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Research Article

Pseudo Almost Automorphic Solutions for Differential Equations Involving Reflection of the Argument

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By means of the fixed point methods and the properties of the pseudo almost automorphic functions, the existence and uniqueness of pseudo almost automorphic solutions are obtained for differential equations involving reflection of the argument. For the nonscalar, case we use the exponential dichotomy properties.

1. Introduction

The existence, uniqueness, and stability of periodic, almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo almost automorphic solutions has been one of the most attractive topics in the qualitative theory of ordinary or functional differential equations for its significance in the physical sciences, mathematical biology, control theory, and others.

The differential equations involving reflection of argument have many applications in the study of stability of differential-difference equations, see Šarkovskiĭ [1], and such equations show very interesting properties by themselves, so many authors have worked on this category of equations. Wiener and Aftabizadeh [2] initiated to study boundary value problems involving reflection of the argument. Gupta in [3, 4] investigated two point boundary value problems for this kind of equations and Aftabizadeh and Huang [5] studied the existence of unique bounded solution of

$$x'(t) = f(t, x(t), x(-t)). \tag{1.1}$$

They proved that $x(t)$ is almost periodic by assuming the existence of bounded solution. In [6], Piao considers the case of pseudo almost periodic solution. This work is motivated by the last reference and devoted to investigate the existence and uniqueness of pseudo almost automorphic solution in the scalar and vectorial case.

The concept of almost automorphic functions, which was introduced by Bochner as an extension of one of the almost periodic functions, has recently caught the attention of many mathematicians (see, e.g., [7–11]...). In [12], Zhang has introduced an extension of the almost periodic functions, the so-called pseudo almost periodic functions. For more details on this notion, we can refer to [12–18]. Then the combination between pseudo almost periodic and almost automorphic leads to the pseudo almost automorphic functions, which is considered in this work.

The theory of exponential dichotomy has played a central role in the study of ordinary differential equations and diffeomorphisms for finite dimensional dynamic systems. This theory, which addresses the issue of strong transversality in dynamic systems, originated in the pioneering works of Lyapunov (1892) and Poincaré (1890). During the last few years, one finds an ever growing use of exponential dichotomies to study the dynamic structures of various partial differential delay equations, for more details, we refer to [19, 20].

This paper is organized as follows. In Section 2, we recall some preliminary results which is divided in two sections, in the first one we give some results on the exponential dichotomy theory, and in the second one, we give some definitions of pseudo almost automorphic functions. The main results are announced and discussed in Section 3. In the last section, we give some illustrated examples.

2. Preliminaries

Throughout the paper $C_b := \{f : \mathbb{R} \to \mathbb{R}^n, f$ continuous and bounded$\}$ and for $f \in C_b$, $|f| = \sup\{|f(t)| : t \in \mathbb{R}\}$.

2.1. Exponential Dichotomy

In the sequel, A denotes a continuous mapping from $\mathbb{R}$ to $M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ is the space of square matrices with real coefficients.

Definition 2.1. Let $A(t)$ be a continuous square matrix on an interval J and let $X(t)$ be a fundamental matrix of the following system:

$$\frac{dx}{dt}(t) = A(t)x(t), \tag{2.1}$$

satisfying $X(0) = I$, where I is the unit matrix.

The system of differential equations (2.1) is said to possess an exponential dichotomy on the interval J, if there exists a projection matrix P (i.e., $P^2 = P$) and constants $k > 1$, $\alpha > 0$, such that

$$\begin{gathered}\left\| X(t)PX^{-1}(s) \right\| \leq k\exp(-\alpha(t-s)), \quad \text{for } s \leq t, \text{ with } s,t \in J \\ \left\| X(t)(I-P)X^{-1}(s) \right\| \leq k\exp(-\alpha(s-t)), \quad \text{for } t \leq s, \text{ with } s,t \in J.\end{gathered} \tag{2.2}$$

We denote by (P, k, α) the triple of elements associated to an exponential dichotomy.

Remark 2.2. When $A(t) = A$ is constant, the system (2.1) has an exponential dichotomy on an infinite interval, if and only if the eigenvalues of A have a nonzero real part. When $A(t)$ is periodic, (2.1) has an exponential dichotomy on an infinite interval, if and only if the Floquet multipliers lie off the unit circle.

For the properties of exponential dichotomies, one may refer to [13, 19–22].

Remark 2.3. Putting $\widetilde{A}(t) = -A(-t)$. Then equation

$$y' = \widetilde{A}(t)y \tag{2.3}$$

has as fundamental matrix $Y(t) = X(-t)$. Let J be one of the following intervals $\mathbb{R}^+$, $\mathbb{R}^-$. Equation (2.3) admits an exponential dichotomy with parameters (P, k, α) on J, if and only if (2.1) has an exponential dichotomy on $-J$ with parameters $(I - P, k, \alpha)$. On the other hand, x is a solution of

$$x' = A(t)x + f(t) \tag{2.4}$$

if and only if $y : t \to x(-t)$ is a solution of

$$y' = \widetilde{A}(t)y + \widetilde{f}(t), \tag{2.5}$$

where $\widetilde{f}(t) = -f(-t)$.

Theorem 2.4 (see [21]). *Assume that the following differential equation:*

$$\frac{d}{dt}x(t) = A(t)x(t) \tag{2.6}$$

has an exponential dichotomy on $\mathbb{R}^+$($\mathbb{R}^-$, $\mathbb{R}$, resp.) with parameters (P, α, k). Let $B : \mathbb{R} \to M_n(\mathbb{R})$ be a bounded continuous function such that $\delta = \sup_{t\in\mathbb{R}^+}|B(t)| < \alpha/4k^2$($\mathbb{R}^-$, $\mathbb{R}$, resp.). Then the perturbed equation

$$\frac{d}{dt}x(t) = [A(t) + B(t)]x(t) \tag{2.7}$$

has an exponential dichotomy on $\mathbb{R}^+(\mathbb{R}^-,\ \mathbb{R},\ resp.)$ *with parameters* $(Q,\ \alpha - 2k\delta,\ k_1 = (5/2)k^2)$, *where* Q *is a projection with the same kernel (range, resp.) as the one of* P. *Moreover, if* $Y(t)$ *is the fundamental matrix of* (2.7) *satisfying* $Y(0) = I$, *then*

$$\left| Y(t)QY^{-1}(t) - X(t)PX^{-1}(t) \right| \le \frac{4}{\alpha}\delta k^3 \quad \forall t \text{ on } \mathbb{R}^+(\textit{resp. } \mathbb{R}^-,\ \mathbb{R}). \tag{2.8}$$

Lemma 2.5 (see [21]). *Let* t_0, τ *be real constants,* $\tau < t_0 (\tau > t_0,\ resp.)$. *If* (2.6) *has an exponential dichotomy on* $[t_0, +\infty[$ *respectively* $]-\infty, t_0]$, *then it has one on* $[\tau, +\infty[$ *respectively* $]-\infty, \tau]$, *with the same exponents and the same projection* P.

