

Research Article

Odd-Ary Approximating Subdivision Schemes and RS Strategy for Irregular Dense Initial Data

Muhammad Aslam and W. P. Abeyasinghe

Department of Mathematics, Lock Haven University, Lock Haven, PA 17745, USA

Correspondence should be addressed to Muhammad Aslam, maslam@lhup.edu

Received 29 January 2012; Accepted 22 April 2012

Academic Editors: A. Bastos and S. Zhang

Copyright © 2012 M. Aslam and W. P. Abeyasinghe. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the implementation of approximating subdivision schemes on noisy or irregular initial control data. Presence of noise in the initial data generates oscillatory curves by subdivision schemes. To reduce or completely eliminate these oscillations, we combine subdivision schemes with other noise removal schemes such as variational regularization method. This setup will allow us to produce the limit curve with less oscillations and still stay as close as possible to the initial data points.

1. Introduction

Subdivision has been a very important tool for computer-aided geometric design, image reconstruction, and computer animation. Subdivision schemes can be divided into two categories, interpolating subdivision schemes and approximating subdivision schemes. In interpolating subdivision schemes, more data points are added between the initial or existing data points at each level of subdivision. However, in approximating subdivision schemes existing points are replaced by their approximations and new points are inserted at each level of refinement. As a result we see that approximating schemes generate smoother curve as compared to interpolating subdivision schemes.

A general form [1] of univariate n -ary subdivision scheme S which maps set of data points $f^k = \{f_i^k\}_{i \in \mathbb{Z}}$ into the next refinement level of data points $f^{k+1} = \{f_i^{k+1}\}_{i \in \mathbb{Z}}$ is defined as

$$f_{ni+s}^{k+1} = \sum_{j \in \mathbb{Z}} \beta_{nj+s} f_{i-j}^k, \quad s = 0, 1, 2, \dots, n-1, \quad (1.1)$$

where the set of coefficients $\{\beta_i \mid i \in Z\}$ is called the mask and n represents the arity of the subdivision scheme. The previous equation can also be expressed as $f^{k+1} = S f^k$. A necessary condition for the uniform convergence [2] of n -ary subdivision scheme is

$$\sum_{j \in Z} \beta_{nj+s} = 1, \quad s = 0, 1, 2, \dots, n-1. \quad (1.2)$$

During the last several years, many subdivision schemes have been developed, and analyses regarding their smoothness and error bounds have been performed [3–7]. It is noted that if initial control data is dense and corrupted with noise then subdivision schemes will generate oscillatory curves and oscillations in the curve depend on the noise level present in the initial data points. There is not much work done on this problem, and we try to address this issue in this paper. In Section 2, we revisit some of the odd point odd-ary approximating subdivision schemes. In Section 3, we measure the performance of subdivision schemes on noisy initial data. Overview of total variational regularization is given in Section 4, and RS strategy and its performance are discussed in Section 5.

2. Odd Point Odd-Ary Subdivision Schemes

We have recently introduced a general procedure of constructing a $(2n-1)$ -point $(2b+1)$ -ary parametric approximating subdivision schemes for any integers $n \geq 2, b \geq 1$. These schemes have order of smoothness up to $C^{2n-2+[(n-1)/b]}$, where $[(n-1)/b]$ is the largest integer less than $(n-1)/b$. Brief description of construction procedure is given below.

Consider a family of polynomials

$$P_{2n-1,t}^{2b+1}(z) = \left(1 + z + z^2 + \dots + z^{2b}\right)^t \sum_{i=0}^{(2b+1)(2n-1)-2bt-1} u_i z^i, \quad (2.1)$$

where $t = 1, 2, 3, \dots, (2n-1) + [(n-1)/b]$ and $[(n-1)/b]$ is the largest integer less than or equal to $(n-1)/b$

$$u_{(2b+1)n-b(t+1)-1-i} = u_{(2b+1)n-b(t+1)-1+i}, \quad i = 1, 2, \dots, (2b+1)n-b(t+1)-1, \quad (2.2)$$

$$u_{(2b+1)n-b(t+1)-1} = \frac{1}{(2b+1)^{t-1} - 2} \sum_{i=0}^{(2b+1)n-b(t+1)-2} u_i.$$

