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Research Article

Bifurcation of Sign-Changing Solutions for m-Point Boundary Value Problems

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With the help of bifurcation techniques, some multiplicity results and global structure for sign-changing solutions of some m-point boundary value problems are obtained when the nonlinear term is sublinear at 0.

1. Introduction and Main Results

In this paper, we consider the m-point boundary value problems:

$$u'' + \lambda f(u) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.2}$$

where integer $m \geq 3$, $\eta_i \in (0,1)$, λ is a positive parameter, and $f \in C^1(\mathbb{R}, \mathbb{R})$. We study the multiplicity and global structure of sign-changing solutions of (1.1) and (1.2) under the assumptions:

(A1) $\alpha_i > 0$ for $i = 1, \ldots, m-2$ with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;

(A2) $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $sf(s) > 0$ for $s \neq 0$;

(A3) $f_0 := \lim_{|s| \to 0} f(s)/s = 0$;

(A4) $f_\infty := \lim_{|s| \to \infty} f(s)/s = 0$.

Multipoint boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. The existence of solutions of second-order multipoint boundary value problems has been extensively studied in the literature, see [1–4] and the references therein. Particularly, many authors have studied the existence of sign-changing solutions for various nonlinear boundary value problems, see for example [5–10].

Recently, the global structure of solutions of nonlinear multipoint boundary value problems has also been investigated by several authors using bifurcation methods, see [7–10]. These papers dealt with the case $f_0 \in (0, \infty)$, and relatively little is known about the global structure of solutions when f satisfying $f_0 = 0$. The main reason is that the global bifurcation techniques cannot be used directly in this case. Very recently, [11] investigated the global structure of positive solutions for a class of boundary value problems with $f_0 = 0$. However, to our knowledge there is no paper studying the global structure of sign-changing solutions for nonlinear multipoint boundary value problems under the assumption $f_0 = 0$. The purpose of present paper is to fill this gap.

In this paper, we consider the global structure of nodal solutions of (1.1) and (1.2), a kind of sign-changing having a given number of zeros, when $f_0 = f_\infty = 0$. We find that the discussion is more complicated, when sign-changing solutions are concerned. Eigenvalue theory and Sturm's comparison theorem play important roles in our discussion.

Now, we introduce some notations as follows.

Let $Y = C[0,1]$ with the norm $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$. Let $X = \{u \in C^1[0,1] \mid u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$, and $E = \{u \in C^2[0,1] \mid u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$ equipped with the norm:

$$\|u\|_X = \max\{\|u\|_\infty,\ \|u'\|_\infty\}, \qquad \|u\| = \max\{\|u\|_\infty,\ \|u'\|_\infty,\ \|u''\|_\infty\}. \tag{1.3}$$

For any C^1 function u, if $u(x_0) = 0$ and $u'(x_0) \neq 0$, then x_0 is called a simple zero of u. For any integer $k \geq 1$ and any $\nu \in \{+,-\}$, let $S_k^\nu,\ T_k^\nu \subset C^2[0,1]$ be sets consisting of functions $u \in C^2[0,1]$ satisfying the following conditions:

S_k^ν: (i) $u(0) = 0,\ \nu u'(0) > 0$;

(ii) u has only simple zeros in $[0,1]$ and has exactly $k-1$ zeros in $(0,1)$.

T_k^ν: (i) $u(0) = 0,\ \nu u'(0) > 0$ and $u'(1) \neq 0$;

(ii) u' has only simple zeros in $(0,1)$ and has exactly k zeros in $(0,1)$;

(iii) u has a zero strictly between each two consecutive zeros of u'.

Remark 1.1. If $u \in T_k^\nu$, then $u \in S_k^\nu$ or $u \in S_{k+1}^\nu$. The sets $T_k^\nu (k = 1, 2, \ldots)$ are open in E and disjoint [8].

Lemma 1.2 (See [8]). *Let* (A1) *and* (A2) *hold. If* (μ, u) *is a nontrivial solution of* (1.1) *and* (1.2). *Then,* $u \in T_k^\nu$ *for some* k, ν.

