

Research Article

Boundedness and Compactness of the Mean Operator Matrix on Weighted Hardy Spaces

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We investigate the boundedness and the compactness of the mean operator matrix acting on the weighted Hardy spaces.

1. Introduction

First in the following, we generalize the definitions coming in [1]. Let $\beta = \{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 < p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_{\beta}^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty. \quad (1.1)$$

The notation

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \quad (1.2)$$

will be used whether or not the series converges for any value of z . These are called formal power series and the set of such series is denoted by $H^p(\beta)$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$. Recall that $H^p(\beta)$ is a reflexive Banach space with norm $\|\cdot\|_{\beta}$ and the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$ where $1/p + 1/q = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}$ [2]. For some other sources on this topic see [1–12].

The study of weighted Hardy spaces lies at the interface of analytic function theory and operator theory. As a part of operator theory, research on weighted Hardy spaces is of fairly recent origin, dating back to valuable work of Allen Shields [1] in the mid- 1970s. The mean operator matrix has been the focus of attention for several decades and many of its properties have been studied. Some of basic and useful works in this area are due to Browein et al. [13–16], which are pretty large works that contain a number of interesting results and indeed they are mainly of auxiliary nature. Also, some properties of mean operator matrices have been studied recently by Lashkaripour on weighted sequence spaces [17–20]. In this paper, we have given conditions under which the mean operator matrix is bounded and compact as an operator acting on weighted Hardy spaces. More details of our works are as follows: the idea of Theorem 2.6 comes from [16]. In Theorem 2.9, we extend the method used in [20, Theorem 1.2] to show the boundedness of the mean operator matrix acting on the weighted Hardy spaces. Some inequalities are useful to find a bound for the mean operator matrix acting on weighted Hardy spaces [21–26]. For example the inequality proved in [26, Theorem 8] is used in the proof of Theorem 2.11.

2. Main Results

In this section we define an operator acting on $H^p(\beta)$ and then we will investigate its boundedness and compactness on $H^p(\beta)$.

Definition 2.1. Let $\{a_n\}$ be a sequence of positive numbers and define

$$A_n = \sum_{i=0}^n a_i \beta(i)^p. \quad (2.1)$$

The mean operator matrix associated with the sequence $\{a_n\}$ is represented by the matrix $A = [a_{nk}]_{n,k}$ and is defined by

$$a_{nk} = \begin{cases} \frac{a_k \beta(n)^p}{A_n}, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases} \quad (2.2)$$

From now on, by A we denote the mean operator matrix associated with the fixed sequence $\{a_n\}$ as in Definition 2.1.

Theorem 2.2 (see [12, Theorem 1]). *If $0 < a_n \leq a_{n+1}$ for all integers $n \geq 0$, then A is a bounded operator on $H^p(\beta)$.*

Theorem 2.3 (see [12, Theorem 2]). *Let $1/p + 1/q = 1$ and $b_n > 0$ for $n = 0, 1, \dots$. If*

$$\begin{aligned} M_1 &= \sup_{n \geq 0} \sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} \left(\frac{b_k}{b_n} \right)^{1/p} < \infty, \\ M_2 &= \sup_{k \geq 0} \sum_{n=k}^{\infty} \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} \left(\frac{b_n}{b_k} \right)^{1/q} < \infty, \end{aligned} \quad (2.3)$$

then $A = [a_{nk}]_{n,k}$ is a bounded operator on $H^p(\beta)$ and $\|A\| \leq M_1^{1/q} M_2^{1/p}$.

Recall that if a_n, b_n are two positive sequences, by $a_n \sim b_n$, we mean that $a_n/b_n \rightarrow 1$ whenever $n \rightarrow \infty$. Also, we write $a_n = o(b_n)$, if $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$.

Corollary 2.4. *Let $\lim_{n \rightarrow \infty} na_n/A_n$ be finite and $1/p + 1/q = 1$. If*

$$\begin{aligned} \sup_{n \geq 0} \sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{na_n \beta(k)} \left(\frac{b_k}{b_n} \right)^{1/p} &< \infty, \\ \sup_{k \geq 0} \sum_{n=k}^{\infty} \frac{a_k \beta(n)^{p+1}}{na_n \beta(k)} \left(\frac{b_n}{b_k} \right)^{1/q} &< \infty, \end{aligned} \quad (2.4)$$

then A is a bounded operator on $H^p(\beta)$.