From (2.1), we get $(2n-1) + [(n-1)/b]$ different polynomials $P_{2n-1,t}^{2b+1}(z)$ one for each t . Let $\alpha_{2n-1,t}^{2b+1}$ denote the sets of coefficients of polynomials $P_{2n-1,t}^{2b+1}(z)$, then these set of coefficients will represent the masks of $(2n-1)$ -point $(2b+1)$ -ary approximating schemes.

For $b = 1$, $n = 2$, and $t = 3, 4$ in (2.1), we have the following masks for 3-point ternary subdivision schemes:

$$\begin{aligned}\alpha_{3,3}^3 &= \left\{ u_0, u_0 + \frac{1}{9}, u_0 + \frac{1}{3}, \frac{2}{3} - 2u_0, \frac{7}{9} - 2u_0, \frac{2}{3} - 2u_0, u_0 + \frac{1}{3}, u_0 + \frac{1}{9}, u_0 \right\}, \\ \alpha_{3,4}^3 &= \frac{1}{27} \{1, 4, 10, 16, 19, 16, 10, 4, 1\}.\end{aligned}\quad (2.3)$$

Similarly, for $b = 1$, $n = 3$, and $t = 6, 7$ in (2.1), we have the following masks for 5-point ternary subdivision schemes:

$$\begin{aligned}\alpha_{5,6}^3 &= \left\{ u_0, 4u_0 + \frac{1}{3^5}, 10u_0 + \frac{6}{3^5}, 14u_0 + \frac{21}{3^5}, 11u_0 + \frac{50}{3^5}, \frac{90}{3^5} - 4u_0, \frac{126}{3^5} - 21u_0, \frac{141}{3^5} \right. \\ &\quad \left. - 30u_0, \frac{126}{3^5} - 21u_0, \frac{90}{3^5} - 4u_0, 11u_0 + \frac{50}{3^5}, 14u_0 + \frac{21}{3^5}, 10u_0 + \frac{6}{3^5}, 4u_0 + \frac{1}{3^5}, u_0 \right\}, \\ \alpha_{5,7}^3 &= \frac{1}{3^6} \{1, 7, 28, 77, 161, 266, 357, 393, 357, 266, 161, 77, 28, 7, 1\}.\end{aligned}\quad (2.4)$$

Masks for 3-point and 5-point quinary schemes can be obtained by a similar approach. Given below are two of these masks one for 3-point and one for 5-point quinary scheme:

$$\begin{aligned}\alpha_{3,3}^5 &= \left\{ u_0, u_0 + \frac{1}{25}, u_0 + \frac{3}{25}, u_0 + \frac{6}{25}, u_0 + \frac{10}{25}, \frac{15}{25} - 2u_0, \frac{18}{25} - 2u_0, \frac{19}{25} \right. \\ &\quad \left. - 2u_0, \frac{18}{25} - 2u_0, \frac{15}{25} - 2u_0, u_0 + \frac{10}{25}, u_0 + \frac{6}{25}, u_0 + \frac{3}{25}, u_0 + \frac{1}{25}, u_0 \right\}, \\ \alpha_{5,6}^5 &= \frac{1}{5^5} \{1, 6, 21, 56, 126, 246, 426, 666, 951, 1246, 1506, \\ &\quad 1686, 1751, 1686, 1506, 1246, 951, 666, 426, 246, 126, 56, 21, 6, 1\}.\end{aligned}\quad (2.5)$$

It is to be noted that all the masks given above satisfy the necessary condition for the uniform convergence (1.2) of the subdivision schemes.

3. Performance over Irregular Initial Polygon

Subdivision schemes have been successful in finding smooth curves and surfaces from the regular or noise free data points. However, they do not have the same performance if the initial data or control points are dense and corrupted with noise. Subdivision schemes on irregular or noisy control points will generate oscillatory curves. Figure 1 explains this phenomenon.