Let $\mathbb{X} = \mathbb{R} \times X$ *with the product topology. As in [12], we add the point* $\{(\lambda, \infty) \mid \lambda \in \mathbb{R}\}$ *to the space* $\mathbb{X}$. *Denote* $\theta \in X,\ \theta(t) \equiv 0,\ t \in [0,1]$.

The main results of this paper are as follows.

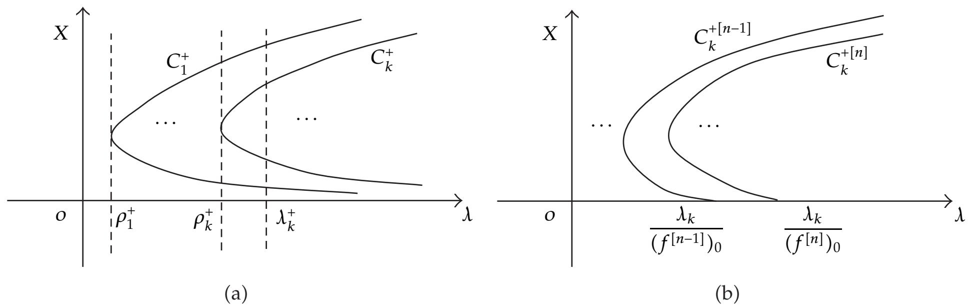

Figure 1

Theorem 1.3. *Let* (A1)–(A4) *hold. Then, there exists a component* $\mathcal{C}_k^\nu \subset (0,\infty) \times T_k^\nu$ *of solutions of* (1.1) *and* (1.2), *which joins* (∞, θ) *to* (∞,∞)*(see Figure 1(a)) such that* $\text{Proj}_{\mathbb{R}}\ \mathcal{C}_k^\nu = [\rho_k^\nu, \infty)$ *for some* $\rho_k^\nu > 0$. *Here,* $\mathcal{C}_k^\nu$ *joins* (∞,θ) *to* (∞,∞) *meaning that:*

$$\lim_{(\lambda,u)\in\mathcal{C}_k^\nu, \|u\|\le 1, \lambda\to+\infty} \|u\| = 0, \qquad \lim_{(\lambda,u)\in\mathcal{C}_k^\nu, \|u\|>1, \lambda\to+\infty} \|u\| = +\infty. \tag{1.4}$$

Corollary 1.4. *Let* (A1)–(A4) *hold. Then, there exists* $\lambda_k^\nu \ge \rho_k^\nu > 0$ *such that* (1.1) *and* (1.2) *have at least two solutions in* T_k^ν *for* $\lambda \in (\lambda_k^\nu, \infty)$.

Remark 1.5. Theorem 1.3 extends the result stated in [11]. Meanwhile, Theorem 1.3 and Corollary 1.4 do not only obtain the multiplicity of nodal solutions of (1.1) and (1.2), but also describe the global structure of these solutions.

2. Preliminary Lemmas

The following definition and lemmas about superior limit and component are important to prove Theorem 1.3.

Definition 2.1 (See [13]). Let W be a Banach space, and $\{C_n \mid n = 1,2,\ldots\}$ be a family of subsets of W. Then, the superior limit $\mathfrak{D}$ of $\{C_n\}$ is defined by:

$$\mathfrak{D} := \limsup_{n\to\infty} C_n = \{x \in W \mid \exists\{n_i\} \subset \mathbb{N},\ x_{n_i} \in C_{n_i},\ \text{such that } x_{n_i} \longrightarrow x\}. \tag{2.1}$$

Lemma 2.2 (See [13]). *Each connected subset of metric space W is contained in a component, and each component of W is closed.*

Lemma 2.3 (See [11]). *Let W be a Banach space and C_n a family of closed connected subsets of W. Assume that:*

(i) there exist $z_n \in C_n$, $n = 1,2,\ldots$, and $z^* \in W$ such that $z_n \to z^*$;
(ii) $r_n = \infty$, where $r_n = \sup\{\|x\| \mid x \in C_n\}$;
(iii) for all $R > 0$, $(\cup_{n=1}^\infty C_n) \cap B_R$ is a relative compact set of W, where

$$B_R = \{x \in W \mid \|x\| \le R\}. \tag{2.2}$$

Then, there exists an unbounded connected component $\mathcal{C}$ in $\mathfrak{D}$ such that $z^* \in \mathcal{C}$.