Proof. Put $\lim_{n \rightarrow \infty} na_n/A_n = \beta$. Then $na_n/\beta \sim A_n$ and so

$$\begin{aligned} \sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} \left(\frac{b_k}{b_n} \right)^{1/p} &\sim \beta \sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{na_n \beta(k)} \left(\frac{b_k}{b_n} \right)^{1/p} \quad \text{as } n \rightarrow \infty, \\ \sum_{n=k}^{\infty} \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} \left(\frac{b_n}{b_k} \right)^{1/q} &\sim \beta \sum_{n=k}^{\infty} \frac{a_k \beta(n)^{p+1}}{na_n \beta(k)} \left(\frac{b_n}{b_k} \right)^{1/q} \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2.5)$$

On the other hand

$$\begin{aligned} \sup_{n \geq 0} \sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{na_n \beta(k)} \left(\frac{b_k}{b_n} \right)^{1/p} &< \infty, \\ \sup_{k \geq 0} \sum_{n=k}^{\infty} \frac{a_k \beta(n)^{p+1}}{na_n \beta(k)} \left(\frac{b_n}{b_k} \right)^{1/q} &< \infty, \end{aligned} \quad (2.6)$$

thus Theorem 2.3 implies that A is a bounded operator on $H^p(\beta)$. \square

Lemma 2.5. *Suppose that $n^c a_n/\beta(n)$ is eventually increasing when the constant $c > 1 - \gamma$, and eventually decreasing when $c < 1 - \gamma$. Let*

$$\begin{aligned} S_1(n) &= \frac{1}{n} \sum_{k=1}^n \frac{a_k \beta(n)}{a_n \beta(k)} \left(\frac{k}{n} \right)^{-1/p}, \\ S_2(k) &= k^{1/q} \sum_{n=k}^{\infty} \frac{a_k \beta(n)}{a_n \beta(k)} \frac{1}{n^{1/(q+1)}}. \end{aligned} \quad (2.7)$$

If $\gamma > 1/p$, then $\lim_{n \rightarrow \infty} S_1(n) = \lim_{k \rightarrow \infty} S_2(k) = 1/(\gamma - 1/p)$.

Proof. Let $1/p + 1/q = 1$ and $c_2 < 1 - \gamma < c_1 < 1$. Then in either case there is a positive integer N such that

$$\left(\frac{k}{n}\right)^{-c_2} < \frac{a_k \beta(n)}{a_n \beta(k)} < \left(\frac{k}{n}\right)^{-c_1} \quad (2.8)$$

for $N \leq k \leq n$. Suppose first that $\gamma > 1/p$, then

$$\lim_{n \rightarrow \infty} \frac{n^{1-1/p} a_n}{\beta(n)} = \infty \quad (2.9)$$

and hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{N-1} \frac{a_k \beta(n)}{a_n \beta(k)} \left(\frac{k}{n}\right)^{-1/p} = 0. \quad (2.10)$$

Therefore

$$\limsup_{n \rightarrow \infty} S_1(n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=N}^n \left(\frac{k}{n}\right)^{-c_1-1/p} = \int_0^1 x^{-c_1-1/p} dx. \quad (2.11)$$

By calculus integral we get

$$\int_0^1 x^{-c-1/p} dx = \frac{1}{1-c-1/p}; \quad c \neq \frac{1}{q}, \quad (2.12)$$

and so

$$\liminf_{n \rightarrow \infty} S_1(n) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=N}^n \left(\frac{k}{n}\right)^{-c_2-\delta} = \int_0^1 x^{-c_2-1/p} dx = \frac{1}{1-c_2-1/p}. \quad (2.13)$$

Letting $c_1 \rightarrow 1 - \gamma$ from the right and $c_2 \rightarrow 1 - \gamma$ from the left, we have

$$\lim_{n \rightarrow \infty} S_1(n) = \frac{1}{\gamma - 1/p}. \quad (2.14)$$

Also note that

$$\begin{aligned} \limsup_{k \rightarrow \infty} S_2(k) &\leq \lim_{k \rightarrow \infty} k^{1/q} \sum_{n=k}^{\infty} \left(\frac{k}{n}\right)^{-c_1} \frac{1}{n^{1/q+1}}, \\ \lim_{k \rightarrow \infty} k^{1/q} \sum_{n=k}^{\infty} \left(\frac{k}{n}\right)^{-c_1} \frac{1}{n^{1/q+1}} &= \frac{1}{1/q - c_1}. \end{aligned} \quad (2.15)$$