In order to measure the performance of subdivision schemes on irregular initial data points, we are defining mathematical error function as follows. Let f represent the initial

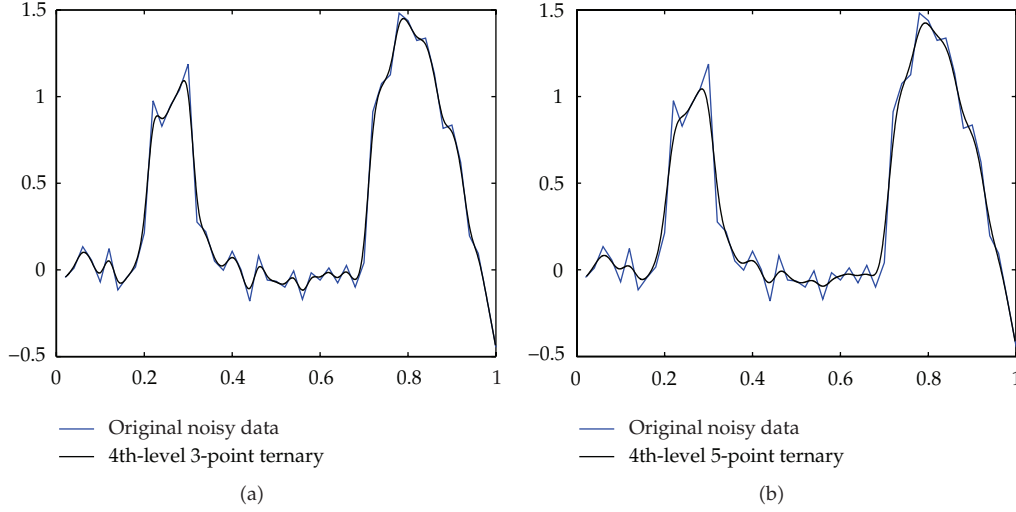


Figure 1: (a) Initial polygon (noisy or irregular data points) and curves generated by 3-point ternary scheme (4th level) in (2.3) with mask $\alpha_{3,4}^3$ and (b) same initial polygon and curve generated by 5-point ternary scheme (4th level) in (2.4) with mask $\alpha_{5,7}^3$. Oscillations at 4th level of subdivision schemes are significant.

data point and g represent the data obtained after applying subdivision transform S , that is, $g = Sf$. We define an error function e_λ as

$$e_\lambda = (1 - \lambda) \|f - g\|_2^2 + \lambda TV(g) \quad (3.1)$$

for some parameter $0 < \lambda < 1$ and $TV(g) = \sum_{i=1}^{n-1} |g(i+1) - g(i)|$, for $g \in R^n$.

Second term in (3.1) represents a measure of smoothness and the first term is a least square error term. Since $g = Sf$, both f and g are vectors of different lengths. For example, if S is ternary subdivision scheme and $f \in R^n$ then $g \in R^{3n}$ and we have $\|f - g\|_2^2 = \sum_{i=1}^n |f(i) - g(3i-1)|^2$ and similarly, if S is quinary then $\|f - g\|_2^2 = \sum_{i=1}^n |f(i) - g(5i-2)|^2$. Least square error term for interpolating subdivision schemes is zero because of the retention of initial data in the new data.

In practice, subdivision schemes with small number of points and smaller arity are preferred, therefore, we limit our analysis to 3-point and 5-point ternary and quinary subdivision schemes. We present comparison of Ternary and Quinary approximating subdivision schemes on their performance on the initial noisy control points. With $\lambda = 1/2$ in (3.1), we have computed error function for ternary and quinary schemes at different subdivision levels to the irregular initial data shown in Figure 1. Our results are shown in Figure 2. It is evident that 5-point schemes have better performance than 3-point subdivision schemes on noisy initial data.