Define a linear operator $L : E \to Y$ by:

$$\mathrm{Lu} := -u'', \quad u \in E. \tag{2.3}$$

We consider the linear eigenvalues problem:

$$Lu = \lambda u, \quad u \in E. \tag{2.4}$$

Let λ_k be the kth eigenvalue of (2.4), and φ_k an eigenfunction corresponding to λ_k. The following lemma or similar result can be found in [7–9].

Lemma 2.4. *Let* (A1) *hold. Then,*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty. \tag{2.5}$$

For each $k \in N$, algebraic multiplicity of λ_k is equal to 1, and the corresponding eigenfunction $\varphi_k \in T_k^+$ and is strictly positive on (0,1).

Define a map $T_\lambda : Y \longrightarrow E$ by:

$$T_\lambda u(t) = \lambda \int_0^1 H(t,s) f(u(s))ds, \tag{2.6}$$

where

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{\sum_{i=1}^{m-2} \alpha_i G(\eta_i, s)}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\, t, \\ G(t,s) &= \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{2.7}$$

It is clear that $T_\lambda : Y \to X$ is completely continuous provided that (A1) and (A2) hold.

Lemma 2.5. *Let* (A1) *and* (A2) *hold, and $\{(\mu_l, y_l)\} \subset (0,\infty) \times T_k^\nu$ be a sequence of solutions of* (1.1) *and* (1.2). *Assume that $\mu_l \le C_0$ for some constant $C_0 > 0$, and $\lim_{l\to\infty} \|y_l\| = \infty$. Then,*

$$\lim_{l\to\infty} \|y_l\|_\infty = \infty. \tag{2.8}$$

Proof. From the relation $y_l(t) = \mu_l \int_0^1 H(t,s) f(y_l(s))ds$, we conclude that $y_l'(t) = \mu_l \int_0^1 H_t(t,s) f(y_l(s))ds$. Then,

$$\|y_l'\|_\infty \le C_0 \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 |f(y_l(s))| ds. \tag{2.9}$$

Equations (2.9) and (1.1) imply that $\{\|y_l'\|_\infty\}, \{\|y_l''\|_\infty\}$ are bounded, whenever $\{\|y_l\|_\infty\}$ is bounded. □

3. Proof of the Main Results

We will construct a sequence of functions $\{f^{[n]}\}$ which is asymptotic linear at 0 and satisfies

$$\lim_{n\to\infty}\sup_{s\in\mathbb{R}}\left|f^{[n]}(s)-f(s)\right|=0,\ \lim_{n\to\infty}\left(f^{[n]}\right)_0:=\lim_{n\to\infty}\left(\frac{\lim_{|s|\to 0}f^{[n]}(s)}{s}\right)=0. \tag{3.1}$$

By means of some corresponding auxiliary equations, we can obtain a sequence of unbounded components $\{C_k^{\nu[n]}\}$ via Rabinowitz's global bifurcation theorem [14]. Based on the sequence, we can find an unbounded component C_k^ν satisfying:

$$C_k^\nu \subset \limsup_{n\to\infty} C_k^{\nu[n]}, \tag{3.2}$$

and joining (∞,θ) with (∞,∞). We do it as follows.

For each $n\in\mathbb{N}$, define $f^{[n]}(s):\mathbb{R}\to\mathbb{R}$ by:

$$f^{[n]}(s)=\begin{cases} f(s), & s\in\left(\dfrac{1}{n},\infty\right)\cup\left(-\infty,-\dfrac{1}{n}\right),\\ nf\left(\dfrac{1}{n}\right)s, & s\in\left[-\dfrac{1}{n},\dfrac{1}{n}\right].\end{cases} \tag{3.3}$$

Then, $f^{[n]}\in C(\mathbb{R},\mathbb{R})\bigcap C^1(\mathbb{R}\setminus\{\pm 1/n\},\mathbb{R})$ with

$$sf^{[n]}(s)>0,\quad \forall s\neq 0,\quad \left(f^{[n]}\right)_0=nf\left(\frac{1}{n}\right). \tag{3.4}$$

By (A3), it follows that

$$\lim_{n\to\infty}\left(f^{[n]}\right)_0=0. \tag{3.5}$$

Now let us consider the auxiliary family of problems:

$$\begin{gathered} u''+\lambda f^{[n]}(u)=0,\quad t\in(0,1),\\ u(0)=0,\qquad u(1)=\sum_{i=1}^{m-2}\alpha_i u(\eta_i). \end{gathered} \tag{3.6}$$

From Proposition 4.1 in [8], we obtain the following.

