{aligned}\int_{x_i}^{x} |a(\tau,x)| d\tau &= \int_{x_i}^{x} (-a(\tau,x)) d\tau \\ &= \frac{(v_i(3-\theta) - w_i(1-\theta))(1-\theta)^3\theta^2 h_i^3}{3q_i(\theta)} + \frac{(w_i - 3v_i)(1-\theta)\theta^2 h_i^3}{3q_i(\theta)} + \frac{v_i\theta^3 h_i^3}{3q_i(\theta)} - \frac{\theta^3 h_i^3}{3}.\end{aligned} \tag{3.6}$$

The value of $a(\tau, x)$ varies from negative to positive on the root τ_1^{**} when $\theta > \theta^*$,

$$\begin{aligned}\int_{x_i}^{x} |a(\tau,x)| d\tau &= \int_{x_i}^{\tau_1^{**}} (-a(\tau,x)) d\tau + \int_{\tau_1^{**}}^{x} a(\tau,x) d\tau \\ &= \frac{2((w_i - v_i)\theta - A)^3\theta^3 h_i^3}{3((1-\theta)u_i + \theta w_i)^3} - \frac{\theta^3 h_i^3}{3} - \frac{2h_i^3}{3q_i(\theta)}\left[(1-\theta) + \frac{\theta((w_i - v_i)\theta - A)}{(1-\theta)u_i + \theta w_i}\right]^3 \\ &\quad \times ((1-\theta)w_i + \theta v_i) + \frac{2h_i^3 v_i \theta^2(1-\theta)}{q_i(\theta)}\left[(1-\theta) + \frac{\theta((w_i - v_i)\theta - A)}{(1-\theta)u_i + \theta w_i}\right]^2.\end{aligned} \tag{3.7}$$

3.2. Part 2

In this part, we discuss the properties of function $b(\tau, x)$. Consider $b(\tau, x), \tau \in [x, x_{i+1}]$ as function of τ. The roots of function $b(\tau, x)$ are similar as $a(\tau, x)$ in Section 3.1 at $\tau = x$. It is easy to show that when $\theta \le \theta^*$, $b(x,x) \le 0$ and $\theta \ge \theta^*$, $b(x,x) \ge 0$. The roots of quadratic function $b(\tau, x) = 0$ are

$$\tau_2^* = x_{i+1}, \qquad \tau_2^{**} = x_{i+1} - \frac{2(1-\theta)v_i h_i}{(1-\theta)w_i + \theta v_i}. \tag{3.8}$$

The function $b(\tau, x)$ varies from negative to positive on the root τ_2^{**} when $\theta \le \theta^*$. Thus,

$$\begin{aligned}\int_x^{x_{i+1}} |b(\tau,x)| d\tau &= \int_x^{\tau_2^{**}} (-b(\tau,x)) d\tau + \int_{\tau_2^{**}}^{x_{i+1}} b(\tau,x) d\tau \\ &= \frac{8\theta^2(1-\theta)^3 v_i{}^3 h_i^3}{3q_i(\theta)((1-\theta)u_i+\theta w_i)^2} + \frac{h_i^3\theta^2(1-\theta)^3}{3q_i(\theta)}(w_i(1-\theta)-v_i(3-\theta)),\end{aligned} \tag{3.9}$$

when $\theta \ge \theta^*$,

$$\begin{aligned}\int_x^{x_{i+1}} |b(\tau,x)| d\tau &= \int_x^{x_{i+1}} b(\tau,x) d\tau \\ &= \frac{h_i^3\theta^2(1-\theta)^3}{3q_i(\theta)}(v_i(3-\theta)-w_i(1-\theta)).\end{aligned} \tag{3.10}$$