2.2. Almost Automorphic Functions

Definition 2.6. A continuous function $g : \mathbb{R} \to E$ is said to be almost automorphic if for every sequence of real numbers $(t'_n)_n$, there exists a subsequence of $(t'_n)_n$, denoted $(t_n)_n$ such that for each $t \in \mathbb{R}$

$$\begin{aligned} &\lim_{n\to+\infty} g(t+t_n) = k(t) \quad \text{exists } \forall t \in \mathbb{R} \\ &\lim_{n\to+\infty} k(t-t_n) = g(t) \quad \text{exists } \forall t \in \mathbb{R}. \end{aligned} \tag{2.9}$$

Denote by $AA(\mathbb{R}, \mathbb{E})$ is the set of all such functions.

If g is almost automorphic, then its range is relatively compact, thus bounded in norm.

By the pointwise convergence, the function k is just measurable and not necessarily continuous.

If the convergence in both limits is uniform, then g is almost periodic. The concept of almost automorphy is then larger than the one of almost periodicity. It was introduced in the literature by Bochner and recently studied by several authors. A complete description of their properties and further applications to evolution equations can be found in the monographs by N'Guérékata [10, 11].

Example 2.7. $f(t) = \sin 1/(2 - \sin t - \sin \pi t)$ is an almost automorphic function, which is not almost periodic, because it is not uniformly continuous.

Definition 2.8 (see [23]). A continuous function $f : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ is said to be almost automorphic in t uniformly with respect to x in $\mathbb{E}$, if the following two conditions hold:

(i) for all $x \in \mathbb{E}$, $f(\cdot, x) \in AA(\mathbb{R}, \mathbb{E})$;

(ii) f is uniformly continuous on each compact subset $K \subset X$ with respect to the second variable x, namely, for each compact subset K in $\mathbb{E}$, and for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x_1, x_2 \in K$, one has

$$\|x_1 - x_2\| \le \delta \Longrightarrow \sup_{t\in\mathbb{R}} \left\| f(t, x_1) - f(t, x_2) \right\| \le \varepsilon. \tag{2.10}$$

Denote by $AA_U(\mathbb{R} \times \mathbb{E}, \mathbb{E})$ is the set of all such functions.

With these definitions, we have the following inclusions:

$$AP(\mathbb{E}) \subset AA(\mathbb{E}), \qquad AP_U(\mathbb{R}\times\mathbb{E}) \subset AA_U(\mathbb{R}\times\mathbb{E}). \tag{2.11}$$

Theorem 2.9 (see [23]). *Let* $f \in AA_U(\mathbb{R}\times\mathbb{E},\mathbb{E})$ *and* $x \in AA(\mathbb{R},\mathbb{E})$. *Then* $[t \to f(t,x(t)) \in AA(\mathbb{R},\mathbb{E})]$.

2.3. *Pseudo Almost Automorphic Functions*

Set

$$\begin{gathered} PAP_0(\mathbb{R},\mathbb{E}) = \left\{\varphi \in C_b(\mathbb{R},\mathbb{E}),\ \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|\varphi(s)\|ds = 0\right\}, \\ PAP_0(\mathbb{R}\times\Omega,\mathbb{E}) = \left\{\begin{matrix}\varphi \in C_b(\mathbb{R}\times\Omega,\mathbb{E}),\ \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|\varphi(s,x)\|ds = 0 \\ \text{uniformly in } x\in\Omega\end{matrix}\right\}. \end{gathered} \tag{2.12}$$

Definition 2.10. $f \in C_b(\mathbb{R},\mathbb{E})(C_b(\mathbb{R}\times\mathbb{E},\mathbb{E})$, resp.) is called pseudo almost-automorphic, if $f = g + \varphi$ with $g \in AA(\mathbb{R},\mathbb{E})(AA(\mathbb{R}\times\mathbb{E},\mathbb{E})$, resp.) and $\varphi \in PAP_0(\mathbb{R},\mathbb{E})(PAP_0(\mathbb{R}\times\mathbb{E},\mathbb{E})$, resp.),$g$ and φ are, respectively, called the almost automorphic component and the ergodic perturbation of f. Denote the set of all such functions by $PAA(\mathbb{R},\mathbb{E})(PAA(\mathbb{R}\times\mathbb{E},\mathbb{E})$, resp.).

It is easy to verify that $PAA(\mathbb{R},\mathbb{E})$ is a translation invariant closed subspace of $C_b(\mathbb{R},\mathbb{E})$ containing the constant functions. Furthermore,

$$PAA(\mathbb{E}) = AA(\mathbb{E}) \oplus PAP_0(\mathbb{E}). \tag{2.13}$$

Definition 2.11. A closed subset K of $\mathbb{R}$ is said to be an ergodic zero set if $\operatorname{meas}(K \cap [-t,t])/2t \to 0$ as $t \to \infty$, where meas is the Lebesgue measure on $\mathbb{R}$.

Remark 2.12. One sees that $\varphi \in C(\mathbb{R},\mathbb{E})$ is in $PAP_0(\mathbb{R},\mathbb{E})$ if and only if for $\varepsilon > 0$, the set $K_\varepsilon = \{t\in\mathbb{R} : \|\varphi(t)\| \geq \varepsilon\}$ is an ergodic zero set in $\mathbb{R}$.

Example 2.13. (a) $f(t) = \sin 1/(2-\sin t - \sin \pi t) + (1/\sqrt{1+t^2})$ is a pseudo almost automorphic function.

(b) A continuous function $f : \mathbb{R} \to E$ satisfying $\lim_{|t|\to+\infty} f(t) = 0$ is an ergodic function. Indeed by hypothesis, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|t| \geq \delta \Rightarrow \|f(t)\| \leq \varepsilon$, then for all $r > \delta$,

$$\left\{\frac{t\in[-r,r]}{(-\delta,\delta)} : \|f(t)\| \geq \varepsilon\right\} = \emptyset. \tag{2.14}$$

We conclude by using Remark 2.12.

