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Research Article

Continuation Criterion for the 2D Liquid Crystal Flows

Jishan Fan[1] and Tohru Ozawa[2]

[1] *Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China*
[2] *Department of Applied Physics, Waseda University, Tokyo 169-8555, Japan*

Correspondence should be addressed to Tohru Ozawa, txozawa@waseda.jp

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We consider the 2D liquid crystal systems, which consists of Navier-Stokes system coupled with wave maps or biharmonic wave maps, respectively. By logarithmic Sobolev inequalities, we obtain a blow-up criterion $\nabla d, \partial_t d \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^2))$ for the case with wave maps, and we prove the existence of a global-in-time strong solutions for the case with biharmonic wave maps.

1. Introduction

First, we consider the following simplified liquid crystal flows in two space dimensions [1]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \sum_k \partial_t d^k \cdot \nabla d^k, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t^2 d + u \cdot \nabla d - \Delta d = d\left(|\nabla d|^2 - |\partial_t d|^2\right), \quad |d| = 1, \tag{1.3}$$

$$(u, d, \partial_t d)(x, 0) = (u_0, d_0, d_1)(x), \quad x \in \mathbb{R}^2,\ |d_0| = 1,\ d_0 \cdot d_1 = 0, \tag{1.4}$$

where u is the velocity, π is the pressure, and d represents the macroscopic average of the liquid crystal orientation field with values in the unit circle.

The first two equations (1.1) and (1.2) are the well-known Navier-Stokes system with the Lorentz force $\sum_k \partial_t d^k \cdot \nabla d^k$. The last equation (1.3) is the well-known wave maps when $u = 0$.

It is a simple matter to show that the system (1.1)–(1.4) has a unique local-in-time smooth solution when $u_0, \nabla d_0,\ d_1 \in H^{1+s}\ (\mathbb{R}^2)$ with $s > 0$, $\operatorname{div} u_0 = 0$, $|d_0| = 1$, $d_0 \cdot d_1 = 0$ in $\mathbb{R}^2$. The aim of this paper is to study the regularity criterion of smooth solutions to the problem (1.1)–(1.4). We will prove the following.

Theorem 1.1. *Let $u_0, \nabla d_0,\ d_1 \in H^{1+s}(\mathbb{R}^2)$ with $s > 0$, $\operatorname{div} u_0 = 0, |d_0| = 1,\ d_0 \cdot d_1 = 0$ in $\mathbb{R}^2$ and let (u, d) be a smooth solution of (1.1)–(1.4) on some interval $[0, T]$ with $0 < T < \infty$. Assume that*

$$\nabla d, \partial_t d \in L^1\left(0, T; \dot{B}^0_{\infty,\infty}\left(\mathbb{R}^2\right)\right). \tag{1.5}$$

Then the solution (u, d) can be extended beyond $T > 0$.

$\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space. We have $L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}$; see Triebel [2].

In the proof of Theorem 1.1, we will use the logarithmic Sobolev inequalities [3–6]:

$$\|u\|_{L^\infty} \le C\|u\|_{H^1}\log^{1/2}(e + \|u\|_{H^{1+s}}), \tag{1.6}$$

$$\|\nabla d\|_{L^\infty} \le C\left(1 + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\nabla d\|_{H^{1+s}})\right), \tag{1.7}$$

$$\|\partial_t d\|_{L^\infty} \le C\left(1 + \|\partial_t d\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\partial_t d\|_{H^{1+s}})\right), \tag{1.8}$$

for $s > 0$, and the Gagliardo-Nirenberg inequalities:

$$\begin{aligned} \|w\|_{L^4} &\le C\|w\|_{L^2}^{\alpha}\left\|\Lambda^{1+s}w\right\|_{L^2}^{1-\alpha}, \\ \|\Lambda^s w\|_{L^4} &\le C\|w\|_{L^2}^{1-\alpha}\left\|\Lambda^{1+s}w\right\|_{L^2}^{\alpha}, \end{aligned} \tag{1.9}$$

with $\Lambda := (-\Delta)^{1/2}$, $\alpha := 1 - (1/2) \cdot (1/(1 + s))$, and $s > 0$, and the product estimate due to Kato-Ponce [7]:

