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Research Article

Some Dense Linear Subspaces of Extended Little Lipschitz Algebras

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Let (X, d) be a compact metric space. In 1987, Bade, Curtis, and Dales obtained a sufficient condition for density of a subspace P of little Lipschitz algebra $\operatorname{lip}(X, \alpha)$ in this algebra and in particular showed that $\operatorname{Lip}(X, 1)$ is dense in $\operatorname{lip}(X, \alpha)$, whenever $0 < \alpha < 1$. Let K be a compact subset of X. We define new classes of Lipchitz algebras $\operatorname{Lip}(X, K, \alpha)$ for $\alpha \in (0, 1]$ and $\operatorname{lip}(X, K, \alpha)$ for $\alpha \in (0, 1)$, consisting of those continuous complex-valued functions f on X such that $f|_K \in \operatorname{Lip}(K, \alpha)$ and $f|_K \in \operatorname{lip}(K, \alpha)$, respectively. In this paper we obtain a sufficient condition for density of a linear subspace P of extended little Lipschitz algebra $\operatorname{lip}(X, K, \alpha)$ in this algebra and in particular show that $\operatorname{Lip}(X, K, 1)$ is dense in $\operatorname{lip}(X, K, \alpha)$, whenever $0 < \alpha < 1$.

1. Introduction

Let Ω be a locally compact Hausdorff space. The linear space of all continuous (bounded continuous) complex-valued functions on Ω is denoted by $C(\Omega)$ $(C^b(\Omega))$. It is known that $C^b(\Omega)$ under the uniform norm on Ω, that is,

$$\|h\|_\Omega = \sup\{|h(w)| : w \in \Omega\} \quad \left(h \in C^b(\Omega)\right), \tag{1.1}$$

is a commutative Banach algebra. The set of all f in $C(\Omega)$, which vanish at infinity, is denoted by $C_0(\Omega)$, which is a closed linear subspace of $(C^b(\Omega), \|\cdot\|_\Omega)$. Clearly, $C_0(\Omega) = C^b(\Omega) = C(\Omega)$, whenever Ω is compact. The linear space of all complex regular Borel measures on Ω is denoted by $M(\Omega)$. It is known that $M(\Omega)$ under the norm $\|\mu\| = |\mu|(\Omega)$ $(\mu \in M(\Omega))$ is a Banach space, where $|\mu|$ is the total variation of $\mu \in M(\Omega)$.

The Riesz representation theorem asserts that there exists a linear isometry from $(C_0(\Omega), \|\cdot\|_\Omega)^*$, the dual space $(C_0(\Omega), \|\cdot\|_\Omega)$ onto $(M(\Omega), \|\cdot\|)$. In fact, for each $\Lambda \in (C_0(\Omega), \|\cdot\|_\Omega)^*$, there exists a unique measure $\mu \in M(\Omega)$ with $\|\mu\| = \|\Lambda\|$ such that

$$\Lambda(f) = \int_\Omega f d\mu \quad (f \in C_0(\Omega)). \tag{1.2}$$

Let (X, d) be a compact metric space and $\alpha > 0$. The Lipschitz algebra $\mathrm{Lip}(X, \alpha)$ is defined as the set of all complex-valued functions f on X such that

$$p_\alpha(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d^\alpha(x,y)} : x, y \in X,\ x \neq y \right\} < \infty. \tag{1.3}$$

Then $\mathrm{lip}(X, \alpha)$ is a subalgebra of $C(X)$. The subalgebra $\mathrm{Lip}(X, \alpha)$ of $\mathrm{Lip}(X, \alpha)$ is the set of all those complex-valued functions f on X for which $\lim |f(x) - f(y)|/d^\alpha(x, y) = 0$ as $d(x, y) \to 0$ and is called little Lipschitz algebra of order α.