3. Main Results

In this part of this work, we are concerned with the following differential equation:

$$x'(t) = A(t)x(t) + B(t)x(-t) + f(t, x(t), x(-t)), \tag{3.1}$$

where A and B are two square matrices, and $f(t, x, y)$ is almost automorphic in t uniformly with respect to x and y in any compact subset of $\mathbb{R}^{2n}$.

In the first time, we consider the following scalar and linear differential equation:

$$x'(t) = ax(t) + bx(-t) + g(t), \quad b \neq 0, \; t \in \mathbb{R}, \; |a| > |b|, \tag{3.2}$$

where g is continuous on $\mathbb{R}$. Let $y(t) = x(-t)$, then (3.2) is changed into the following system:

$$\begin{aligned} x'(t) &= ax(t) + by(t) + g(t), \quad t \in \mathbb{R} \\ y'(t) &= -bx(t) - ay(t) - g(-t), \quad t \in \mathbb{R}, \end{aligned} \tag{3.3}$$

which is in a formally Hamilton system with Hamiltonian function as

$$H(x, y, t) = \frac{1}{2}bx^2 + \frac{1}{2}by^2 + axy - g(-t)x - g(t)y. \tag{3.4}$$

So, one may say that some first order scalar differential equations can also generate Hamilton systems.

3.1. Scalar Case

In the scalar case, our main results can be stated as follows.

Theorem 3.1. *For any $g \in PAA(\mathbb{R}, \mathbb{R})$, with $\alpha^2 = a^2 - b^2 > 0$, (3.2) has an unique pseudo almost automorphic solution $x(t)$.*

For the proof of Theorem 3.1, we use the following lemmas.

Lemma 3.2 (see [10]). *If $g \in AA(\mathbb{R}, \mathbb{E})$, then $t \to g(-t) \in AA(\mathbb{R}, \mathbb{E})$.*

Lemma 3.3 (see [15]). *If $\varphi \in PAP_0(\mathbb{R}, \mathbb{E})$, then $t \to \varphi(-t) \in PAP_0(\mathbb{R}, \mathbb{E})$.*

Lemma 3.4 (see [8]). *If $f \in PAA(\mathbb{R}, \mathbb{R})$, and $g \in L^1(\mathbb{R}, \mathbb{R})$, then $h = f * g \in PAA(\mathbb{R}, \mathbb{R})$.*

Proof of Theorem 3.1. Uniqueness. If there is two pseudo almost automorphic solutions $x_1(t)$ and $x_2(t)$ of (3.2), then the difference $x_1(t) - x_2(t)$ should be a solution of the homogeneous equation as

$$x'(t) = ax(t) + bx(-t), \quad b \neq 0, \; t \in \mathbb{R}. \tag{3.5}$$

According to Lemma 2 of [2], one can derive that

$$x_1(t) - x_2(t) = C\left[\frac{\alpha + a}{-b}\exp(\alpha t) + \exp(-\alpha t)\right], \quad t \in \mathbb{R}, \tag{3.6}$$

for some constant C. If $C \neq 0$, $x_1(t) - x_2(t)$ will be unbounded. This is a contradiction to the boundedness of pseudo almost automorphic function. So $x_1(t) = x_2(t)$.

Existence. From Lemmas 2 and 3 of [5] that we can derive to the following solution:

$$\begin{aligned} x(t) = {} & \frac{-1}{2\alpha}\left[\int_{-\infty}^{\infty} \exp -\alpha(s-t)\left[(\alpha + a)g(s) - bg(-s)\right]ds\right] \\ & + \frac{1}{2\alpha}\left[\int_{-\infty}^{\infty} \exp \alpha(s-t)\left[(\alpha + a)g(s) + bg(-s)\right]ds\right] \end{aligned} \tag{3.7}$$

is a particular solution of (3.2) for any $g \in PAA(\mathbb{R}, \mathbb{R})$. Now we show that $x(t) \in PAA(\mathbb{R}, \mathbb{R})$.

Let us go back to the rest of the proof. Now we show $x(t) \in PAA(\mathbb{R}, \mathbb{R})$. Assume that $g(t) = h(t) + \varphi(t), h(t) \in AA(\mathbb{R}, \mathbb{R}),\ \varphi(t) \in PAP_0(\mathbb{R}, \mathbb{R})$. Let

$$\begin{aligned} H(t) = {} & \frac{-1}{2\alpha}\left[\int_{t}^{\infty} \exp -\alpha(s-t)[(\alpha + a)h(s) - bh(-s)]ds\right] \\ & + \frac{1}{2\alpha}\left[\int_{-\infty}^{t} \exp \alpha(s-t)[(\alpha + a)h(s) + bh(-s)]ds\right] \\ \Phi(t) = {} & \frac{-1}{2\alpha}\left[\int_{t}^{\infty} \exp -\alpha(s-t)\left[(\alpha + a)\varphi(s) - b\varphi(-s)\right]ds\right] \\ & + \frac{1}{2\alpha}\left[\int_{-\infty}^{t} \exp \alpha(s-t)\left[(\alpha + a)\varphi(s) + b\varphi(-s)\right]ds\right], \end{aligned} \tag{3.8}$$

then $x(t) = H(t) + \Phi(t)$.

Similar to the proof of Theorem 2.2 in [6], we have $\Phi \in PAP_0(\mathbb{R}, \mathbb{R})$. Now, we prove that $H(t)$ is almost automorphic indeed, let $(s'_n) \subset \mathbb{R}$ be an arbitrary sequence. Since $h \in AA(\mathbb{R}, \mathbb{R})$, then $t \to h(-t)$ is also almost automorphic, consequently $t \to (h(t), h(-t))$ is also almost automorphic, which leads to the fact that we can found a same subsequence (s_n) of (s'_n) and two functions (k, k_1) such that

$$\begin{aligned} & \lim_{n\to\infty} h(t + s_n) = k(t), \quad \lim_{n\to\infty} k(t - s_n) = h(t), \quad \forall t \in \mathbb{R}, \\ & \lim_{n\to\infty} h(t - s_n) = k_1(t), \quad \lim_{n\to\infty} k_1(t + s_n) = h(t), \quad \forall t \in \mathbb{R} \end{aligned} \tag{3.9}$$