Thus, from (3.6) and (3.9), it can be shown that when $0 \le \theta \le \theta^*$,

$$\left|f(x)-S_i(x)\right| \le \frac{1}{2}\left\|f^{(3)}(\tau)\right\| \int_{x_i}^{x_{i+1}} \left|R_x\left[(x-\tau(x-\tau)_+^2\right]\right| d\tau = \left\|f^{(3)}(\tau)\right\| h_i^3 p_1(u_i,v_i,w_i,\theta), \tag{3.11}$$

where

$$\begin{aligned}p_1(u_i,v_i,w_i,\theta) = {}& \frac{(v_i(3-\theta)-w_i(1-\theta))(1-\theta)^3\theta^2}{6\ q_i(\theta)} + \frac{(w_i-3v_i)(1-\theta)\theta^2}{6\ q_i(\theta)} + \frac{v_i\theta^3}{6\ q_i(\theta)} - \frac{\theta^3}{6} \\ &+ \frac{8\theta^2(1-\theta)^3 v_i^3}{6\ q_i(\theta)((1-\theta)u_i+\theta w_i)^2} + \frac{\theta^2(1-\theta)^3}{6\ q_i(\theta)}(w_i(1-\theta)-v_i(3-\theta)),\end{aligned} \tag{3.12}$$

and, from (3.7) and (3.10), it can be shown that when $\theta^* \le \theta \le 1$,

$$\left|f(x)-S_i(x)\right| \le \frac{1}{2}\left\|f^{(3)}(\tau)\right\| \int_{x_i}^{x_{i+1}} \left|R_x\left[(x-\tau)_+^2\right]\right| d\tau = \left\|f^{(3)}(\tau)\right\| h_i^3 p_2(u_i,v_i,w_i,\theta), \tag{3.13}$$

where

$$\begin{aligned}p_2(u_i,v_i,w_i,\theta) = {}& \frac{2((w_i-v_i)\theta-A)^3\theta^3}{6((1-\theta)u_i+\theta w_i)^3} - \frac{\theta^3}{6} - \frac{2}{6q_i(\theta)}\left[(1-\theta)+\frac{\theta((w_i-v_i)\theta-A)}{(1-\theta)u_i+\theta w_i}\right]^3 \\ &\times ((1-\theta)w_i+\theta v_i) + \frac{v_i\theta^2(1-\theta)}{q_i(\theta)}\left[(1-\theta)+\frac{\theta((w_i-v_i)\theta-A)}{(1-\theta)u_i+\theta w_i}\right]^2 \\ &+ \frac{\theta^2(1-\theta)^3}{6q_i(\theta)}(v_i(3-\theta)-w_i(1-\theta)).\end{aligned} \tag{3.14}$$

Theorem 3.1. *For the positive free parameters* $u_i, v_i,$ *and* w_i*, the error of interpolating rational cubic function* $S_i(x)$*, for* $f(x) \in C^3[x_0, x_n]$*, in each subinterval* $I_i = [x_i, x_{i+1}]$ *is*

$$\left|f(x) - S_i(x)\right| \leq \frac{1}{2}\left\|f^{(3)}(\tau)\right\| \int_{x_i}^{x_{i+1}} \left|R_x\left[(x-\tau)_+^2\right]\right| d\tau = \left\|f^{(3)}(\tau)\right\| h_i^3 c_i, \quad c_i = \max_{0\leq\theta\leq 1} p(u_i, v_i, w_i, \theta), \tag{3.15}$$

where

$$p(u_i, v_i, w_i, \theta) = \begin{cases} \max\, p_1(u_i, v_i, w_i, \theta), & 0 \leq \theta \leq \theta^* \\ \max\, p_2(u_i, v_i, w_i, \theta), & \theta^* \leq \theta \leq 1. \end{cases} \tag{3.16}$$

Remark 3.2. It is interesting to note that the rational cubic interpolation (2.1) reduces to standard cubic Hermite interpolation when we adjust the values of parameters as $u_i = 1$, $v_i = 1$ and $w_i = 3$. In this special case, the functions $p_1(u_i, v_i, w_i, \theta)$ and $p_2(u_i, v_i, w_i, \theta)$ are

$$p_1(u_i, v_i, w_i, \theta) = \frac{4\theta^2(1-\theta)^3}{3(3-2\theta)^2}, \quad 0 \leq \theta \leq \frac{1}{2}, \tag{3.17}$$

$$p_2(u_i, v_i, w_i, \theta) = \frac{4\theta^3(1-\theta)^2}{3(1+2\theta)^2}, \quad \frac{1}{2} \leq \theta \leq 0, \tag{3.18}$$

respectively. Since $c_i = \max\{\max_{0\leq\theta\leq 0.5} p_1(u_i, v_i, w_i, \theta), \max_{0.5\leq\theta\leq 0} p_2(u_i, v_i, w_i, \theta)\} = 1/96$. This is the standard result for standard cubic Hemite spline interpolation.