Lemma 3.1. *Let* (A1) *and* (A2) *hold. If* (μ,u) *is a nontrivial solution of* (3.6). *Then,* $u\in T_k^\nu$ *for some* k,ν.

Let $g^{[n]} \in C(\mathbb{R}, \mathbb{R})$ such that:

$$f^{[n]}(u) = \left(f^{[n]}\right)_0 u + g^{[n]}(u) = nf\left(\frac{1}{n}\right)u + g^{[n]}(u). \tag{3.7}$$

Note that

$$\lim_{|s|\to 0} \frac{g^{[n]}(s)}{s} = 0. \tag{3.8}$$

Let us consider

$$Lu - \lambda\left(f^{[n]}\right)_0 u = \lambda g^{[n]}(u), \tag{3.9}$$

as a bifurcation problem from the trivial solution $u \equiv \theta$.

Equation (3.9) can be converted to the equivalent form:

$$\begin{aligned} u(t) &= \int_0^1 H(t,s)\left[\lambda\left(f^{[n]}\right)_0 u(s) + \lambda g^{[n]}(u(s))\right]ds \\ &:= \lambda L^{-1}\left[\left(f^{[n]}\right)_0 u(\cdot)\right](t) + \lambda L^{-1}\left[g^{[n]}(u(\cdot))\right](t). \end{aligned} \tag{3.10}$$

Note that $\|L^{-1}[g^{[n]}(u)]\| = o(\|u\|)$ for u near θ in X. Applying Lemma 2.4, the global bifurcation result of Rabinowitz [14] for (3.9) can be stated as follows: for each integer $k \geq 1$, $\nu \in \{+, -\}$, there exists a continuum $C_k^{\nu[n]}$ of solutions of (3.9) joining $(\lambda_k/(f^{[n]})_0, \theta)$ to infinity in $\mathbb{X}$. Moreover, $C_k^{\nu[n]} \setminus (\lambda_k/(f^{[n]})_0, \theta) \subset (0,\infty) \times T_k^{\nu}$.

For properties of $C_k^{\nu[n]}$, we give the following lemmas.

Lemma 3.2. *Let* (A1)–(A4) *hold. Then for each fixed n, $C_k^{\nu[n]}$ joins $(\lambda_k/(f^{[n]})_0, \theta)$ to (∞, ∞) in $\mathbb{X}$ (see Figure 1(b)).*

Proof. We divide the proof into two steps.

Step 1. We show that $\sup\{\lambda \mid (\lambda, u) \in C_k^{\nu[n]}\} = \infty$. Assume on the contrary that $\sup\{\lambda \mid (\lambda, u) \in C_k^{\nu[n]}\} =: C_0 < \infty$. Let $\{(\mu_l, y_l)\} \subset C_k^{\nu[n]}$ be such that:

$$|\mu_l| + \|y_l\| \to \infty. \tag{3.11}$$

Similar to the argument of Lemma 2.5, we conclude that $\|y_l\|_\infty \to \infty$.

Since $(\mu_l, y_l) \in C_k^{\nu[n]}$, we have

$$\begin{aligned} &y_l''(t) + \mu_l f^{[n]}(y_l(t)) = 0, \quad t \in (0,1), \\ &y_l(0) = 0, \qquad y_l(1) = \sum_{i=1}^{m-2} \alpha_i y_l(\eta_i). \end{aligned} \tag{3.12}$$

Set $v_l(t) = y_l(t)/\|y_l\|_\infty$. Then, $\|v_l\|_\infty = 1$, and

$$v_l''(t) + \mu_l \frac{f^{[n]}(y_l(t))}{\|y_l\|_\infty} = 0, \quad t \in (0,1). \tag{3.13}$$