We define

$$K(t) = \frac{-1}{2\alpha}\left[\int_t^{\infty} \exp -\alpha(s-t)[(\alpha+a)k(s) - bk_1(-s)]ds\right] + \frac{1}{2\alpha}\left[\int_{-\infty}^{t} \exp \alpha(s-t)[(\alpha+a)k(s) + bk_1(-s)]ds\right]. \tag{3.10}$$

Now, consider the following:

$$\begin{aligned} H(t+s_n) &= \frac{-1}{2\alpha}\left[\int_{t+s_n}^{\infty} \exp -\alpha(s-s_n-t)[(\alpha+a)h(s) - bh(-s)]ds\right] \\ &\quad + \frac{1}{2\alpha}\left[\int_{-\infty}^{t+s_n} \exp \alpha(s-t-s_n)[(\alpha+a)h(s) + bh(-s)]ds\right] \\ &= \frac{-1}{2\alpha}\left[\int_t^{\infty} \exp -\alpha(s-t)[(\alpha+a)h(s+s_n) - bh(-s-s_n)]ds\right] \\ &\quad + \frac{1}{2\alpha}\left[\int_{-\infty}^{t} \exp \alpha(s-t)[(\alpha+a)h(s+s_n) + bh(-s-s_n)]ds\right]. \end{aligned} \tag{3.11}$$

Note that

$$\begin{gathered} \left|e^{-\alpha(s-t)}[(\alpha+a)h(s+s_n) - bh(-s-s_n)]\right| \le |h|_\infty(|\alpha+a|+|b|)e^{-\alpha(t-s)}, \\ \left|e^{\alpha(s-t)}[(\alpha+a)h(s+s_n) + bh(-s-s_n)]\right| \le |h|_\infty(|\alpha+a|+|b|)e^{-\alpha(t-s)}, \\ e^{-\alpha(s-t)}[(\alpha+a)h(s+s_n) - bh(-s-s_n)] \underset{n\to\infty}{\longrightarrow} e^{-\alpha(s-t)}[(\alpha+a)k(s) - bk_1(-s)] \quad \forall t\in\mathbb{R}, \\ e^{\alpha(s-t)}[(\alpha+a)h(s+s_n) + bh(-s-s_n)] \underset{n\to\infty}{\longrightarrow} e^{\alpha(s-t)}[(\alpha+a)k(s) + bk_1(-s)] \quad \forall t\in\mathbb{R}. \end{gathered} \tag{3.12}$$

Then by the Lebesgue dominated convergence theorem, $\lim_{n\to\infty}H(t+s_n) = K(t)$, for all $t\in\mathbb{R}$. In similar, way we can show that $\lim_{n\to\infty}K(t-s_n) = H(t)$ for all $t\in\mathbb{R}$, which ends the proof. □

3.2. The Vectorial Case

Let us consider the following equation with reflection:

$$\frac{dx}{dt} = A(t)x(t) + B(t)x(-t) + g(t), \tag{3.13}$$

where $t \to A(t)$, $B(t) \in M_n(\mathbb{R})$, $x : \mathbb{R} \to M_{n,1}(\mathbb{R})$, and $g : \mathbb{R} \to M_{n,1}(\mathbb{R})$ bounded continuous functions.

Putting $y(t) = x(-t)$, one has

$$y'(t) = -x'(-t) = -A(-t)x(-t) - B(-t)x(t) - g(-t) = -A(-t)y(t) - B(-t)x(t) - g(-t). \tag{3.14}$$

If we put $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, then $X(t)$ is a solution of the following system:

$$\frac{dX(t)}{dt} = M(t)X(t) + G(t), \tag{3.15}$$

where

$$M(t) = \begin{pmatrix} A(t) & B(t) \\ -B(-t) & -A(-t) \end{pmatrix}, \qquad G(t) = \begin{pmatrix} g(t) \\ -g(-t) \end{pmatrix}. \tag{3.16}$$

Theorem 3.5. *If the system*

$$\frac{dx}{dt} = A(t)x(t) \tag{3.17}$$

has a fundamental matrix $X(t)$ and has an exponential dichotomy with parameters (P, α, k), then the following system:

$$\frac{dx}{dt} = M_0(t)x(t), \tag{3.18}$$

where $M_0(t) = \begin{pmatrix} A(t) & 0 \\ 0 & -A(-t) \end{pmatrix}$, has a fundamental matrix $Y(t) = \begin{pmatrix} X(t) & 0 \\ 0 & X(-t) \end{pmatrix}$ and admits an exponential dichotomy with parameters $(Q, \alpha, 2k)$, where

$$Q = \begin{pmatrix} P & 0 \\ 0 & I - P \end{pmatrix}. \tag{3.19}$$

Proof. If $X(t)$ is a fundamental matrix of the system (3.17),

$$\begin{gathered} \frac{d}{dt}[X(t)] = \frac{dX}{dt}(t) = A(t)X(t), \\ \frac{d}{dt}[X(-t)] = -\frac{dX}{dt}(-t) = -A(-t)X(-t). \end{gathered} \tag{3.20}$$

Consequently,

$$\begin{gathered} \frac{dY}{dt}(t) = \begin{pmatrix} A(t) & 0 \\ 0 & -A(-t) \end{pmatrix}\begin{pmatrix} X(t) & 0 \\ 0 & X(-t) \end{pmatrix} = M_0(t)Y(t) \\ \text{with } Y(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad Y^{-1}(t) = \begin{pmatrix} X^{-1}(t) & 0 \\ 0 & [X(-t)]^{-1} \end{pmatrix}. \end{gathered} \tag{3.21}$$

Furthermore, since (3.17) has an exponential dichotomy, then there exist a projection P and positive constants α, k such that

$$\begin{gathered}\left\|X(t)PX^{-1}(s)\right\| \leq k\exp(-\alpha(t-s)), \quad \text{for } s \leq t,\\ \left\|X(t)(I-P)X^{-1}(s)\right\| \leq k\exp(-\alpha(s-t)), \quad \text{for } t \leq s.\end{gathered} \tag{3.22}$$