4. Regularization Method

In this section, we revisit variational regularization method which is widely used for noise removal in image processing and other applications. Suppose we are given a noisy signal

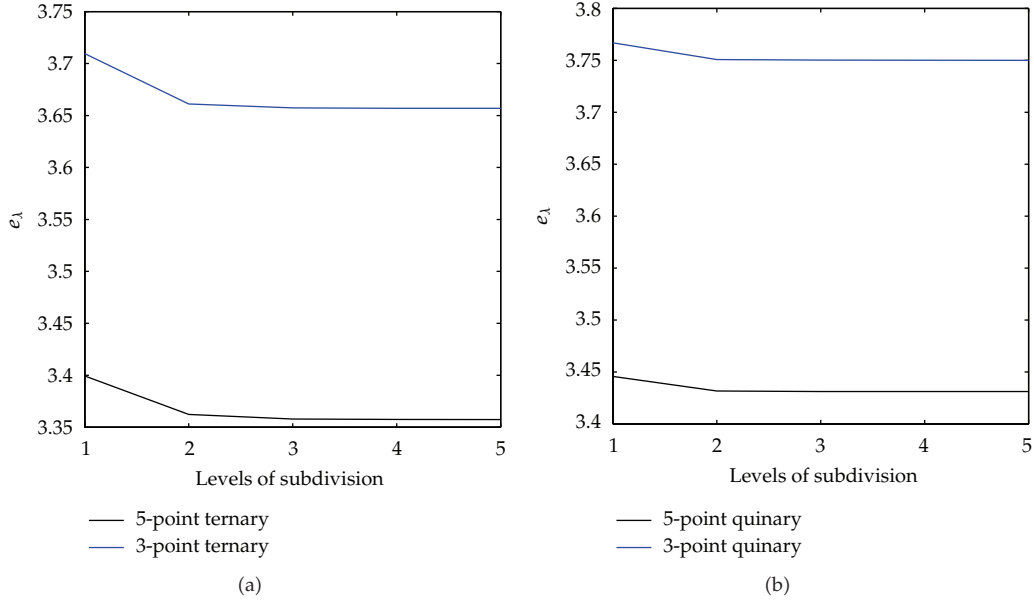


Figure 2: Comparison: graphs of error function e_1 are shown against subdivision levels for the initial polygon (noisy initial data) shown in Figure 1. We used $\lambda = 1/2$, masks for ternary schemes $\alpha_{3,4}^3$ and $\alpha_{3,4}^3$ in (2.3) and (2.4) and masks for quinary schemes $\alpha_{3,3}^5$ (with $u_0 = 1/50$) and $\alpha_{5,6}^5$ in (2.5).

f and we are interested in finding its noise-free approximation. The most successful and commonly used method to solve this problem is total variation regularization approach [8–10] and is given by

$$g_\theta = \arg \min_{g \in R^n} \left\{ \frac{1}{2} \|f - g\|_2^2 + \theta TV(g) \right\} \quad (4.1)$$

for some parameter $\theta > 0$, where g_θ is the approximation of f and subject to the penalty function $TV(g)$. Here $TV(g)$ approximates the total variation of the function g and it is defined in the previous section.

First term in (4.1) describes the closeness of regularized solutions with f and the second term penalizes those solutions which have high oscillations. For our computations, we used Newton's method to find g_θ and L -curve method to select parameter θ . For more details about implementation and parameter selection processes, we may refer to [9]. This method is extensively explored and implemented for noise removal, image restoration, and other applications [8–10].

Figure 3(a) shows a noisy signal and its approximation by regularization method with $\theta = .1$. Noise in the approximated signal by regularization method is almost gone, but resulting curve is not smooth at certain locations where it appears to have big jumps. On the other hand curves obtained by subdivision schemes as shown in Figure 1 are smoother but effects of noise can be seen in the form of oscillations.