We know that $\mathrm{Lip}(X, 1)$ separates the points of X, $1 \in \mathrm{Lip}(X, 1)$ and $\mathrm{Lip}(X, \beta) \subseteq \mathrm{lip}(X, \alpha) \subseteq \mathrm{Lip}(X, \alpha)$, where $0 < \alpha < \beta \leqslant 1$. Also, if X is infinite and $0 < \alpha < 1$, then $\mathrm{lip}(X, \alpha) \neq \mathrm{Lip}(X, \alpha)$. The algebras $\mathrm{Lip}(X, \alpha)$ for $\alpha \leqslant 1$ and $\mathrm{lip}(X, \alpha)$ for $\alpha < 1$ are Banach function algebras on X under the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$. Since these algebras are self-adjoint, they are uniformly dense in $C(X)$, by the Stone-Weierstrass theorem. We know that if A is a Banach function algebra on a compact Hausdorff space X such that A is self-adjoint and $1/f \in A$ whenever $f \in A$ and $f(x) \neq 0$ for each $x \in X$, then A is natural, that is, the maximal ideal space of A coincides with X. Hence, if X is infinite, then the Lipschitz algebras $\mathrm{Lip}(X, \alpha)$ for $\alpha \leqslant 1$ and $\mathrm{lip}(X, \alpha)$ for $\alpha < 1$, are natural.

Extensive study of Lipschitz algebras started with Sherbert [1, 2]. Honary and Moradi introduced new classes of analytic Lipschitz algebras on compact plane sets and determined their maximal ideal spaces [3].

Bade et al. have obtained a sufficient condition for density of a linear subspace P of $\mathrm{lip}(X, \alpha)$ in this algebra as follows.

Theorem 1.1 (see [4, Theorem 3.6]). *Let (X, d) be a compact metric space, and let P be a linear subspace of $\mathrm{lip}(X, \alpha)$. Suppose that there is a constant C such that for each finite subset E of X and each $f \in \mathrm{lip}(X, \alpha)$, there exists $g \in P$ with $g|_E = f|_E$ and with $\|g\|_\alpha \leqslant C\|f\|_\alpha$. Then P is dense in $\mathrm{lip}(X, \alpha)$.*

They also showed that $\mathrm{Lip}(X, 1)$ is dense in $\mathrm{lip}(X, \alpha)$ [4, Corollary 3.7]. We extend the above results for the more general classes of the Lipschitz algebras by generalizing and using some results that have been given by them.

Throughout this work we always assume that (X, d) is a compact metric space, K is nonempty compact subset of X, and α is a positive number.

Definition 1.2. The algebra of all continuous complex-valued functions f on X for which

$$p_{\alpha,K} = \sup\left\{ \frac{|f(x) - f(y)|}{d^\alpha(x,y)} : x, y \in K,\ x \neq y \right\} < \infty \tag{1.4}$$

is denoted by $\mathrm{Lip}(X,K,\alpha)$, and the subalgebra of those $f \in \mathrm{Lip}(X,K,\alpha)$ for which $|f(x) - f(y)|/d^{\alpha}(x,y) \to 0$ as $d(x,y) \to 0$, when $x, y \in K$, is denoted by $\mathrm{lip}(X,K,\alpha)$. The algebras $\mathrm{Lip}(X,K,\alpha)$ and $\mathrm{lip}(X,K,\alpha)$ are called *extended Lipschitz algebra* and *extended little Lipschitz algebra* of order α on (X,d) with respect to K, respectively.

It is easy to see that these extended Lipschitz algebras are both Banach algebras under the norm $\|f\|_{\alpha,K} = \|f\|_X + P_{\alpha,K}(f)$. In fact, $\mathrm{lip}(X,K,\alpha)$ is a Banach function algebra on X for $\alpha \in (0,1]$, and $\mathrm{lip}(X,K,\alpha)$ is a Banach function algebra on X for $\alpha \in (0,1)$. Note that if $0 < \alpha < \beta \leqslant 1$, then $\mathrm{Lip}(X,K,\beta) \subseteq \mathrm{lip}(X,K,\alpha)$. We always assume that $0 < \alpha \leqslant 1$ for Lip and $0 < \alpha < 1$ for lip. Note that $\mathrm{lip}(X,K,\alpha)$ is a proper subalgebra of $\mathrm{Lip}(X,K,\alpha)$ when K is infinite. Because if $y \in K$, then function $f : X \to \mathbb{C}$ defined by $f(x) = d^{\alpha}(x,y)$ is an element of $\mathrm{Lip}(X,K,\alpha)$ but does not belong to $\mathrm{lip}(X,K,\alpha)$.

It is clear that whenever $K = X$, the new classes of Lipschitz algebras coincide with the standard Lipschitz algebras. Also $\mathrm{Lip}(X,K,\alpha) = \mathrm{lip}(X,K,\alpha) = C(X)$, whenever K is finite. Hence, we may assume that K is infinite.