If we put $Q = \left(\begin{smallmatrix} P & 0 \\ 0 & I-P \end{smallmatrix}\right)$, then it is easy to see that Q is a projection, and that

$$\begin{aligned}
\left\|Y(t)QY^{-1}(s)\right\| &= \left\|\begin{pmatrix} X(t) & 0 \\ 0 & X(-t) \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & I-P \end{pmatrix}\begin{pmatrix} X^{-1}(s) & 0 \\ 0 & [X(-s)]^{-1} \end{pmatrix}\right\| \\
&= \left\|\begin{pmatrix} X(t)P & 0 \\ 0 & X(-t)(I-P) \end{pmatrix}\begin{pmatrix} X^{-1}(s) & 0 \\ 0 & [X(-s)]^{-1} \end{pmatrix}\right\| \\
&= \left\|\begin{pmatrix} X(t)PX^{-1}(s) & 0 \\ 0 & X(-t)(I-P)[X(-s)]^{-1} \end{pmatrix}\right\| \\
&\leq \left\|X(t)PX^{-1}(s)\right\| + \left\|X(-t)(I-P)[X(-s)]^{-1}\right\| \\
&\leq k\exp(-\alpha(t-s)) + k\exp(-\alpha(-s+t)) \\
&\leq 2k\exp(-\alpha(t-s)), \quad \text{for } t \geq s, \\
\left\|Y(t)(I-Q)Y^{-1}(s)\right\| &= \left\|\begin{pmatrix} X(t) & 0 \\ 0 & X(-t) \end{pmatrix}\begin{pmatrix} I-P & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} X^{-1}(s) & 0 \\ 0 & [X(-s)]^{-1} \end{pmatrix}\right\| \\
&= \left\|\begin{pmatrix} X(t)(I-P) & 0 \\ 0 & X(-t)P \end{pmatrix}\begin{pmatrix} X^{-1}(s) & 0 \\ 0 & [X(-s)]^{-1} \end{pmatrix}\right\| \\
&= \left\|\begin{pmatrix} X(t)(I-P)X^{-1}(s) & 0 \\ 0 & X(-t)P[X(-s)]^{-1} \end{pmatrix}\right\| \\
&\leq \left\|X(t)(I-P)X^{-1}(s)\right\| + \left\|X(-t)P[X(-s)]^{-1}\right\| \\
&\leq k\exp(-\alpha(s-t)) + k\exp(-\alpha(-t+s)) \\
&\leq 2k\exp(-\alpha(s-t)), \quad \text{for } s \geq t,
\end{aligned} \tag{3.23}$$

which ends the proof. □

Lemma 3.6. *If the system*

$$\frac{dx}{dt} = A(t)x(t) \tag{3.24}$$

has an exponential dichotomy with parameters (P, α, k) *and if* $B(t)$ *is continuous and uniformly bounded in* t *such that*

(H_1) $\delta = \sup_{t\in\mathbb{R}^+}|B(t)| < \alpha/(4k^2)$, *then the system*

$$\frac{dx}{dt} = M(t)x(t) \tag{3.25}$$

has an exponential dichotomy.

Proof. The proof is a direct application of Theorems 2.4 and 3.5. □

Corollary 3.7. *If the system*

$$\frac{dx}{dt} = A(t)x(t) \tag{3.26}$$

has an exponential dichotomy with parameters (P, α, k), *and if* $B(t)$ *is continuous and uniformly bounded in* t, *such that*

(H_2) $\lim_{r\to\infty}(1/2r)\int_{-r}^{r}|B(t)|dt = 0$, *then the system*

$$\frac{dx}{dt} = M(t)x(t) \tag{3.27}$$

has an exponential dichotomy too.

Proof. The proof is a direct consequence of Corollary 3.3 in Ait Dads and Arino [13] and Theorem 3.5. □

Theorem 3.8. *Under the hypothesis* (H_1) *or* (H_2), *if moreover* g *is pseudo almost automorphic, then* (3.13) *has a unique pseudo almost automorphic solution.*

Proof. The proof is a direct application of the two following results. □

Lemma 3.9 (see [7]). *If* g *is pseudo almost automorphic, then* $G(t) = \begin{pmatrix} g(t) \\ -g(-t) \end{pmatrix}$ *is also pseudo almost automorphic.*

Lemma 3.10. *If the system* (2.6) *has an exponential dichotomy and if* f *is pseudo almost automorphic, then the system* $(dx/dt)(t) = A(t)x(t) + f(t)$ *has an unique pseudo almost automorphic solution.*

Proof (The unique bounded solution, when we consider f bounded). A solution $x(t)$ is represented as follows (see [24]):

$$x(t) = \int_{-\infty}^{\infty} G(t,s) f(s) ds, \tag{3.28}$$

where

$$G(t,s) = \begin{cases} X(t) P X^{-1}(s), & \text{for } t \geq s \\ -X(t)(I-P)X^{-1}(s), & \text{for } t \leq s. \end{cases} \tag{3.29}$$

$G(t,s)$ is a piecewise continuous function on the (t,s) plane. If $f(t) = g(t) + \varphi(t)$, where g is almost automorphic and φ is an ergodic perturbation, then

$$x(t) = \int_{-\infty}^{\infty} G(t,s) g(s) ds + \int_{-\infty}^{\infty} G(t,s) \varphi(s) ds. \tag{3.30}$$

Moreover, it is known that $\int_{-\infty}^{\infty} G(t,s)\varphi(s)ds$ is an ergodic perturbation [13]. It remains to be prove that $\int_{-\infty}^{\infty} G(t,s)g(s)ds$ is almost automorphic. For this, we use the following result. □

Proposition 3.11 (see [9]). *Let $A : \mathbb{R} \to M_n(\mathbb{R})$ be continuous function and assume that the equation $dx/dt = A(t)x(t)$ has an exponential dichotomy on $\mathbb{R}$, then for $f \in AA(\mathbb{R}, \mathbb{R}^n)$, the unique bounded solution of $dx/dt = A(t)x(t) + f(t)$ is almost automorphic.*

Corollary 3.12. *If A and B are ω-periodic with the same period, such that the Floquet multipliers of $A(t)$ lie of the unit circle and B verifies the condition (H_1) or (H_2), then the system* (3.27) *has an exponential dichotomy. Moreover, if g is ω periodic, then* (3.13) *has an unique ω periodic solution.*

3.3. *Autonomous Case*

Definition 3.13. For $A \in M_n(\mathbb{R})$, the spetrum of A denoted by

$$\text{sp}(A) = \{\lambda \in \mathbb{C}, \text{ such that there exists } x \in \mathbb{M}_{n,1}(\mathbb{C}), x \neq 0 \text{ with } Ax = \lambda x\}. \tag{3.31}$$

Proposition 3.14. *In the autonomous case, If $sp(A-B)(A+B) \cap \mathbb{R}^- = \emptyset$, and g is pseudo almost automorphic, then* (3.13) *has an unique pseudo almost automorphic solution.*