Using $\lim_{|u|\to 0} f(u)/u = 0$, we can show that

$$\lim_{l\to\infty} \frac{\left|f^{[n]}(y_l(t))\right|}{\|y_l\|_\infty} = 0. \tag{3.14}$$

The proof is similar to that of Theorem 1 in [12], and therefore we omit it. Equations (3.13) and (3.14) imply that $\|v_l''\|_\infty \leq M$ for some constant $M > 0$, independent of l. Hence, $\{v_l\}$ has a convergent subsequence in X. Without loss of generality, we assume that there exists $(\mu_*, v_*) \in [0, C_0] \times X$ with:

$$\|v_*\|_\infty = 1, \tag{3.15}$$

such that

$$\lim_{l\to\infty} (\mu_l, v_l) = (\mu_*, v_*), \quad \text{in } \mathbb{R} \times Y. \tag{3.16}$$

Note that (3.12) is equivalent to

$$v_l(t) = \mu_l \int_0^1 H(t,s) \frac{f^{[n]}(y_l(s))}{\|y_l\|_\infty} ds, \quad t \in (0,1). \tag{3.17}$$

Combining this with (3.16) and using (3.14) and the Lebesgue dominated convergence theorem, we have

$$v_*(t) = \mu_* \int_0^1 H(t,s)\ 0\, ds = 0, \quad t \in (0,1). \tag{3.18}$$

This contradicts (3.15). Therefore,

$$\sup\left\{\lambda \mid (\lambda, y) \in C_k^{\nu[n]}\right\} = \infty. \tag{3.19}$$

Step 2. We show that $\sup\{\|u\|_\infty \mid (\lambda, u) \in C_k^{\nu[n]}\} = \infty$. On the contrary, assume that $\sup\{\|u\|_\infty \mid (\lambda, u) \in C_k^{\nu[n]}\} = M_0 < \infty$. Then, there exists a sequence $\{(\mu_l, y_l)\} \subset C_k^{\nu[n]}$ such that

$$\mu_l \longrightarrow \infty, \quad \|y_l\|_\infty \leq M_0. \tag{3.20}$$

From Remark 1.1, we can take a subsequences of $\{(\mu_l, y_l)\}$, still denoted by $\{(\mu_l, y_l)\}$, such that $\{y_l\} \subset T_k^\nu \bigcap S_k^\nu$ or $\{y_l\} \subset T_k^\nu \bigcap S_{k+1}^\nu$. Without loss of generality, we suppose that $\{y_l\} \subset T_k^\nu \bigcap S_k^\nu$. When $\{y_l\} \subset T_k^\nu \bigcap S_{k+1}^\nu$ is considered, the proof is similar. We omit it.

Note that (μ_l, y_l) satisfies the autonomous equation:

$$y_l'' + \mu_l f^{[n]}(y_l) = 0, \quad t \in (0,1). \tag{3.21}$$

Therefore, the graph of y_l consists of a sequence of positive and negative bumps, together with a truncated bump at the right end of the interval $[0,1]$, with the following properties (ignoring the truncated bump) (see [8]): all the positive (respectively, negative) bumps (i) have the same shape (the shapes of the positive and negative bumps may be different); (ii) attain the same maximum (minimum) value.

Let

$$0 = \tau_l^0 < \tau_l^1 < \cdots < \tau_l^{k-1} \tag{3.22}$$

denote the zeros of y_l in $[0,1]$. Then, after taking a subsequence if necessary, $\lim_{l\to\infty}\tau_l^j := \tau_\infty^j, j \in \{0,1,\ldots,k-1\}$. Clearly, $\tau_\infty^0 = 0$. Set $\tau_\infty^k = 1$. We can choose at least one subinterval $(\tau_\infty^j, \tau_\infty^{j+1}) \triangleq I_\infty^j$ which is of length at least $1/k$ for some $j \in \{0,1,\ldots,k-1\}$. Then, for this j, $\tau_l^{j+1} - \tau_l^j > 3/4k$ if l is large enough. Put $(\tau_l^j, \tau_l^{j+1}) \triangleq I_l^j$.