4. Shape Preserving 2D Convex Data Rational Cubic Spline Interpolation

The piecewise rational cubic function (2.1) does not guarantee to preserve the shape of convex data. So, it is required to assign suitable constraints on the free parameters by some mathematical treatment to preserve the convexity of convex data.

Theorem 4.1. *The* C^1 *piecewise rational cubic function* (2.1) *preserves the convexity of convex data if in each subinterval* $I_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n$*, the free parameters satisfy the following sufficient conditions:*

$$w_i > \max\left\{0, \frac{d_{i+1}v_i}{(d_{i+1}-\Delta_i)}, \frac{d_{i+1}v_i}{(\Delta_i - d_i)}, \frac{2u_i v_i(d_{i+1}-\Delta_i)}{(d_{i+1}v_i - \Delta_i u_i)}, \frac{2u_i v_i(\Delta_i - d_i)}{(\Delta_i v_i - d_i u_i)}, \frac{u_i v_i(d_{i+1}-d_i)}{\Delta_i(u_i + v_i)}\right\},$$

$$u_i, v_i > 0. \tag{4.1}$$

The above constraints are rearranged as

$$w_i = l_i + \max\left\{0, \frac{d_{i+1}v_i}{(d_{i+1}-\Delta_i)}, \frac{d_{i+1}v_i}{(\Delta_i - d_i)}, \frac{2u_iv_i(d_{i+1}-\Delta_i)}{(d_{i+1}v_i - \Delta_i u_i)}, \frac{2u_iv_i(\Delta_i - d_i)}{(\Delta_i v_i - d_i u_i)}, \frac{u_iv_i(d_{i+1}-d_i)}{\Delta_i(u_i+v_i)}\right\},$$
$$l_i \geq 0, \qquad u_i, v_i > 0. \tag{4.2}$$

Proof. Let $\{(x_i, f_i),\ i = 0,1,2,\ldots,n\}$ be the given set of convex data. For the strictly convex set of data, so

$$\Delta_1 < \Delta_2 < \Delta_3 < \cdots < \Delta_{n-1}. \tag{4.3}$$

In similar way for the concave set of data, we have

$$\Delta_1 > \Delta_2 > \Delta_3 > \cdots > \Delta_{n-1}. \tag{4.4}$$

Now, for a convex interpolation $S_i(x)$, necessary conditions on derivatives parameters d_i should be in the form such that

$$d_1 < \Delta_1 < \cdots < \Delta_{i-1} < d_i < \Delta_i < \cdots < \Delta_{n-1} < d_n. \tag{4.5}$$

Similarly, for concave interpolation,

$$d_1 > \Delta_1 > \cdots > \Delta_{i-1} > d_i > \Delta_i > \cdots > \Delta_{n-1} > d_n. \tag{4.6}$$

The necessary conditions for the convexity of data are

$$\Delta_i - d_i \geq 0, \qquad d_{i+1} - \Delta_i \geq 0. \tag{4.7}$$

Now a piecewise rational cubic interpolation $S_i(x)$ is convex if and only if $S_i^{(2)}(x) \geq 0,\ \forall x \in [x_1, x_n]$, for $x \in [x_i, x_{i+1}]$ after some simplification it can be shown that;

$$S_i^{(2)}(x) = \frac{\sum_{k=1}^{8} \theta^{k-1}(1-\theta)^{8-k} C_{ik}}{h_i\left(q_i(\theta)\right)^3}, \tag{4.8}$$