Remark 3.15. If the matrices A and B are constant, the system (3.15) has an exponential dichotomy if and only if the eigenvalues of the matrix M have nonzero real part. One has

$$M^2 = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} = \begin{pmatrix} C & D \\ D & C \end{pmatrix}, \tag{3.32}$$

with $C = A^2 - B^2$ and $D = AB - BA$. Let $P = \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}$. One has $P^{-1} = (1/2)\begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}$ and

$$M^2 = P\begin{pmatrix} C+D & 0 \\ 0 & C-D \end{pmatrix}P^{-1}. \tag{3.33}$$

On the other hand,

$$sp(M) \cap i\mathbb{R} = \emptyset \Longleftrightarrow sp\left(M^2\right) \cap \mathbb{R}^- = \emptyset \Longleftrightarrow sp(C+D) \cap \mathbb{R}^- = \emptyset, \tag{3.34}$$

and $sp(C-D) \cap \mathbb{R}^- = \emptyset$, so, we have

$$\begin{gathered} C + D = (A-B)(A+B), \qquad C - D = (A+B)(A-B), \\ sp(A-B)(A+B) = sp(A+B)(A-B), \end{gathered} \tag{3.35}$$

then

$$sp(M) \cap i\mathbb{R} = \emptyset \Longleftrightarrow sp((A-B)(A+B)) \cap \mathbb{R}^- = \emptyset. \tag{3.36}$$

In the sequel, we suppose that

$$sp((A-B)(A+B)) \cap \mathbb{R}^- = \emptyset. \tag{3.37}$$

Then, (3.40) has an unique bounded solution denoted by $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

One has

$$\begin{gathered} x'(t) = Ax(t) + By(t) + g(t) \\ y'(t) = -Bx(t) - Ay(t) - g(-t). \end{gathered} \tag{3.38}$$

Putting

$$Z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \quad \text{with} \begin{cases} z_1(t) = y(-t) \\ z_2(t) = x(-t), \end{cases} \tag{3.39}$$

one has

$$\begin{gathered} z_1'(t) = Bx(-t) + Ay(-t) + g(t) = Az_1(t) + Bz_2(t) + g(t), \\ z_2'(t) = -Ax(-t) - By(-t) - g(-t) = -Bz_1(t) - Az_2(t) - g(-t). \end{gathered} \tag{3.40}$$

Remark 3.16. $Z(t)$ is a bounded solution of (3.40), by the uniqueness of bounded solution, we have that $Z(t) = X(t)$ in the sequel $y(t) = x(-t)$, consequently, x is a bounded solution of (3.13). Finally (3.38) has a unique bounded solution, and the application $x(t) \to \begin{pmatrix} x(t) \\ x(-t) \end{pmatrix}$ is a bijective from the set of bounded solutions of (3.13) to the set of bounded solutions of (3.40).

3.4. Goal Result of Nonlinear Case

Consider the following equation involving reflection of the argument:

$$x'(t) = A(t)x(t) + B(t)x(-t) + f(t, x(t), x(-t)). \tag{3.41}$$

If we put $y(t) = x(-t)$, then (3.41) is changed into system

$$X'(t) = M(t)X(t) + F(t, X(t)), \tag{3.42}$$

where $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, and $M(t)$ is defined as in (3.16) and

$$F : \left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right) \longrightarrow \begin{pmatrix} f(t, x, y) \\ -f(-t, y, x) \end{pmatrix}. \tag{3.43}$$

We assume that there exists $\ell_1 \in L^p(\mathbb{R}) \cap C(\mathbb{R})$, with $1 < p \leq +\infty$ such that

$$\|f(t, x, y) - f(t, x', y')\| \leq \ell_1(t)\left[|x - x'| + |y - y'|\right], \ \forall t \in \mathbb{R}, \text{ as } x, x', y, y' \in \mathbb{R}^n. \tag{3.44}$$

In what follows, let us put $\ell(t) = (1/2)[\ell_1(t) + \ell_1(-t)]$.

Remark 3.17. If f satisfies (3.44), then F satisfies

$$\begin{aligned} |F(t, x, y) - F(t, x', y')| &= \left|\begin{pmatrix} f(t, x, y) - f(t, x', y') \\ -f(-t, y, x) + f(-t, y', x') \end{pmatrix}\right| \\ &\leq 2\ell(t)\left[|x - x'| + |y - y'|\right], \quad \forall t \in \mathbb{R}, \text{ as } x, x', y, y' \in \mathbb{R}^n. \end{aligned} \tag{3.45}$$

Theorem 3.18. *Assume that* $f \in PAA(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n)$ *and satisfies the Lipschitz condition* (3.44)*, if the system*

$$\frac{dx}{dt} = A(t)x(t) \tag{3.46}$$

has an exponential dichotomy with parameters (P, α, k) *and if* $B(t)$ *is continuous and uniformly bounded in* t*, such that* (H_1) *or* (H_2) *holds. If*

$$\|\ell\|_p \leq \frac{(\alpha q)^{1/q}}{2k\left(1 + [e - 1]^{1/p}\right)}, \tag{3.47}$$

then (3.41) *has an unique pseudo almost automorphic solution.*

For the proof, we need the following preliminary result.

Lemma 3.19. *Let $c > 0$, and put*

$$\mu(t) = 2^p \int_{-\infty}^{t} \ell^p(s)ds, \\ \|\varphi\|_c = \sup_{t\in\mathbb{R}} \exp(-c\mu(t))|\varphi(t)|. \tag{3.48}$$

Then $\|\cdot\|_c$ is an equivalent norm to the uniform convergence norm.

Proof. In fact,

$$\|\varphi\|_\infty \exp\left(-2^p c \int_{\mathbb{R}} \ell^p(s)ds\right) \le \|\varphi\|_c \le \|\varphi\|_\infty. \tag{3.49}$$

□

Proof of Theorem 3.18. $PAA(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n)$ is a Banach space with the supremum norm. If $f(t,x,y) \in PAA(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n)$, then for any $\varphi \in PAA(\mathbb{R}, \mathbb{R}^n)$, $f(t,\varphi(t),\varphi(-t))$ is also pseudo almost automorphic. For $\varphi \in PAA(\mathbb{R}, \mathbb{R}^n)$, the following differential equation:

$$x'(t) = A(t)x(t) + B(t)x(-t) + f(t,\varphi(t),\varphi(-t)), \quad B \neq 0,\ t \in \mathbb{R} \tag{3.50}$$

has a unique pseudo almost automorphic solution, denoted by $T\varphi(t)$, then we define a mapping as

$$K : PAA\left(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n\right) \longrightarrow PAA\left(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n\right), \\ (K\Phi)(t) = \int_{-\infty}^{\infty} G(t,s)F(s,\Phi(s))ds. \tag{3.51}$$

Thus, $K\Phi \in PAA(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n)$, so that K is well defined.