Obviously, for the above given k, ν, and j, $y_l(t)$ have the same sign on I_l^j for all l. Without loss of generality, we assume

$$y_l(t) > 0, \quad t \in I_l^j. \tag{3.23}$$

Armed with the information on the shape of y_l, it is easy to show that for the above given I_l^j, $\|y_l\|_{I_l^j,\infty} := \max_{I_l^j} y_l(t) \leq M_0$, $l = 1,2,\ldots$.

Let σ be a constant with $0 < \sigma < 3/8k$. Since y_l is concave on I_l^j, we have

$$y_l(t) \geq \sigma \|y_l\|_{I_l^j,\infty}, \quad \forall\, t \in \left[\tau_l^j + \sigma, \tau_l^{j+1} - \sigma\right]. \tag{3.24}$$

Then, there must exist constants α, β with $[\alpha, \beta] \subset I_\infty^j$ and l_0 such that

$$y_l(t) \geq \sigma \|y_l\|_{I_l^j,\infty} > 0, \quad \text{uniformly for } t \in [\alpha, \beta] \text{ and } l > l_0. \tag{3.25}$$

On the other hand, note that

$$\frac{f^{[n]}(y_l(t))}{y_l(t)} \geq \inf\left\{\frac{f^{[n]}(s)}{s} \mid 0 < s \leq M_0\right\} > 0, \quad t \in \left(\tau_l^j, \tau_l^{j+1}\right). \tag{3.26}$$

Using the relation:

$$y_l''(t) + \mu_l \frac{f^{[n]}(y_l(t))}{y_l(t)} y_l(t) = 0, \quad t \in \left(\tau_l^j, \tau_l^{j+1}\right), \tag{3.27}$$

and Sturm's comparison theorem, we deduce that y_l must change its sign on (α, β) if l is sufficiently large, contradicting (3.25). Therefore,

$$\lim_{l\to\infty} \|y_l\|_\infty = \infty. \tag{3.28}$$

Hence, $C_k^{\nu[n]}$ joins $(\lambda_k/(f^{[n]})_0, \theta)$ to (∞, ∞) in $\mathbb{X}$. □

Lemma 3.3. *Let* (A1)–(A4) *hold. Then, there exists* $\rho_k^\nu > 0$ *such that*

$$\left(\bigcup_{n=1}^{\infty} C_k^{\nu[n]}\right) \bigcap \left((0, \rho_k^\nu) \times X\right) = \emptyset. \tag{3.29}$$

Proof. The proof is similar to that of Lemma 4.3 in [11]. We omit it. □

Lemma 3.4. *Let* (A1)–(A4) *hold, and let* ρ_k^ν *be as in Lemma 3.3. Then, there exist* $n_0 \in \mathbb{N}$ *and* $\widehat{\lambda_k^\nu} \geq \rho_k^\nu > 0$ *such that for any* $\lambda > \widehat{\lambda_k^\nu}$ *and* $u \in C_k^{\nu[n]}$:

$$C_k^{\nu[n]} \cap \left\{(\lambda, u) \mid \lambda \geq \widehat{\lambda_k^\nu};\ \|u\|_\infty = 1\right\} = \emptyset, \quad \forall\, n > n_0. \tag{3.30}$$

Proof. Suppose on the contrary that there exists $\{(\mu_l, y_l)\} \subset (\bigcup_{n=1}^{\infty} C_k^{\nu[n]}) \bigcap ((0,\infty) \times X)$ such that

$$\lim_{l\to\infty} \mu_l = \infty, \qquad \|y_l\|_\infty = 1. \tag{3.31}$$

Now, the method used in the proof of Lemma 3.2, Step 2, is still valid. Let σ be a constant with $0 < \sigma < 3/8k$. Taking subsequences again if necessary, still denoted by $\{(\mu_l, y_l)\}$, such that $\{y_l\} \subset T_k^\nu \bigcap S_k^\nu$. Without loss of generality, we can also derive an interval $[\alpha, \beta] \subset I_\infty^j$ and l_0 such that

$$1 \geq y_l(t) \geq \sigma \|y_l\|_{I_l^j,\infty} = \sigma, \quad \text{uniformly for } t \in [\alpha, \beta] \text{ and } l > l_0. \tag{3.32}$$