By the Stone-Weierstrass theorem, $\mathrm{Lip}(X,K,\alpha)$ and $\mathrm{lip}(X,K,\alpha)$ are both uniformly dense in $C(X)$.

Let A be $\mathrm{Lip}(X,K,\alpha)$ or $\mathrm{lip}(X,K,\alpha)$. For each $f \in A$ and for all $n \in \mathbb{N}$, we have

$$\|f^n\|_{\alpha,K} \leqslant \|f^n\|_X + n(\|f\|_X)^{n-1} p_{\alpha,K}(f). \tag{1.5}$$

By the spectral radius theorem,

$$\begin{aligned} \left\|\widehat{f}\right\|_{\mathcal{M}(A)} &= \lim_{n\to\infty} \left(\|f^n\|_{\alpha,K}\right)^{1/n} \\ &\leqslant \lim_{n\to\infty} \|f\|_X \left(1 + \frac{p_{\alpha,K}(f)}{\|f\|_X} n\right)^{1/n} \\ &= \|f\|_X, \end{aligned} \tag{1.6}$$

where $\mathcal{M}(A)$ is the maximal ideal space of A and $\widehat{f}$ is the Gelfand transform of f on $\mathcal{M}(A)$. Hence, by applying the main theorem in [5], we can show that A is natural, that is, $\mathcal{M}(A)$ coincides with X. We can prove this fact with another way. Since every self-adjoint inverse-closed Banach function algebra A on a compact Hausdorff space X is natural, and Banach function algebras $\mathrm{Lip}(X,K,\alpha)$ and $\mathrm{lip}(X,K,\alpha)$ have the mentioned properties, they are natural.

In this paper we obtain a sufficient condition for density of a linear subspace P of $\mathrm{lip}(X,K,\alpha)$ that is dense in this algebra by generalizing some results in [4]. In particular, we show that $\mathrm{Lip}(X,K,1)$ is dense in $(\mathrm{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})$.

2. Representing Measure

We denote

$$\begin{aligned} \Delta(K) &= \{(x,y) \in K \times K : x = y\}, \qquad V(K) = (K \times K) \setminus \Delta(K), \\ W(X,K) &= X \cup V(K), \qquad W(K) = W(K,K). \end{aligned} \tag{2.1}$$

Obviously, $W(X, K)$ is a locally compact Hausdorff space. We define the norm $|||\cdot|||$ on $C^b(W(X, K))$ by

$$|||h||| = \|h|_X\|_X + \left\|h|_{V(K)}\right\|_{V(K)}. \tag{2.2}$$

Then $C^b(W(X, K)$ is a Banach space under the norm $|||\cdot|||$, since

$$\|h\|_{W(X,K)} \leqslant |||h||| \leqslant 2\|h\|_{W(X,K)}, \tag{2.3}$$

for all $h \in C^b(W(X, K))$. Moreover, $C_0(W(X, K))$ is a closed linear subspace of $(C^b(W(X, K)), |||\cdot|||)$.

We define the norm $|||\cdot|||$ on $M(W(X, K))$ by

$$\left|\left|\left|\mu\right|\right|\right| = \max\{|\mu|(X), |\mu|(V(K))\}. \tag{2.4}$$

Then $M(W(X, K))$ is a Banach space under the norm $|||\cdot|||$, since

$$\|\mu\| \leqslant |||\mu||| \leqslant 2\|\mu\|, \tag{2.5}$$

for all $\mu \in M(W(X, K))$. By applying the Riesz representation theorem, we obtain the following result which is a generalization of Theorem A in [6], and one can prove it by the same method.

Theorem 2.1. *For each $\Psi \in (C_0(W(X, K)), |||\cdot|||)^*$, there exists a unique measure $\mu \in M(W(X, K))$ with $|||\mu||| = |||\Psi|||$ such that*

$$\Psi(h) = \int_{W(X,K)} h\, d\mu \quad (h \in C_0(W(X, K))), \tag{2.6}$$

where $|||\Psi||| = \sup\{|\Psi(h)| : h \in C_0(W(X, K)), |||h||| \leqslant 1\}$.

Definition 2.2. For $f \in C(X)$, the function $T_{X,K}(f) : W(X, K) \to \mathbb{C}$ defined by

$$\begin{gathered} T_{X,K}(f)(x) = f(x) \quad (x \in X), \\ T_{X,K}(f)(x, y) = \frac{f(x) - f(y)}{d^\alpha(x, y)} \quad ((x, y) \in V(K)) \end{gathered} \tag{2.7}$$

is called Leeuw's extension of f on $W(X, K)$.