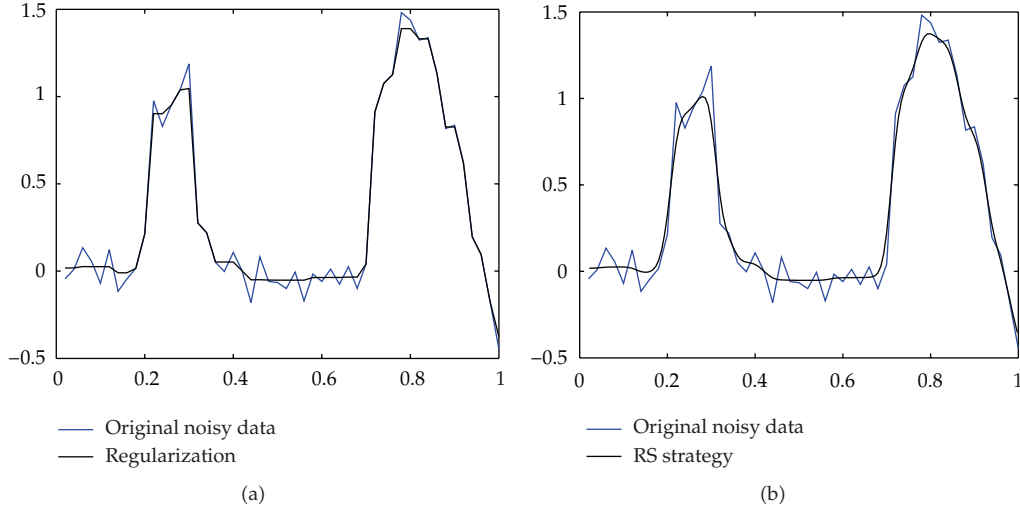


Figure 3: Comparison: (a) initial polygon (noisy) and approximation curve by regularization method and (b) same initial polygon and approximation curve by RS strategy, that is, regularization combined with subdivision scheme (5-point ternary at 1st level with mask $\alpha_{3,4}^3$).

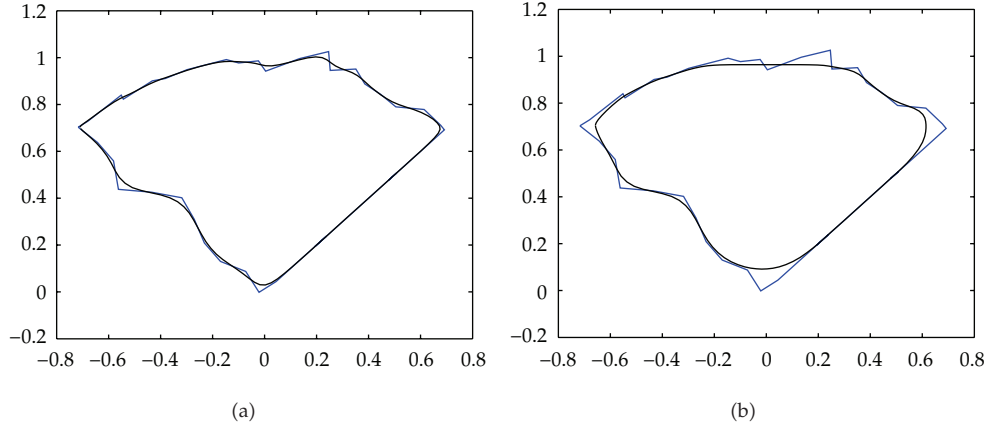


Figure 4: Comparison: (a) Initial irregular polygon and approximation curve generated by 5-point ternary subdivision scheme at 1st level; (b) Initial irregular polygon and approximation curve generated by RS strategy (with 5-point ternary subdivision for one level).

5. RS Strategy and Numerical Results

We have noted that subdivision schemes produce oscillatory curves for the noisy initial data and regularization approach lacks smoothness at certain points. Therefore, we have combined these two approaches into the following two-step strategy and it is called regularization + subdivision or RS strategy.

Let f be the given initial noisy polygon.

Regularization Step: Find approximation g_θ of f as given in (4.1);

Subdivision Step: Apply subdivision transform S to g_θ , that is find Sg_θ .