where

$$\begin{gathered}C_{i1} = 2v_i^2(w_i(d_{i+1}-\Delta_i)+d_iu_i-d_{i+1}v_i), \qquad C_{i2} = 4\,C_{i1}+6v_i^2(d_{i+1}v_i-\Delta_iu_i),\\ C_{i3} = (C_{i2}-C_{i1})+6v_i\{w_i(d_{i+1}v_i-\Delta_iu_i)-2u_iv_i(d_{i+1}-\Delta_i)\},\\ C_{i4} = (C_{i3}+C_{i1}-C_{i2})+2w_i\{w_i(\Delta_i(u_i+v_i))-u_iv_i(d_{i+1}-d_i)\}+14u_iv_i(d_{i+1}v_i-d_iu_i),\\ C_{i5} = (C_{i6}+C_{i8}-C_{i7})+2w_i\{w_i(\Delta_i(u_i+v_i))-u_iv_i(d_{i+1}-d_i)\}+14u_iv_i(d_{i+1}v_i-d_iu_i),\\ C_{i6} = (C_{i7}-C_{i8})+6u_i\{w_i(\Delta_iv_i-d_iu_i)-2u_iv_i(\Delta_i-d_i)\},\\ C_{i7} = 4\,C_{i8}+6u_i^2(\Delta_iv_i-d_iu_i), \qquad C_{i8} = 2u_i^2(w_i(\Delta_i-d_i)+d_iu_i-d_{i+1}v_i).\end{gathered} \tag{4.9}$$

All C_{ik}'s are the expression involving the parameters $d_i's$, $\Delta_i's$, $u_i's$, $v_i's$, and $w_i's$.

A C^1 piecewise rational cubic interpolant (2.1) preserves the convexity of data if $S_i^{(2)}(x) \geq 0$.

$S_i^{(2)}(x) > 0$ if both $\sum_{k=1}^{8}\theta^{k-1}(1-\theta)^{8-k}C_{i\,k} > 0$ and $h_i(q_i(\theta))^3 > 0$.

Since u_i, v_i, w_i are positive free parameters, so $h_i(q_i(\theta))^3 > 0$ must be positive

$$\sum_{k=1}^{8}\theta^{k-1}(1-\theta)^{8-k}C_{ik} > 0 \quad \text{if } C_{ik} > 0,\ k = 1,2,3,4,5,6,7,8. \tag{4.10}$$

Hence, $C_{ik} > 0$, $k = 1,2,3,4,5,6,7,8$ if we have the following sufficient conditions on parameter w_i:

$$w_i > \max\left\{0, \frac{d_{i+1}v_i}{(d_{i+1}-\Delta_i)}, \frac{d_{i+1}v_i}{(\Delta_i-d_i)}, \frac{2u_iv_i(d_{i+1}-\Delta_i)}{(d_{i+1}v_i-\Delta_iu_i)}, \frac{2u_iv_i(\Delta_i-d_i)}{(\Delta_iv_i-d_iu_i)}, \frac{u_iv_i(d_{i+1}-d_i)}{\Delta_i(u_i+v_i)}\right\}. \tag{4.11}$$

The above constraints are rearranged as

$$w_i = l_i + \max\left\{0, \frac{d_{i+1}v_i}{(d_{i+1}-\Delta_i)}, \frac{d_{i+1}v_i}{(\Delta_i-d_i)}, \frac{2u_iv_i(d_{i+1}-\Delta_i)}{(d_{i+1}v_i-\Delta_iu_i)}, \frac{2u_iv_i(\Delta_i-d_i)}{(\Delta_iv_i-d_iu_i)}, \frac{u_iv_i(d_{i+1}-d_i)}{\Delta_i(u_i+v_i)}\right\}, \quad l_i \geq 0, \tag{4.12}$$

where $\Delta_i = (f_{i+1}-f_i)/h_i$. □

5. Determination of Derivatives

Usually, the derivative values at the knots are not given. These values are derived either at the given data set $\{(x_i, f_i),\ i = 0,1,2,\ldots,n\}$ or by some other means. In this paper, these values are determined by following arithmetic mean method for data in such a way that the smoothness of the interpolant (2.1) is maintained.