If $c_1 \rightarrow 1 - \gamma$, then $1/q - c_1 \rightarrow \gamma - 1/p$ and similarly we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} S_2(k) &\geq \lim_{k \rightarrow \infty} k^{1/q-c_2} \sum_{n=k}^{\infty} \left(\frac{1}{n}\right)^{1-1/q-c_2}, \\ \lim_{k \rightarrow \infty} k^{1/q-c_2} \sum_{n=k}^{\infty} \left(\frac{1}{n}\right)^{1-1/q-c_2} &= \frac{1}{1/q-c_2}. \end{aligned} \quad (2.16)$$

If $c_2 \rightarrow 1 - \gamma$, then $1/q - c_2 \rightarrow \gamma - 1/p$. This completes the proof. \square

Theorem 2.6. Let $\lim_{n \rightarrow \infty} na_n \beta(n)^p / A_n = \gamma$, $n^c a_n \beta(n)^p$ be eventually monotonic for any constant c , and $\{\beta(n)\}$ be bounded. Then A is a bounded operator if $1/\gamma < p$.

Proof. Let $\delta_n = na_n \beta(n)^p / A_n$ and suppose first that $0 \leq \gamma < \infty$. Then

$$n(\log(A_n) - \log(A_{n-1})) = -n \log\left(1 - \frac{\delta_n}{n}\right) \rightarrow \gamma \quad (2.17)$$

as $n \rightarrow \infty$, and hence

$$\log(A_n) - \log(A_1) = -n \sum_{k=2}^n \log\left(1 - \frac{\delta_k}{k}\right) = \epsilon_n \log n, \quad (2.18)$$

where $\epsilon_n \rightarrow \gamma$. Consequently $A_n = A_1 n^{\epsilon_n}$. Now suppose that $\gamma = \infty$, then for $n \geq 2$,

$$\log(A_n) - \log(A_{n-1}) = -\log\left(1 - \frac{\delta_n}{n}\right) \geq \frac{\delta_n}{n} \quad (2.19)$$

since $\delta_n \rightarrow \infty$. If $M > 0$, then there is $N_1 \in \mathbb{N}$ such that $\delta_n \geq M + 1$ for all $n \geq N_1$.

Without loss of the generality suppose that there is a positive real number $a > 0$ such that $\delta_n > a$ for $n \leq N_1$. Note that

$$\sum_{k=2}^{N_1} \frac{1}{k} = \log N_1 + c + o(1) - 1. \quad (2.20)$$

If $n > N_1$, then

$$\sum_{k=2}^n \frac{1}{k} = \log n + c + o(1) - 1, \quad \sum_{k=N_1+1}^n \frac{1}{k} = \log n - \log N_1. \quad (2.21)$$

Also,

$$\begin{aligned}
 \frac{\sum_{k=2}^n \delta_k/k}{\log n} &\geq \frac{a\left(\sum_{k=2}^{N_1} 1/k\right) + (M+1)\left(\sum_{k=N_1+1}^n 1/k\right)}{\log n}, \\
 \frac{\sum_{k=2}^{N_1} 1/k + \sum_{k=N_1+1}^n 1/k}{\log n} &\geq \frac{M_1 + 1(\log n - \log N_1) + a(\log N_1 + c + o(1) - 1)}{\log n} \quad (2.22) \\
 &= M_1 + 1 + \frac{(a - M_1 - 1) \log N_1 + a(c + o(1) - 1)}{\log n},
 \end{aligned}$$

for large amount of n last equality greater than M_1 . Hence

$$\log A_n \geq \sum_{k=2}^n \frac{\delta_k}{k} = \gamma_n \log n, \quad (2.23)$$

where $\gamma_n \rightarrow \infty$. It follows that, for any real number c , $n^c A_n = n^{c+\gamma_n}$. Since