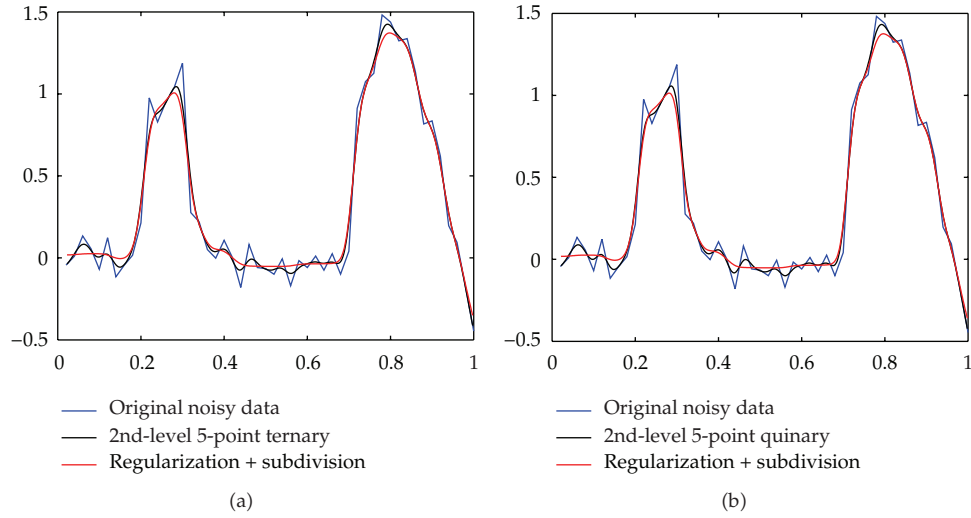


Figure 5: Comparison: (a) Initial irregular polygon and two approximation curves (one generated with 5-point ternary at 2nd level and second approximation curve generated by RS strategy); (b) Initial irregular polygon and two approximation curves (one generated with 5-point quinary at 2nd level and second approximation curve generated by RS strategy).

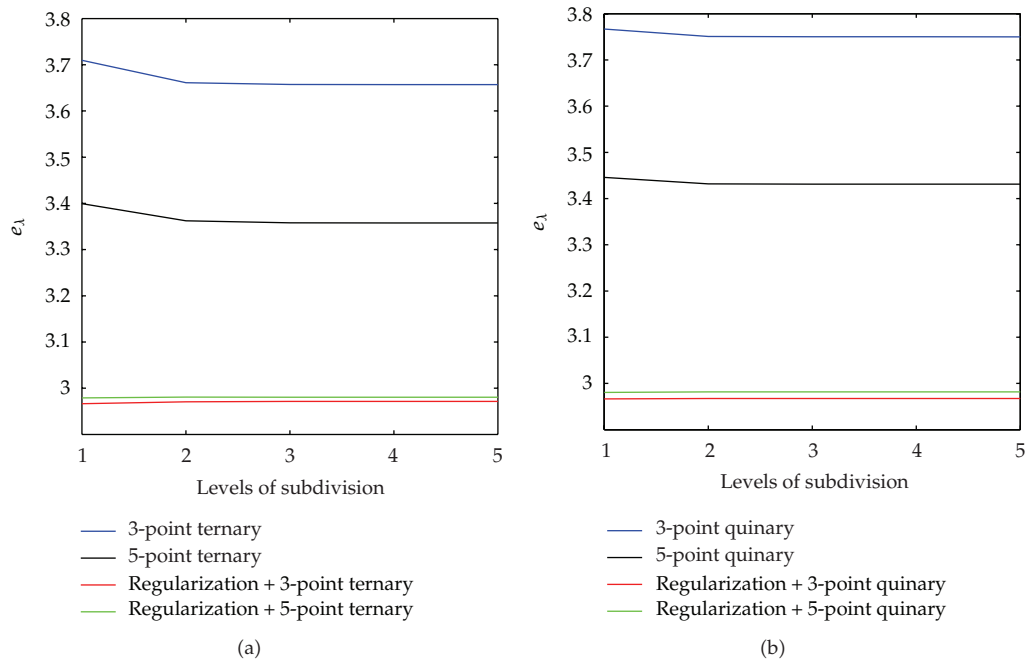


Figure 6: Comparison: (a) graphs of error function e_λ (with $\lambda = 1/2$) are given for 3-point, 5-point ternary schemes, and RS strategy (for both 3-point and 5-point ternary subdivision schemes); (b) graphs of error function e_λ (with $\lambda = 1/2$) are given for 3-point, 5-point quinary schemes, and RS strategy (for both 3-point and 5-point quinary schemes).