Now for $\Phi_1, \Phi_2 \in PAA(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n)$, we have

$$\begin{aligned}
\|(K\Phi_1 - K\Phi_2)(t)\| \le{}& \int_{-\infty}^{t} k\exp(-\alpha(t-s))2\ell(s)\|\Phi_1(s) - \Phi_2(s)\|ds \\
&+ \int_{t}^{\infty} k\exp(-\alpha(s-t))2\ell(s)\|\Phi_1(s) - \Phi_2(s)\|ds \\
\le{}& k\|\Phi_1 - \Phi_2\|_c \left[\int_{-\infty}^{t} \exp(-\alpha(t-s))2\ell(s)\exp(c\mu(s))ds \right.\\
&\left. + \int_{t}^{\infty} \exp(-\alpha(s-t))2\ell(s)\exp(c\mu(s))ds\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq k\|\Phi_1-\Phi_2\|_c\left(\int_{-\infty}^{t}\exp(pc\mu(s))2^p\ell^p(s)ds\right)^{1/p}\left(\int_{-\infty}^{t}\exp(-\alpha q(t-s))ds\right)^{1/q}\\
&\quad+k\|\Phi_1-\Phi_2\|_c\left(\int_{t}^{\infty}\exp(pc\mu(s))2^p\ell^p(s)ds\right)^{1/p}\left(\int_{-\infty}^{t}\exp(-\alpha q(s-t))ds\right)^{1/q}\\
&\leq\frac{k}{(\alpha q)^{1/q}}\|\Phi_1-\Phi_2\|_c\left[\left(\int_{-\infty}^{t}\exp(pc\mu(s))\mu'(s)ds\right)^{1/p}\right.\\
&\qquad\left.+\left(\int_{t}^{\infty}\exp(pc\mu(s))\mu'(s)ds\right)^{1/p}\right].
\end{aligned}
\tag{3.52}
$$

Let us put

$$
I(t)=\left(\int_{-\infty}^{t}\exp(pc\mu(s))\mu'(s)ds\right)^{1/p}+\left(\int_{t}^{\infty}\exp(pc\mu(s))\mu'(s)ds\right)^{1/p},
\tag{3.53}
$$

then one has

$$
\|K\Phi_1(t)-K\Phi_2(t)\|\leq\frac{k}{(\alpha q)^{1/q}}\|\Phi_1-\Phi_2\|_c I(t).
\tag{3.54}
$$

On the other hand,

$$
\begin{aligned}
I(t)&=\left(\int_{-\infty}^{t}\exp(pc\mu(s))\mu'(s)ds\right)^{1/p}+\left(\int_{t}^{+\infty}\exp(pc\mu(s))\mu'(s)ds\right)^{1/p}\\
&=\left(\frac{1}{pc}\right)^{1/p}\left(\exp(c\mu(t))+\left[\exp\left(pc2^p\|\ell\|_p^p\right)-\exp(pc\mu(t))\right]^{1/p}\right),
\end{aligned}
\tag{3.55}
$$

which leads to

$$
\begin{aligned}
I(t)\exp(-c\mu(t))&=\left(\frac{1}{pc}\right)^{1/p}\left(1+\left[\exp\left(pc\left(2^p\|\ell\|_p^p-\mu(t)\right)\right)-1\right]^{1/p}\right)\\
&=\left(\frac{1}{pc}\right)^{1/p}\left(1+\left[\exp\left(pc2^p\int_{t}^{+\infty}\ell^p(s)ds\right)-1\right]^{1/p}\right).
\end{aligned}
\tag{3.56}
$$

Hence,

$$
\sup_{t\in\mathbb{R}}I(t)\exp(-c\mu(t))=\left(\frac{1}{pc}\right)^{1/p}\left(1+\left[\exp\left(pc2^p\|\ell\|_p^p\right)-1\right]^{1/p}\right),
\tag{3.57}
$$

therefore,

$$\|K\Phi_1 - K\Phi_2\|_c \leq \frac{k}{(\alpha q)^{1/q}} \left(\frac{1}{pc}\right)^{1/p} \left(1 + \left[\exp\left(pc2^p\|\ell\|_p^p - 1\right)\right]^{1/p}\right) \|\Phi_1 - \Phi_2\|_c. \tag{3.58}$$

If we choose c such that

$$pc2^p\|\ell\|_p^p = 1, \tag{3.59}$$

then,

$$\begin{aligned} &\frac{k}{(\alpha q)^{1/q}} \left(\frac{1}{pc}\right)^{1/p} \left(1 + \left[\exp\left(pc2^p\|\ell\|_p^p\right) - 1\right]^{1/p}\right) < 1 \\ &\Longleftrightarrow \frac{k}{(\alpha q)^{1/q}} 2\|\ell\|_p \left(1 + [e-1]^{1/p}\right) < 1 \Longleftrightarrow \|\ell\|_p < \frac{(\alpha q)^{1/q}}{2k\left(1 + [e-1]^{1/p}\right)}. \end{aligned} \tag{3.60}$$

Then, K will be a contraction, which proves that K is continuous. So by the Banach fixed point theorem, there exists a unique $u \in PAA(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n)$, such that $Ku = u$, that is,

$$u(t) = \int_{-\infty}^{\infty} G(t,s)F(s,u(s))ds. \tag{3.61}$$

The proof is complet. □

Proposition 3.20. *Assume that $f \in PAA(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}^n)$ and satisfies the Lipschitz condition as*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|). \tag{3.62}$$

If the system

$$\frac{dx}{dt} = A(t)x(t) \tag{3.63}$$

has an exponential dichotomy with parameters (P, α, k) and if $B(t)$ is continuous and uniformly bounded in t, such that (H_1) or (H_2) holds, then (3.41) *has an unique pseudo almost automorphic solution provided that $(4k/\alpha)L < 1$.*