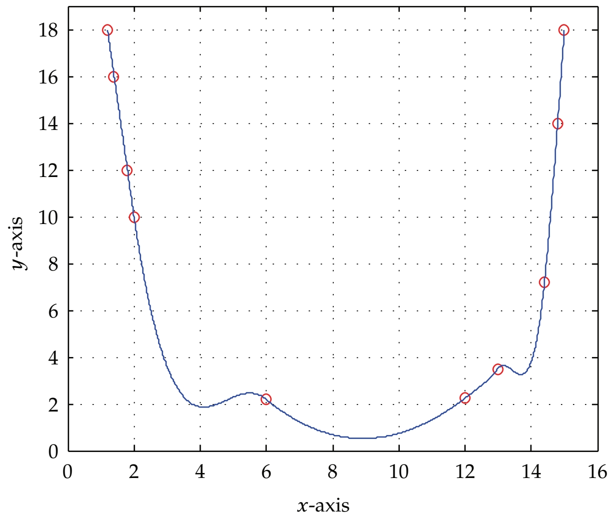

Figure 1: Cubic Hermite spline scheme.

Table 1: Convex data set.

i	1	2	3	4	5	6	7	8	9	10
x	1.2	1.4	1.8	2	6	12	13	14.4	14.8	15
$y = f(x)$	18	16	12	10	2.2	2.23	3.5	7.2	14	18

5.1. Arithmetic Mean Method

This method is the three point difference approximation with

$$d_i = \begin{cases} 0 & \text{if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0, \\ \dfrac{h_i \Delta_{i-1} + h_{i-1}\Delta_i}{h_i + h_{i-1}} & \text{otherwise}, \ i = 2, 3, \dots n-1, \end{cases} \tag{5.1}$$

and the end conditions are given as

$$d_1 = \begin{cases} 0 & \text{if } \Delta_1 = 0 \text{ or } \operatorname{sgn}(d_1) \neq \operatorname{sgn}(\Delta_1), \\ \dfrac{\Delta_1 + (\Delta_1 - \Delta_2)h_1}{h_1 + h_2} & \text{otherwise}, \end{cases}$$
$$d_n = \begin{cases} 0 & \text{if } \Delta_{n-1} = 0 \text{ or } \operatorname{sgn}(d_n) \neq \operatorname{sgn}(\Delta_{n-1}), \\ \dfrac{\Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})h_{n-1}}{h_{n-1} + h_{n-2}} & \text{otherwise}. \end{cases} \tag{5.2}$$

6. Numerical Examples

In this section, a numerical demonstration of convexity-preserving scheme given in Section 4 is presented.

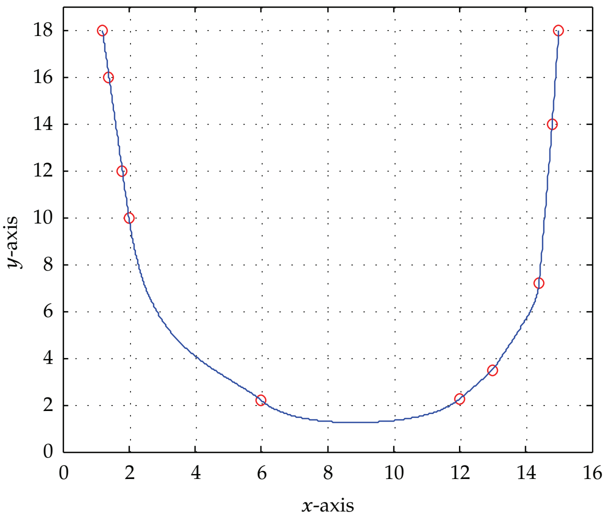

Figure 2: Convexity shape-preserving rational cubic interpolation.

Table 2: Numerical results of Figure 2.

i	1	2	3	4	5	6	7	8	9	10
d_i	−10	−10	−10	−9.6167	−1.168	1.0893	1.842	13.81	19	21
Δ_i	−10	−10	−10	−1.95	0.005	1.27	2.6429	17	20	—
u_i	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
v_i	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
w_i	0.11	5.6295e013	1.0827e014	0.14932	0.12009	0.30386	0.44488	0.29	0.52	0.11