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}}), \tag{1.10}$$

with $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

Motivated by the problem (1.1)–(1.4), we consider the following liquid crystal flows:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \sum_k \partial_t d^k \cdot \nabla d^k, \tag{1.11}$$

$$\operatorname{div} u = 0, \tag{1.12}$$

$$\partial_t^2 d + u \cdot \nabla d + (-\Delta)^2 d = -\lambda d, \qquad |d| = 1, \tag{1.13}$$

$$\lambda := |\partial_t d|^2 + |\Delta d|^2 + \Delta|\nabla d|^2 + 2\sum_k \partial_k d \cdot \Delta\partial_k d, \tag{1.14}$$

$$(u, d, \partial_t d)(x, 0) = (u_0, d_0, d_1)(x, 0), \qquad x \in \mathbb{R}^2,\ |d_0| = 1. \tag{1.15}$$

The last two equations (1.13) and (1.14) are the biharmonic wave maps. It is also a simple matter to show that the problem (1.11)–(1.15) has at least one local-in-time strong solution. The aim of this paper is to prove the global-in-time regularity. We obtain the following.

Theorem 1.2. *Let* $u_0 \in H^2$, $(\nabla d_0, d_1) \in H^3 \times H^2$ *with* $\operatorname{div} u_0 = 0$, $|d_0| = 1$, $d_0 \cdot d_1 = 0$ *in* $\mathbb{R}^2$. *Then there exists at least a global-in-time smooth solution:*

$$(u, \nabla d, \partial_t d) \in L^\infty\left(0, T; H^2\right) \times L^\infty\left(0, T; H^3\right) \times L^\infty\left(0, T; H^2\right) \tag{1.16}$$

for any $T > 0$.

Remark 1.3. We are unable to prove the uniqueness of strong solutions in Theorem 1.2.

2. Proof of Theorem 1.1

We only need to prove a priori estimates.

Testing (1.1) by u, using (1.2), we see that

$$\frac{1}{2}\frac{d}{dt}\int u^2 dx + \int |\nabla u|^2 dx = \int (u \cdot \nabla) d \cdot \partial_t d \, dx. \tag{2.1}$$

Testing (1.3) by $\partial_t d$, using $|d| = 1$ and $d \cdot \partial_t d = 0$, we find that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 + |\partial_t d|^2 dx = -\int (u \cdot \nabla) d \cdot \partial_t d \, dx. \tag{2.2}$$

Summing up (2.1) and (2.2), we get

$$\frac{1}{2}\frac{d}{dt}\int u^2 + |\nabla d|^2 + |\partial_t d|^2 dx + \int |\nabla u|^2 dx = 0, \tag{2.3}$$

from which we get

$$\int u^2 + |\nabla d|^2 + |\partial_t d|^2 dx + \int_0^T \int |\nabla u|^2 dx \, dt \le C. \tag{2.4}$$

Applying Λ^{1+s} to (1.1), testing by $\Lambda^{1+s} u$, using (1.2) and (1.10), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left|\Lambda^{1+s} u\right|^2 dx + \int \left|\Lambda^{2+s} u\right|^2 dx \\
&\quad = -\int \Lambda^{1+s} \operatorname{div}(u \otimes u) \cdot \Lambda^{1+s} u \, dx + \int \Lambda^{1+s} (\partial_t d \cdot \nabla d) \cdot \Lambda^{1+s} u \, dx \\
&\quad \le C \|u\|_{L^\infty} \left\|\Lambda^{2+s} u\right\|_{L^2} \left\|\Lambda^{1+s} u\right\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+C\left(\|\partial_t d\|_{L^\infty}\left\|\Lambda^{2+s}d\right\|_{L^2}+\|\nabla d\|_{L^\infty}\left\|\Lambda^{1+s}\partial_t d\right\|_{L^2}\right)\left\|\Lambda^{1+s}u\right\|_{L^2}\\
\le{}& \frac{1}{2}\left\|\Lambda^{2+s}u\right\|_{L^2}^2+C\|u\|_{L^\infty}^2\left\|\Lambda^{1+s}u\right\|_{L^2}^2\\
&+C\|(\partial_t d,\nabla d)\|_{L^\infty}\left(y^2+\left\|\Lambda^{1+s}u\right\|_{L^2}^2\right),
\end{aligned}
\tag{2.5}
$$

where

$$
y^2:=\left\|\Lambda^{1+s}\partial_t d\right\|_{L^2}^2+\left\|\Lambda^{2+s}d\right\|_{L^2}^2. \tag{2.6}
$$