It is easy to find an integer $n_0 \in \mathbb{N}$ such that $1/n_0 < \sigma$. This implies that

$$\inf\left\{\frac{f^{[n]}(s)}{s} \mid \sigma < s \le 1\right\} = \inf\left\{\frac{f(s)}{s} \mid \sigma < s \le 1\right\}, \quad \forall\, n > n_0. \tag{3.33}$$

Note that for $n > n_0$,

$$\frac{f^{[n]}(y_l(t))}{y_l(t)} \ge \inf\left\{\frac{f(s)}{s} \mid \sigma < s \le 1\right\} > 0, \quad \text{uniformly for } t \in (\alpha,\beta) \text{ and } l > l_0. \tag{3.34}$$

Combining these facts and the relation:

$$y_l''(t) + \mu_l \frac{f^{[n]}(y_l(t))}{y_l(t)} y_l(t) = 0, \quad t \in (\alpha,\beta), \tag{3.35}$$

and Sturm's comparison theorem, we conclude that y_l must change its sign on (α,β) if l is large enough. This contradicts (3.32), and the proof is done. □

Lemma 3.5. *Let* (A1)–(A4) *hold, and let* n_0 *be as in Lemma 3.4. Then, there exist* $\lambda_k^\nu \ge \rho_k^\nu > 0$ *and* $\epsilon \in (0, 1/2)$ *such that for any* $\lambda > \lambda_k^\nu$ *and* $u \in C_k^{\nu[n]}$:

$$C_k^{\nu[n]} \cap \{(\lambda,u) \mid \lambda \ge \lambda_k^\nu;\ 1 - 2\epsilon \le \|u\|_\infty \le 1 + 2\epsilon\} = \emptyset, \quad \forall\, n > n_0. \tag{3.36}$$

Proof. Similar to the proof of Lemma 3.4, we can find a constant $\widetilde{\lambda_k^\nu} > 0$ such that $\|u\|_\infty \ne 1$ provided that $(\lambda, u) \in (\widetilde{\lambda_k^\nu}, \infty) \times T_k^\nu$ being a solution of (1.1) and (1.2).

Let $\lambda_k^\nu = \max\{\widehat{\lambda_k^\nu}, \widetilde{\lambda_k^\nu}\} + 1$. We claim that there exists $\epsilon \in (0,\ 1/2)$ such that (3.36) holds. Suppose on the contrary that there exists

$$\{(\mu_l, y_l)\} \subset \left(\bigcup_{n=1}^{\infty} C_k^{\nu[n]}\right) \bigcap \left((\lambda_k^\nu, \infty) \times X\right), \tag{3.37}$$

satisfying

$$\lim_{l\to\infty} \mu_l = \mu^* \ge \lambda_k^\nu \qquad \lim_{l\to\infty} \|y_l\|_\infty = 1. \tag{3.38}$$

We can discuss two cases.

Case 1. If $\mu^* < \infty$. $\{y_l\}$ is compact in X implies that there exists a subsequence, still denoted by $\{y_l\}$, such that

$$\lim_{l\to\infty} y_l = y^* \in T_k^\nu, \qquad \|y^*\|_\infty = 1. \tag{3.39}$$

Obviously, (μ^*, y^*) is a solution of (1.1) and (1.2). It is impossible.

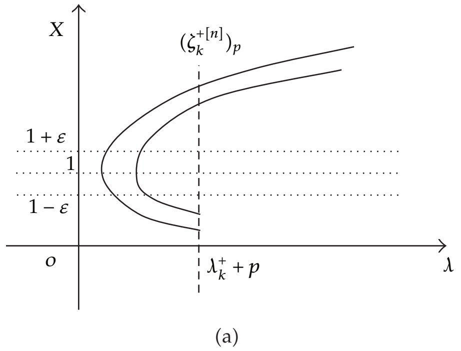

Figure 2

Case 2. If $\mu^* = \infty$. Taking subsequences again if necessary, still denoted by $\{(\mu_l, y_l)\}$, such that $1/2 \leq \|y_l\|_\infty \leq 3/2$. Using the same argument as Lemma 3.4, we can find a contradiction. □