We choose two examples to explain the results by RS strategy: one example with open irregular initial control polygon shown in Figure 3(b) and second example of closed irregular initial control polygon shown in Figure 4(b). In our implementation, we used $\theta = .1$ in the regularization step and 5-point ternary subdivision schemes at 1st level in subdivision step. Improvement by RS strategy can be easily seen by comparing Figures 4(a) and 4(b) and similarly Figures 1(b) and 3(b). Subdivision step of RS strategy is repeated for the curve obtained in Figure 3(a), with 5-point ternary and 5-point quinary subdivision schemes. Results are shown in Figure 5.

We have done performance analysis of RS strategy by using the error function defined by (3.1) and compared it with ternary and quinary subdivision schemes. Results obtained at different levels of subdivision (for the initial noisy data shown in Figure 1) are shown in Figure 6. Error function for RS strategy gives the same value for all levels of subdivision. It means that for RS strategy only one level of subdivision is enough and its performance is much better than subdivision schemes.

6. Conclusion

In this paper, we presented a criterion for performance analysis of subdivision schemes over noisy initial control polygon and showed that 5-point ternary and quinary schemes performed better than 3-point ternary and quinary subdivision schemes. Further we defined RS strategy by combining total variational regularization with subdivision schemes. Our results showed that RS strategy outperformed subdivision schemes for noisy and dense initial data points.

References

- [1] G. Mustafa and N. A. Rehman, "The mask of $(2b+4)$ -point n -ary subdivision scheme," *Computing. Archives for Scientific Computing*, vol. 90, no. 1-2, pp. 1–14, 2010.
- [2] N. Aspert, *Non-linear subdivision of univariate signals and discrete surfaces*, EPFL thesis, 2003.
- [3] N. Dyn, M. S. Floater, and K. Hormann, "A C^2 four-point subdivision scheme with fourth order accuracy and its extensions," in *Mathematical Methods for Curves and Surfaces*, M. Daehlen, K. Morken, and L. L. Schumaker, Eds., pp. 145–156, Nashboro Press, Tromso, Norway, 2005.
- [4] M. F. Hassan and N. A. Dodgson, A. Cohen, J. L. Marrien, and L. L. Schumaker, "Ternary and three-point univariate subdivision schemes," in *Curve and Surface Fitting (Saint-Malo, 2002)*, pp. 199–208, Nashboro Press, Brentwood, Tenn, USA, 2003.
- [5] J.-A. Lian, "On a -ary subdivision for curve design. III. $2m$ -point and $(2m+1)$ -point interpolatory schemes," *Applications and Applied Mathematics*, vol. 4, no. 2, pp. 434–444, 2009.
- [6] G. Mustafa and F. Khan, "A new 4-point C^3 quaternary approximating subdivision scheme," *Abstract and Applied Analysis*, vol. 2009, Article ID 301967, 14 pages, 2009.
- [7] M. Aslam, G. Mustafa, and A. Ghaffar, " $(2n-1)$ -point ternary approximating and interpolating subdivision schemes," *Journal of Applied Mathematics*, Article ID 832630, 12 pages, 2011.
- [8] L. Rudin, S. Osher, and E. Fatemi, "Nonlinear total variation based noise removal algorithms," *Physica D*, vol. 60, pp. 259–268, 1992.
- [9] C. R. Vogel, *Computational Methods for Inverse Problems*, vol. 23 of *Frontiers in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 2002.
- [10] S. Osher, M. Burger, D. Goldfarb, J. Xu, and W. Yin, "An iterative regularization method for total variation-based image restoration," *Multiscale Modeling & Simulation*, vol. 4, no. 2, pp. 460–489, 2005.