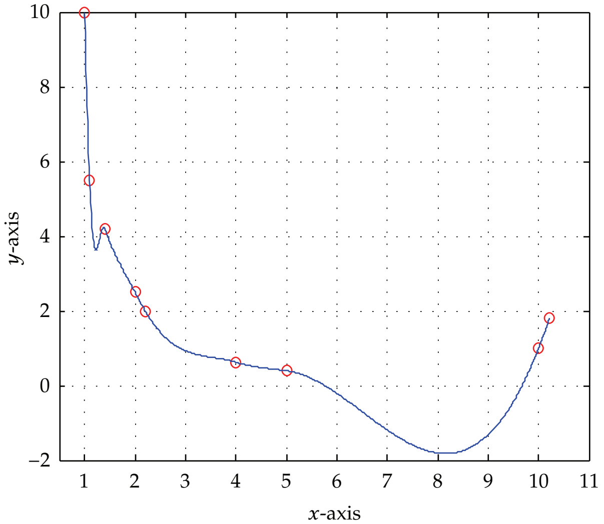

Figure 3: Cubic hermite spline scheme.

Table 3: Convex data set [11].

i	1	2	3	4	5	6	7	8	9
x	1	1.1	1.4	2	2.2	4	5	10	10.22
$y = f(x)$	10	5.5	4.2	2.5	2	0.625	0.4	1	1.8

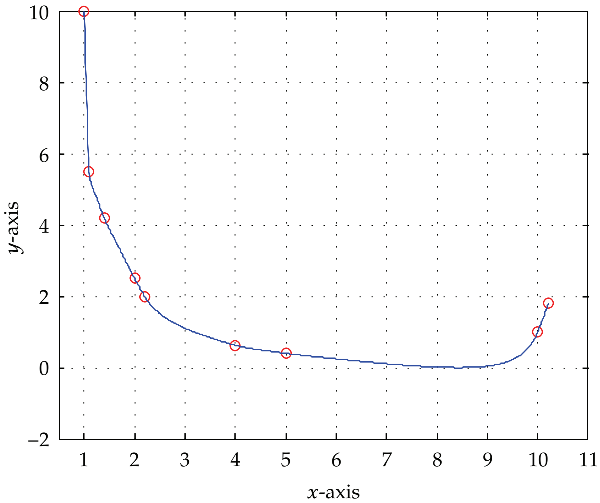

Figure 4: Convexity shape-preserving rational cubic Interpolation.

Example 6.1. Consider convex data set taken in Table 1. Figure 1 is produced by cubic Hermite spline. We remark that Figure 1 does not preserve the shape of convex data. To overcome this flaw, Figure 2 is produced by the convexity-preserving rational cubic spline interpolation developed in Section 4 with the values of free parameters $u_i = 0.02$, $v_i = 0.02$ to preserve the shape of convex data. Numerical results of Figure 2 are determined by developed convexity preserving rational cubic spline interpolation shown in Table 2.

Example 6.2. Consider convex data set taken in Table 3. Figure 3 is produced by cubic Hermite spline, and it is easy to see that Figure 3 does not preserve the shape of convex data. Figure 4 is produced by the convexity-preserving rational cubic spline interpolation developed in Section 4 with the values of free parameters $u_i = 0.02$, $v_i = 0.02$ to preserve the shape of convex data. Numerical results of Figure 4 are determined by developed convexity preserving rational cubic spline interpolation shown in Table 4.

7. Conclusion

In this paper, we have constructed a C^1 piecewise rational cubic function with three free parameters. Data-dependent constraints are derived to preserve the shape of convex data. Remaining two free parameters are left free for user's choice to refine the convexity-preserving shape of the convex data as desired. No extra knots are inserted in the interval when the curve loses the convexity. The developed curve scheme has been tested through different numerical examples, and it is shown that the scheme is not only local and computationally economical but also visually pleasant.

Table 4: Numerical results of Figure 4.

i	1	2	3	4	5	6	7	8	9
d_i	−55.16	−34.83	−3.83	−2.58	−2.32	−0.41	−0.16	3.48	3.78
Δ_i	−45	−4.33	−2.83	−2.5	−0.76	−0.22	0.12	3.63	—
u_i	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
v_i	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
w_i	0.14279	0.87	0.38	0.4368	0.11125	0.15086	0.26265	0.74	0.14279

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