It is obvious that $T_{X,K}(f) \in C^b(W(X, K))$ for each $f \in \mathrm{Lip}(X, K, \alpha)$.

Theorem 2.3. *Take $0 < \alpha < 1$.*

(i) *$T_{X,K}$ is a linear isometry from the extended Lipschitz algebra $(\mathrm{Lip}(X, K, \alpha), \|\cdot\|_{\alpha,K})$ into $(C^b(W(X, K)), |||\cdot|||)$.*

(ii) $T_{X,K}(\mathrm{Lip}(X,K,\alpha))$ *is a closed linear subspace of* $(C_0(W), |||\cdot|||)$.

(iii) *For each* $\Phi \in (\mathrm{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})^*$, *there exists* $\mu \in M(W(X,K))$ *such that*

$$\Phi(f) = \int_{W(X,K)} T_{X,K}(f)\,d\mu \quad (f \in \mathrm{lip}(X,K,\alpha)),$$
$$\| |\mu| \| = \|\Phi\|. \tag{2.8}$$

Proof. (i) It is immediate.

(ii) Since $\mathrm{lip}(X,K,\alpha)$ is a linear subspace $\mathrm{Lip}(X,K,\alpha)$, $T_{X,K}(\mathrm{lip}(X,K,\alpha))$ is a linear subspace of $C^b(W(X,K))$ by (i). Let $f \in \mathrm{lip}(X,K,\alpha)$, and let ε be an arbitrary positive number. There exists $\delta > 0$ such that

$$\frac{|f(x) - f(y)|}{d^{\alpha}(x,y)} < \varepsilon, \tag{2.9}$$

for all $x, y \in K$ with $0 < d(x,y) < \delta$. Set $E_\varepsilon = X \cup \{(x,y) \in K : d(x,y) \geqslant \delta\}$. Clearly, E_ε is a compact subset of $W(X,K)$ and

$$|T_{X,K}(f)(x,y)| < \varepsilon, \tag{2.10}$$

for all $(x,y) \in W(X,K) \setminus E_\varepsilon$. It follows that $T_{X,K}(f) \in C_0(W(X,K))$. Therefore, $T_{X,K}(\mathrm{lip}(X,K,\alpha))$ is a subset $C_0(W(X,K))$. Since $T_{X,K}$ is a linear isometry and $C_0(W(X,K))$ is a closed linear subspace of $(C^b(W(X,K)), |||\cdot|||)$, we conclude that $T_{X,K}(\mathrm{lip}(X,K,\alpha))$ is a closed linear subspace of $(C_0(W(X,K)), |||\cdot|||)$.

(iii) Let $\Phi \in (\mathrm{Lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})^*$ and define $\eta_{X,K} : \mathrm{lip}(X,K,\alpha) \to T_{X,K}(\mathrm{lip}(X,K,\alpha))$ by

$$\eta_{X,K}(f) = T_{X,K}(f) \quad (f \in \mathrm{lip}(X,K,\alpha)). \tag{2.11}$$

Then $\Phi o(\eta_{X,K})^{-1} \in (T_{X,K}(\mathrm{lip}(X,K,\alpha)), \|\cdot\|)^*$. By the Hahn-Banach extension theorem, there exists $\Psi \in C_0(W(X,K), |||\cdot|||)^*$ with $|||\Psi||| = |||\Phi o(\eta_{X,K})^{-1}|||$ such that

$$\Psi(h) = \Phi o(\eta_{X,K})^{-1}(h)(h \in T_{X,K}(\mathrm{lip}(X,K,\alpha))). \tag{2.12}$$

By Theorem 2.1, there exists $\mu \in M(W(X,K))$ with $|||\mu||| = |||\Psi|||$ such that

$$\Psi(h) = \int_{W(X,K)} h\,d\mu \quad (h \in C_0(W(X,K))). \tag{2.13}$$

Therefore,

$$\begin{aligned}\Phi(f) &= \left(\Phi o(\eta_{X,K})^{-1}\right)(T_{X,K}(f)) \\ &= \int_{W(X,K)} T_{X,K}(f)d\mu \quad (f \in \text{lip}(X,K,\alpha)), \\ \||\mu|\| &= \left\|\left|\Phi o(\eta_{X,K})^{-1}\right|\right\|.\end{aligned} \tag{2.14}$$