Corollary 3.21. *In the scalar case, if $A = a$, $B = b$ and $f \in PAA(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ and satisfies the lipschitz condition as*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|), \tag{3.64}$$

for any $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, where $L < \alpha^2/(|\alpha - a| + |\alpha + a| + 2|b|)$, $\alpha^2 = a^2 - b^2 > 0$, and $\alpha > 0$, then (3.41) *has an unique pseudo almost automorphic solution.*

Proof. Thanks to Lemmas 3.2, 3.3, and 3.6, and from Theorem 3.1, for any $\varphi \in PAA(\mathbb{R},\mathbb{R})$, we see that the following equation:

$$x'(t) = ax(t) + bx(-t) + f\left(t, \varphi(t), \varphi(-t)\right), \tag{3.65}$$

has a unique pseudo almost automorphic solution, which we denote by $(K_0\varphi)(t)$. Then, if we consider the operator $K_0 : PAA(\mathbb{R},\mathbb{R}) \rightarrow PAA(\mathbb{R},\mathbb{R})$. Now we can show that K_0 is a contraction. Indeed, for φ and $\phi \in PAA(\mathbb{R},\mathbb{R})$, the following equation:

$$x'(t) = ax(t) + bx(-t) + f\left(t, \varphi(t), \varphi(-t)\right) - f\left(t, \phi(t), \phi(-t)\right), \quad b \neq 0,\ t \in \mathbb{R}, \tag{3.66}$$

has a unique pseudo almost automorphic solution $(K_0\varphi - K_0\phi)(t)$. Moreover,

$$\begin{aligned}
(K_0\varphi - K_0\phi)(t) = &-\frac{1}{2\alpha}\int_t^{\infty} \exp -\alpha(s-t)(\alpha + a)\left[f\left(t, \varphi(s), \varphi(-s)\right) - f\left(s, \phi(s), \phi(-s)\right)\right] ds \\
&+ \frac{b}{2\alpha}\int_t^{\infty} \exp -\alpha(s-t)\left[f\left(-s, \varphi(-s), \varphi(s)\right) - f\left(-s, \phi(-s), \phi(s)\right)\right] ds \\
&+ \frac{1}{2\alpha}\int_{-\infty}^{t} \exp \alpha(s-t)(\alpha - a)\left[f\left(t, \varphi(s), \varphi(-s)\right) - f\left(s, \phi(s), \phi(-s)\right)\right] ds \\
&- \frac{b}{2\alpha}\int_{-\infty}^{t} \exp \alpha(s-t)\left[f\left(t, \varphi(-s), \varphi(s)\right) - f\left(-s, \phi(-s), \phi(s)\right)\right] ds.
\end{aligned} \tag{3.67}$$

So

$$\left\| K_0\varphi - K_0\phi \right\| \leq \frac{|\alpha + a| + |\alpha - a| + 2|b|}{2\alpha^2} 2L \left\| \varphi - \phi \right\|. \tag{3.68}$$

Since

$$\frac{|\alpha + a| + |\alpha - a| + 2|b|}{2\alpha^2} 2L < 1, \tag{3.69}$$

so K_0 is a contraction mapping, and so K_0 has a unique fixed point in $PAA(\mathbb{R},\mathbb{R})$, which proves that (3.41) has a unique pseudo almost automorphic solution. □

3.5. Examples

3.5.1. Scalar Case

Consider the following equation:

$$x'(t) = -x(t) + \frac{1}{1+t^2} x(-t) + \sin \frac{1}{2 - \sin t - \sin \pi t} + \frac{1}{\sqrt{1+t^2}}. \tag{3.70}$$

In this situation, $x'(t) = -x(t)$ admits exponential dichotomy and the function $t \to 1/(1+t^2)$ satisfies that $\lim_{t\to\infty} 1/(1+t^2) = 0$, so the condition (ii) in Theorem 3.18 is satisfied, and the function $t \to \sin 1/(2 - \sin t - \sin \pi t) + 1/(\sqrt{1+t^2})$ is a pseudo almost automorphic function, so all the hypotheses of Theorem 3.18 hold, and so (3.70) has an unique pseudo almost automorphic solution.

3.6. Vectorial Case

Let us consider the following example of Markus and Yamabe:

$$\frac{dx}{dt} = A(t)x(t) + B(t)x(-t) + G(t), \tag{3.71}$$

where

$$A(t) = \begin{pmatrix} -1 + \dfrac{3}{2}\cos^2 t & -1 + \dfrac{3}{2}\cos t \sin t \\ -1 - \dfrac{3}{2}\cos t \sin t & -1 + \dfrac{3}{2}\sin^2 t \end{pmatrix}, \tag{3.72}$$

$$B(t) = \begin{pmatrix} \dfrac{1}{1+t^2} & 0 \\ 0 & \dfrac{1}{1+t^2} \end{pmatrix}, \qquad G(t) = \begin{pmatrix} \sin \dfrac{1}{2 - \sin t - \sin \pi t} + \dfrac{1}{1+t^2} \\ \sin \pi t + \dfrac{1}{\sqrt{1+t^2}} \end{pmatrix}, \tag{3.73}$$

The matrix $A(t)$ is π-periodic and the eigenvalues $\lambda_1(t)$, $\lambda_2(t)$ of $A(t)$ are

$$\lambda_1(t) = \frac{-1 + i\sqrt{7}}{4}, \qquad \lambda_2(t) = \frac{-1 - i\sqrt{7}}{4}, \tag{3.74}$$

and, in particular, the real parts of the eigenvalues are negative. If $\rho_j = \exp(\lambda_j \omega)$, $j = 1, 2 \ldots, n$ are the characteristic multipliers of $A(t)$, where ω is the minimal period and λ_j the eigenvalues of $A(t)$, then, we have

$$\prod_{j=1}^{n} \rho_j = \exp \int_0^{\omega} \operatorname{trace} A(s)ds, \qquad \sum_{j=1}^{n} \lambda_j = \frac{1}{\omega} \int_0^{\omega} \operatorname{trace} A(s)ds \left(\operatorname{mod} \frac{2\pi i}{\omega} \right). \tag{3.75}$$

One of the characteristic multipliers is $\exp(\pi/2)$. The other multiplier is $\exp(-\pi)$, since the product of the multipliers is $\exp(-\pi/2)$. So, the system $dx/dt = A(t)x(t)$ has an exponential dichotomy. The matrix $B(t)$ is an ergodic function. G is pseudo almost automorphic, hence, (3.71) has an unique pseudo almost automorphic solution.

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