$$n^{c-1} A_n \sim \frac{1}{\gamma} n^c a_n \beta(n)^p, \quad (2.24)$$

thus $n^c a_n \beta(n)^p$ is eventually increasing for $c > 1 - \gamma$, and eventually decreasing for $c < 1 - \gamma$. But $\{\beta(n)\}_n$ is bounded, so there are $M_1, M_2 > 0$ such that $M_1 < \beta(n) < M_2$, and

$$\begin{aligned}
 \frac{n^c a_n}{\beta(n)} &= \frac{n^c a_n \beta(n)^p}{\beta(n)^{p+1}}, \\
 \frac{n^c a_n \beta(n)^p}{\beta(n)^{p+1}} &\geq \frac{n^c a_n \beta(n)^p}{M_1^{p+1}}. \quad (2.25)
 \end{aligned}$$

This implies that $n^c a_n / \beta(n)$ is eventually increasing for $c > 1 - \gamma$. Similarly $n^c a_n / \beta(n)$ is eventually decreasing for $c < 1 - \gamma$. Thus

$$\sum_{k=1}^n \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} \left(\frac{k}{n}\right)^{-1/p} \sim \frac{\gamma}{n} \sum_{k=1}^n \frac{a_k \beta(n)^{p+1}}{n a_n \beta(k)} \left(\frac{k}{n}\right)^{-1/p}. \quad (2.26)$$

By Lemma 2.5

$$\frac{\gamma}{n} \sum_{k=1}^n \frac{a_k \beta(n)^{p+1}}{n a_n \beta(k)} \left(\frac{k}{n}\right)^{-1/p} \quad (2.27)$$

is bounded and so

$$\sum_{k=1}^n \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} \left(\frac{k}{n}\right)^{-1/p} \quad (2.28)$$

is bounded. We can see that

$$\sum_{n=k}^{\infty} \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} \left(\frac{k}{n} \right)^{1/q} \quad (2.29)$$

is also bounded. Now by Theorem 2.3, A is a bounded operator and so the proof is complete. \square

Lemma 2.7. *Let $\{a_n\}, \{t_n\}$ be nonnegative sequences with $t_{-1} = 0$. Then for all $n \in \mathbb{N}$ one has*

$$\sum_{k=0}^n (t_k a_k) \leq \left\{ \max_{0 \leq k \leq n} \left(\frac{1}{n-k+1} \sum_{j=k}^n a_j \right) \right\} \left(\sum_{k=1}^n (n-k+1)(t_k - t_{k-1})^+ + t_0(n+1) \right). \quad (2.30)$$

Proof. Employing the summation by parts, we get

$$\begin{aligned} \sum_{k=0}^n (t_k a_k) &= \sum_{k=0}^n \left(\sum_{j=k}^n a_j \right) (t_k - t_{k-1}) \\ &\leq \sum_{k=0}^n \left(\sum_{j=k}^n a_j \frac{1}{n-k+1} \right) (t_k - t_{k-1})^+ (n-k+1). \end{aligned} \quad (2.31)$$

So

$$\sum_{k=0}^n (t_k a_k) \leq \left\{ \max_{0 \leq k \leq n} \left(\frac{1}{n-k+1} \sum_{j=k}^n a_j \right) \right\} \left(\sum_{k=1}^n (n-k+1)(t_k - t_{k-1})^+ + t_0(n+1) \right), \quad (2.32)$$

and at this time the proof is complete. \square

Theorem 2.8 (see [26, Theorem 8]). *Let $1/p + 1/q = 1$, $\{x_n\}$ be a positive sequence, then*

$$\sum_{j=0}^{\infty} \max_{0 \leq i \leq j} \left(\frac{1}{j-i+1} \sum_{k=i}^j x_k \right)^p \leq q^p \left(\sum_{k=0}^{\infty} x_k^p \right). \quad (2.33)$$

Theorem 2.9. *Let $\{a_n\}$ be a positive sequence and*

$$M_3 = \sup_{n \geq 0} \left(\sum_{k=1}^n \frac{n-k+1}{A_n} \left(\frac{a_k}{\beta(k)} - \frac{a_{k-1}}{\beta(k-1)} \right)^+ \beta(n)^{p+1} + \frac{(n+1)a_0}{A_n \beta(0)} \beta(n)^{p+1} \right) \quad (2.34)$$

be finite. Then A is bounded and $\|A\| \leq M_3 q$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in H^p(\beta)$, thus

$$A(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{a_k \beta(n)^p}{A_n} \hat{f}(k) \right) z^n. \quad (2.35)$$