On the other hand,

$$\begin{aligned}\left\|\left|\Phi o(\eta_{X,K})^{-1}\right|\right\| &= \sup\{|(\Phi(f))| : f \in \text{lip}(X,K,\alpha),\ \||T_{X,K}(f)|\| \leqslant 1\} \\ &= \|\Phi\|.\end{aligned} \tag{2.15}$$

It follows that $\||\mu|\| = \|\Phi\|$. This completes the proof. □

Note that the map $T_{X,K}$ is not an algebra homomorphism and that its image is not a subalgebra of $C^b(W(X,K))$.

Definition 2.4. For $\Phi \in (\text{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})^*$, a measure $\mu \in M(W(X,K))$ with $\||\mu|\| = \|\Phi\|$ and with

$$\Phi(f) = \int_{W(X,K)} T_{X,K}(f)d\mu \quad (f \in \text{lip}(X,K,\alpha)) \tag{2.16}$$

is called a representing measure for Φ on $W(X,K)$.

Note that a representing measure for Φ on $W(X,K)$ is not unique.

3. Main Results

In this section, by generalizing Theorem 1.1, we obtain a sufficient condition for which a linear subspace of $\text{lip}(X,K,\alpha)$ is dense in this algebra. In particular, we show that $\text{Lip}(X,K,1)$ is dense in $(\text{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})$.

Theorem 3.1. *let P be a linear subspace of* $\text{lip}(X,K,\alpha)$ *which satisfies the following conditions:*

(a) *if $h \in C(X)$ with $h|_K = 0$, then $h \in \overline{P}$, where $\overline{P}$ is the closure of P in* $(\text{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})$.

(b) *there is a constant C such that for each finite subset E of K and each $f \in \text{lip}(X,K,\alpha)$, there exists $g \in P$ with $g|_E = f|_E$ and with $\|g\|_{\alpha,K} \leqslant C\|f\|_{\alpha,K}$.*

Then P is dense in $(\text{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})$.

Proof. We first show that if $P_K = \{g \in C(K) : g = f|_K$ for some $f \in P\}$, then P_K is dense in the little Lipschitz algebra $\text{lip}(K,\alpha)$. Clearly, P_K is a linear subspace of $\text{lip}(K,\alpha)$. Let E be a Finite subset of K, and let $f \in \text{lip}(K,\alpha)$. By Tietze's extension theorem [7, Theorem 20.4], there exists $F \in C(X)$ such that $F|_K = f$ and $\|F\|_X = \|f\|_K$. Clearly, $F \in \text{lip}(X,K,\alpha)$. by (b), there exists $G \in P$ such that $G|_E = F|_E$ and $\|G\|_{\alpha,K} \leqslant C\|F\|_{\alpha,K}$. We define $g = G|_K$. Then $g \in P_K$, $g|_E = f|_E$ and

$$\begin{aligned} \|g\|_\alpha &= \|g\|_K + p_\alpha(g) \leqslant \|G\|_X + p_\alpha(g) \\ &= \|G\|_X + p_{\alpha,K}(G) = \|G\|_{\alpha,K} \\ &\leqslant C\|F\|_{\alpha,K} = C(\|F\|_X + p_{\alpha,K}(F)) \\ &= C(\|f\|_K + p_\alpha(f)) = C\|f\|_\alpha. \end{aligned} \tag{3.1}$$

Thus P_K is dense in $(\mathrm{lip}(K,\alpha), \|\cdot\|_\alpha)$ by Theorem 1.1.

To prove the density of P in $(\mathrm{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})$, it is enough to show that if $\Phi \in (\mathrm{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})^*$ with $\Phi(f) = 0$ for all $f \in P$, then $\Phi(f) = 0$ for all $f \in \mathrm{lip}(X,K,\alpha)$.