Proof of Theorem 1.3. We will prove that the superior limit of $C_k^{\nu[n]}$ contains an unbounded component $\mathcal{C}_k^\nu \subset (0,\infty) \times T_k^\nu$ of solutions of (1.1) and (1.2), which joins (∞, θ) to (∞, ∞). For $r > 0$, introduce

$$\Omega_r = \{u \in Y \mid \|u\|_\infty < r\}. \tag{3.40}$$

Set

$$\begin{aligned} \Gamma_k^\nu &:= \left([0,\infty) \times T_k^\nu\right) \setminus \left\{(\eta, u) \mid \eta \geq \lambda_k^\nu;\ u \in T_k^\nu,\ \|u\|_\infty \leq 1+\epsilon\right\}, \\ \Sigma_k^\nu &:= \left\{(\eta, u) \mid \eta \geq \lambda_k^\nu;\ u \in T_k^\nu,\ \|u\|_\infty \leq 1-\epsilon\right\}. \end{aligned} \tag{3.41}$$

Let n_0 and ϵ be as in Lemma 3.5. Firstly, for each given nonnegative integer $p = 0, 1, 2, \ldots$, and $n \geq n_0$ with $(\lambda_k/(f^{[n]}\ )_0) \geq \lambda_k^\nu + p$, we define the connected subset, $(\zeta_k^{\nu[n]})_p$, in $C_k^{\nu[n]}$ satisfying (see Figure 2(a)):

(i) $(\zeta_k^{\nu[n]})_p \subset (C_k^{\nu[n]} \setminus (\lambda_k^\nu + p, \infty) \times \Omega_{1-\epsilon})$;

(ii) $(\zeta_k^{\nu[n]})_p$ joins $\{\lambda_k^\nu + p\} \times \Omega_{1-\epsilon}$ with infinity in Γ_k^ν.

By Lemmas 2.2 and 2.3, $\limsup_{n\to\infty} (\zeta_k^{\nu[n]})_p$ contains a component $(\zeta_k^\nu)_p$ joining $\{\lambda_k^\nu + p\} \times \Omega_{1-\epsilon}$ with infinity in Γ_k^ν (see Figure 2(b)).

It is easy to verify that if $(\lambda, u) \in (\zeta_k^\nu)_p (p = 0, 1, 2, \ldots)$, then (λ, u) is a solution of (1.1) and (1.2), and $u \in T_k^\nu$.

Next, by using Lemma 2.3 and the method in [11] (see (4.22)–(4.30) in [11]), we can find a component $\mathcal{C}_k^\nu$ in $\limsup_{p\to\infty} (\zeta_k^\nu)_p$, which is unbounded both in Γ_k^ν and Σ_k^ν.

Finally, we show that $\mathcal{C}_k^\nu$ joins (∞, θ) with (∞, ∞). This will be done by the following three steps.

Step 1. We show that $\lim_{\lambda\to+\infty} \|u\|_\infty = 0$ for $(\lambda, u) \in (\mathcal{C}_k^\nu \cap \Sigma_k^\nu)$.

Suppose on the contrary that there exists $\{(\mu_l, y_l)\} \subset \mathcal{C}_k^\nu$ with $\|y_l\|_\infty \leq 1-\epsilon$, and

$$\mu_l \longrightarrow +\infty, \quad \left\|y_l\right\|_\infty \geq a, \tag{3.42}$$

for some constant $a > 0$. Applying the method of proving Lemma 3.2, we can deduce a contradiction.

Step 2. We show that $\sup\{\lambda \mid (\lambda, u) \in (\mathcal{C}_k^{\nu} \cap \Gamma_k^{\nu})\} = \infty$. By a similar argument as Lemma 3.2, we can get the conclusion.

Step 3. We show that $\lim_{\lambda \to +\infty} \|u\|_{\infty} = +\infty$ for $(\lambda, u) \in (\mathcal{C}_k^{\nu} \cap \Gamma_k^{\nu})$.

On the contrary, suppose that there exists $\{(\mu_l, y_l)\} \subset (\mathcal{C}_k^{\nu} \cap \Gamma_k^{\nu})$ with

$$\mu_l \to +\infty, \quad 1 < \|y_l\|_{\infty} \leq M, \tag{3.43}$$

for some constant $M > 0$. The proof can be done by the same argument as Lemma 3.2.

This completes the proof of Theorem 1.3. □

Proof of Corollary 1.4. The result can be directly obtained by Theorem 1.3. □

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