Taking Λ^{1+s} to (1.3), testing by $\Lambda^{1+s}\partial_t d$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}y^2={}&\int\Lambda^{1+s}\left[d\left(|\nabla d|^2-|\partial_t d|^2\right)\right]\cdot\Lambda^{1+s}\partial_t d\,dx\\
&-\int\Lambda^{1+s}(u\cdot\nabla d)\cdot\Lambda^{1+s}\partial_t d\,dx=:I_1+I_2.
\end{aligned}
\tag{2.7}
$$

By using (1.10), (2.4), and (1.9), I_1 can be bounded as follows:

$$
\begin{aligned}
I_1\le{}& C\Big[\|d\|_{L^\infty}\left\|\Lambda^{1+s}\left(|\nabla d|^2-|\partial_t d|^2\right)\right\|_{L^2}\\
&+\left\|\Lambda^{1+s}d\right\|_{L^4}\left(\|\nabla d\|_{L^\infty}\|\nabla d\|_{L^4}+\|\partial_t d\|_{L^\infty}\|\partial_t d\|_{L^4}\right)\Big]\left\|\Lambda^{1+s}\partial_t d\right\|_{L^2}\\
\le{}& C\Big[\left(\|\nabla d\|_{L^\infty}\left\|\Lambda^{2+s}d\right\|_{L^2}+\|\partial_t d\|_{L^\infty}\left\|\Lambda^{1+s}\partial_t d\right\|_{L^2}\right)\\
&+cy^\alpha\left(\|\nabla d\|_{L^\infty}y^{1-\alpha}+\|\partial_t d\|_{L^\infty}y^{1-\alpha}\right)\Big]\left\|\Lambda^{1+s}\partial_t d\right\|_{L^2}\\
\le{}& C\|(\partial_t d,\nabla d)\|_{L^\infty}y^2.
\end{aligned}
\tag{2.8}
$$

By using (1.10), I_2 can be bounded as

$$
\begin{aligned}
I_2\le{}& C\left(\|u\|_{L^\infty}\left\|\Lambda^{2+s}d\right\|_{L^2}+\|\nabla d\|_{L^\infty}\left\|\Lambda^{1+s}u\right\|_{L^2}\right)\left\|\Lambda^{1+s}\partial_t d\right\|_{L^2}\\
\le{}& C\|u\|_{L^\infty}y^2+C\|\nabla d\|_{L^\infty}\left(y^2+\left\|\Lambda^{1+s}u\right\|_{L^2}^2\right).
\end{aligned}
\tag{2.9}
$$

Combining (2.5), (2.7), (2.8), and (2.9) and using (1.6), (1.7), (1.8), and the Gronwall lemma, we arrive at

$$
\begin{aligned}
\|u\|_{L^\infty(0,T;H^{1+s})}+\|u\|_{L^2(0,T;H^{2+s})}&\le C,\\
\|(\nabla d,\partial_t d)\|_{L^\infty(0,T;H^{1+s})}&\le C.
\end{aligned}
\tag{2.10}
$$

This completes the proof.

3. Proof of Theorem 1.2

For simplicity, we only present a priori estimates.

First, we still have (2.1).

Testing (1.13) by $\partial_t d$, using $d \cdot \partial_t d = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2 + |\partial_t d|^2 dx = -\int (u \cdot \nabla) d \cdot \partial_t d \, dx. \tag{3.1}$$