Let $\Phi \in (\mathrm{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})^*$ such that $\Phi(f) = 0$ for all $f \in P$. Continuity of Φ implies that $\Phi(f) = 0$ for all $f \in \overline{P}$. By Theorem 2.3, there exists $\mu \in M(W(X,K))$ such that

$$\Phi(F) = \int_{W(X,K)} T_{X,K}(F)\,d\mu \quad (F \in \mathrm{lip}(X,K,\alpha)), \tag{3.2}$$

where $T_{X,K}(F)$ is Leeuw's extension of $F \in \mathrm{lip}(X,K,\alpha)$ on $W(X,K)$. We claim that

$$\Phi(F) = \int_{W(K)} T_{X,K}(F)\,d\mu \quad (F \in \mathrm{lip}(X,K,\alpha)). \tag{3.3}$$

Let $F \in \mathrm{lip}(X,K,\alpha)$. We define the sequence $\{Y_n\}_{n=1}^{\infty}$ of the subsets of X by

$$Y_n = \left\{x \in X : d(x,K) \geqslant \frac{1}{n}\right\}. \tag{3.4}$$

Then Y_n is a compact subset of X for each $n \in \mathbb{N}$, $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq X \setminus K$, and $\bigcup_{n=1}^{\infty} Y_n = X \setminus K$. Let $n \in \mathbb{N}$. By Urysohn's lemma, there exists $F_n \in C(X)$ such that $\|F_n\|_X = 1$, $F_n|_K = 0$, and $F_n|_{Y_n} = 1$. Define $G_n = F_n F$. Then $G_n \in C(X)$ and $G_n|_K = 0$. Hence, $G_n \in \overline{P}$ by (a) and so $\Phi(G_n) = 0$. Thus

$$\int_{W(X,K)} T_{X,K}(G_n)\,d\mu = 0. \tag{3.5}$$

Let $\tilde{\chi}_{X\setminus K}$ be the characteristic function of $X \setminus K$ on $W(X,K)$. It is easy to see that

$$\lim_{n\to\infty} T_{X,K}(G_n)(w) = (T_{X,K}(F) \cdot \tilde{\chi}_{X\setminus K})(w), \tag{3.6}$$

for all $w \in W(X,K)$. Since $\|T_{X,K}(G_n)\|_{W(X,K)} = 1$ for all $n \in \mathbb{N}$ and $1 \in L^1(W(X,K), |\mu|)$, we conclude that

$$\lim_{n\to\infty} \int_{W(X,K)} T_{X,K}(G_n)\,d\mu = \int_{W(X,K)} T_{X,K}(F) \cdot \tilde{\chi}_{X\setminus K}\,d\mu, \tag{3.7}$$

by Lebesgue's dominated convergence theorem. Thus

$$\int_{W(X,K)} T_{X,K}(F) \cdot \tilde{\chi}_{X\setminus K} d\mu = 0, \tag{3.8}$$

by (3.5) and (3.7). It follows that

$$\int_{X\setminus K} T_{X,K}(F) d\mu = 0. \tag{3.9}$$

Thus (3.3) is justified, by (3.2) and (3.9).

We now define the function $\Psi : \operatorname{lip}(K, \alpha) \to \mathbb{C}$, by

$$\Psi(g) = \int_{W(K)} T_{K,K}(g) d\mu|_{W(K)}. \tag{3.10}$$

Clearly, Ψ is a linear functional on $\operatorname{lip}(K, \alpha)$. Since

$$|\Psi(g)| \leqslant \|T_{K,K}(g)\|_{W(K)} |\mu|(W(K)) \leqslant \|g\|_{\alpha} |\mu|(W(K)), \tag{3.11}$$

for all $g \in \operatorname{lip}(K, \alpha)$, we deduce that $\Psi \in (\operatorname{lip}(K, \alpha), \|\cdot\|_{\alpha})^{*}$. We claim that $\Psi(g) = 0$ for all $g \in P_K$. If $g \in P_K$, there exists $f \in P$ such that $f|_K = g$, and so

$$\begin{aligned} \Psi(g) &= \int_{W(K)} T_{K,K}(g) d\mu|_{W(K)} = \int_{W(K)} T_{K,K}(f) d\mu \\ &= \Phi(f) = 0. \end{aligned} \tag{3.12}$$

Therefore, our claim is justified. It follows that $\Psi(g) = 0$ for all $g \in \operatorname{lip}(K, \alpha)$, by the density of P_K in $(\operatorname{lip}(X, K, \alpha), \|\cdot\|_{\alpha,K})$ and continuity of Ψ on $\operatorname{lip}(K, \alpha)$. Let $f \in \operatorname{lip}(X, K, \alpha)$. If $g = f|_K$, then $g \in \operatorname{lip}(K, \alpha)$, and so $\Psi(g) = 0$. Therefore,

$$\begin{aligned} \Phi(f) &= \int_{W(K)} T_{X,K}(f) d\mu = \int_{W(K)} T_{K,K}(g) d\mu|_{W(K)} \\ &= \Psi(g) = 0, \end{aligned} \tag{3.13}$$

by (3.3). This completes the proof. □

By applying the above result, we show that $\operatorname{Lip}(X, K, 1)$ is dense in $(\operatorname{lip}(X, K, \alpha), \|\cdot\|_{\alpha,K})$.

Bade et al. obtained the following result.

Lemma 3.2 (see [4, Lemma 3.3]). *For each finite subset E of X and each $h \in \operatorname{Lip}(X, \alpha)$, there exists $f \in \operatorname{Lip}(X, 1)$ with $f|_E = h|_E$ and with $\|f\|_{\alpha} \leqslant 2\|h\|_{\alpha}$.*

We now generalize the above lemma by applying it and Tietze's extension theorem as follows.

Lemma 3.3. *For each finite subset E of X and each $h \in \operatorname{Lip}(X,K,\alpha)$, there exists $f \in \operatorname{Lip}(X,K,1)$ with $f|_E = h|_E$ and with $\|f\|_{\alpha,K} \leqslant 3\|h\|_{\alpha,K}$.*

Proof. Let E be a finite subset of X, and let $h \in \operatorname{Lip}(X,K,\alpha)$. Define $g = h|_K$. Then $g \in \operatorname{Lip}(K,\alpha)$. By Lemma 3.2, there exists $g_0 \in \operatorname{Lip}(K,1)$ with $g_0|_{E\cap K} = g|_{E\cap K}$ and with $\|g_0\|_\alpha \leqslant 2\|g\|_\alpha$. We now define the function $g_1 : E \cup K \to \mathbb{C}$ by

$$g_1(x) = \begin{cases} g_0(x), & x \in K, \\ h(x), & x \in E \setminus K. \end{cases} \tag{3.14}$$

Clearly, $E\cup K$ is a compact subset of X and $g_1 \in C(E\cup K)$. By Tietze's extension theorem, there exists $f \in C(X)$ such that $f|_{E\cup K} = g_1$ and $\|f\|_X = \|g_1\|_{E\cup K}$. It follows that $f \in \operatorname{Lip}(X,K,1)$ and $f|_E = h|_E$. Furthermore,

$$\begin{aligned} \|f\|_{\alpha,K} &= \|g_1\|_{E\cup K} + p_\alpha(g_0) \\ &\leqslant \|g_1\|_K + \|g_1\|_{E\setminus K} + P_\alpha(g_0) \\ &= \|g_0\|_K + \|h\|_{E\setminus K} + p_\alpha(g_0) \\ &= \|g_0\|_\alpha + \|h\|_{E\setminus K} \\ &\leqslant 2\|g\|_\alpha + \|h\|_{E\setminus K} \\ &= \left(\|g\|_K + p_\alpha(g)\right) + \|h\|_{E\setminus K} \\ &\leqslant 2\|h\|_X + \|h\|_{E\setminus K} + 2P_{\alpha,K}(h) \\ &\leqslant 3\|h\|_{\alpha,K}. \end{aligned} \tag{3.15}$$

This completes the proof. □

Theorem 3.4. $\operatorname{Lip}(X,K,1)$ *is dense in* $(\operatorname{lip}(X,K,\alpha), \|\cdot\|_{\alpha,K})$.

Proof. Take $P = \operatorname{Lip}(X,K,1)$. Then P is a linear subspace of $\operatorname{lip}(X,K,\alpha)$ and $h \in P$ for all $h \in C(X)$ with $h|_K = 0$. Let E be a finite subset of K and $f \in \operatorname{lip}(X,K,\alpha)$. By Lemma 3.3, there exists $g \in P$ with $g|_E = f|_E$ and with $\|g\|_{\alpha,K} \leqslant 3\|f\|_{\alpha,K}$. Therefore, P is dense in $(\operatorname{lip}(X,K,\alpha), \ \|\cdot\|_{\alpha,K})$, by Theorem 3.1. □

Corollary 3.5. $\operatorname{Lip}(X,1)$ *is dense in* $(\operatorname{lip}(X,\alpha), \|\cdot\|_\alpha)$.

Proof. It is enough to take $K = X$ in Theorem 3.4. □

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