Summing up (2.1) and (3.1), we get

$$\begin{gathered}\int u^2 + |\Delta d|^2 + |\partial_t d|^2 dx + \int_0^T \int |\nabla u|^2 dx\, dt \le C, \\ \frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx = -\int \Delta d \cdot \partial_t d \, dx \le \|\Delta d\|_{L^2}\|\partial_t d\|_{L^2} \le C,\end{gathered} \tag{3.2}$$

which yields

$$\int |\nabla d|^2 dx \le C. \tag{3.3}$$

Applying Δ to (1.11), testing by Δu, using (1.2) and (1.10), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta u|^2 dx + \int |\nabla \Delta u|^2 dx \\
&\quad = -\int \Delta \operatorname{div}(u \otimes u) \cdot \Delta u \, dx + \int \Delta(\partial_t d, \nabla d) \cdot \Delta u \, dx \\
&\quad \le C\|u\|_{L^\infty}\|\nabla \Delta u\|_{L^2}\|\Delta u\|_{L^2} + C(\|\partial_t d\|_{L^\infty}\|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^\infty}\|\Delta \partial_t d\|_{L^2})\|\Delta u\|_{L^2} \\
&\quad \le \frac{1}{2}\|\nabla \Delta u\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\Delta u\|_{L^2}^2 \\
&\qquad + C\left(\|\partial_t d\|_{L^2}^{1/2}\|\Delta \partial_t d\|_{L^2}^{1/2}\|\Delta d\|_{L^2}^{1/2}\left\|\Delta^2 d\right\|_{L^2}^{1/2} + \|\nabla d\|_{L^\infty}\|\Delta \partial_t d\|_{L^2}\right)\|\Delta u\|_{L^2} \\
&\quad \le \frac{1}{2}\|\nabla \Delta u\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\Delta u\|_{L^2}^2 + Cy^2 + C\|\Delta u\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}\left(y^2 + \|\Delta u\|_{L^2}^2\right),
\end{aligned} \tag{3.4}$$

where

$$y^2 := \|\Delta \partial_t d\|_{L^2}^2 + \left\|\Delta^2 d\right\|_{L^2}^2. \tag{3.5}$$

Applying Δ to (1.13), we have

$$\Delta \partial_t^2 d + \Delta^3 d = -(\lambda \Delta d + 2\nabla \lambda \cdot \nabla d + d\Delta \lambda) - \Delta(u \cdot \nabla d). \tag{3.6}$$

Since

$$0 = \Delta(d\partial_t d) = d\Delta\partial_t d + \partial_t d\Delta d + 2\sum_k \partial_k d\partial_k\partial_t d, \tag{3.7}$$

we easily see that

$$-d\Delta\partial_t d = \partial_t d\Delta d + \partial_t |\nabla d|^2. \tag{3.8}$$

Testing (3.6) by $\Delta\partial_t d$, using (3.8), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}y^2 = &-\int \lambda\Delta d\cdot\Delta\partial_t d + \nabla\lambda[2\nabla d\cdot\Delta\partial_t d + \nabla(\partial_t d\Delta d + 2\nabla d\cdot\nabla\partial_t d)]dx \\ &-\int \Delta(u\cdot\nabla d)\cdot\Delta\partial_t d\, dx =: J_1 + \mathrm{J}_2.\end{aligned} \tag{3.9}$$

By the same calculations as those in [8], we have

$$J_1 \le C(1 + \|\nabla d\|_{L^\infty})y^2 \le C\big(1 + \|\nabla d\|_{H^1}\log(e+y)\big)y^2. \tag{3.10}$$

By using (1.10), J_2 can be bounded as

$$\begin{aligned}J_2 &\le C(\|\Delta u\|_{L^2}\|\nabla d\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla\Delta d\|_{L^2})\|\Delta\partial_t d\|_{L^2} \\ &\le C\|\nabla d\|_{L^\infty}\Big(y^2 + \|\Delta u\|_{L^2}^2\Big) + C\|u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\Delta d\|_{L^2}^{1/2}\Big\|\Delta^2 d\Big\|_{L^2}^{1/2}\|\Delta\partial_t d\|_{L^2} \\ &\le C\|\nabla d\|_{L^\infty}\Big(y^2 + \|\Delta u\|_{L^2}^2\Big) + Cy^2 + C\|\Delta u\|_{L^2}^2.\end{aligned} \tag{3.11}$$

Combining (3.4), (3.9), (3.10), and (3.11) and using (1.6) and the Gronwall lemma, we conclude that

$$\begin{aligned}\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;H^3)} &\le C, \\ \|\nabla d\|_{L^\infty(0,T;H^3)} + \|\partial_t d\|_{L^\infty(0,T;H^2)} &\le C.\end{aligned} \tag{3.12}$$

This completes the proof.

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