By definition of $\|\cdot\|_{\beta}$, we have

$$\sum_{n \geq 0} \beta(n)^p \left| \sum_{k=0}^n \frac{a_k \beta(n)^p}{A_n} \hat{f}(k) \right|^p \leq \sum_{n \geq 0} \left(\sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} |\hat{f}(k)| \beta(k) \right)^p. \quad (2.36)$$

In Lemma 2.7, consider $t_k = a_k / \beta(k)$ and $a_j = |\hat{f}(j)| \beta(j)$. Then

$$\begin{aligned} & \sum_{n \geq 0} \left(\sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} |\hat{f}(k)| \beta(k) \right)^p \\ & \leq \sum_{n \geq 0} \left\{ \max_{0 \leq k \leq n} \frac{1}{n-k+1} \sum_{j=k}^n |\hat{f}(j)| \beta(j) \right\}^p \\ & \quad \times \left(\sum_{k=1}^n \frac{n-k+1}{A_n} \left(\frac{a_k}{\beta(k)} - \frac{a_{k-1}}{\beta(k-1)} \right)^+ \beta(n)^{p+1} + \frac{(n+1)a_0}{A_n \beta(0)} \beta(n)^{p+1} \right)^p. \end{aligned} \quad (2.37)$$

Now, Theorem 2.8 implies that

$$\sum_{n \geq 0} \left\{ \max_{0 \leq k \leq n} \left(\frac{1}{n-k+1} \sum_{j=k}^n |\hat{f}(j)| \beta(j) \right) \right\}^p M_3^p \leq M_3^p q^p \sum_{k=1}^{\infty} |\hat{f}(k)|^p \beta(k)^p, \quad (2.38)$$

and so we get $\|Af\| \leq M_3 q \|f\|_{\beta}$ for all $f \in H^p(\beta)$. Thus $A \in B(H^p(\beta))$ and indeed $\|A\| \leq M_3 q$. This completes the proof. \square

Corollary 2.10. Let $1/p + 1/q = 1$, $a_k / \beta(k) \geq a_{k-1} / \beta(k-1)$ and

$$M_4 = \sup_{n \geq 0} \sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{\beta(k) A_n} < \infty. \quad (2.39)$$

Then A is a bounded operator on $H^p(\beta)$ and $\|A\| \leq M_4$.

Proof. Note that

$$\sum_{n \geq 0} \left(\sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{A_n \beta(k)} |\hat{f}(k)| \beta(k) \right)^p \leq \sum_{n \geq 0} \left\{ \max_{0 \leq k \leq n} \frac{1}{n-k+1} \sum_{j=k}^n |\hat{f}(j)| \beta(j) \right\}^p \left(\sum_{k=0}^n \frac{a_k \beta(n)^{p+1}}{\beta(k) A_n} \right)^p. \quad (2.40)$$

Theorem 2.8 implies that

$$\sum_{n \geq 0} \left\{ \max_{0 \leq k \leq n} \left(\frac{1}{n-k+1} \sum_{j=k}^n |\hat{f}(j)| \beta(j) \right) \right\}^p M_4^p \leq M_4^p q^p \sum_{k=1}^{\infty} |\hat{f}(k)|^p \beta(k)^p, \quad (2.41)$$

and so by Theorem 2.9 we obtain $\|Af\| \leq qM_4\|f\|_{\beta}$ for all $f \in H^p(\beta)$. Thus $A \in B(H^p(\beta))$ and indeed $\|A\| \leq M_4q$. This completes the proof. \square

Now, we characterize compactness of subsets of $H^p(\beta)$ and then we will investigate compactness of the mean operator matrix on $H^p(\beta)$.

Theorem 2.11. *Let S be a nonempty subset of $H^p(\beta)$. Then S is relatively compact if and only if the following hold:*

- (i) *there exists $M > 0$, such that for all $\sum_{n=0}^{\infty} \hat{f}(n)z^n \in S$, $|\hat{f}(i)\beta(i)| \leq M$ for all $i \in \mathbb{N} \cup \{0\}$;*
- (ii) *given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \epsilon^p$ for all $\sum_{n=0}^{\infty} \hat{f}(n)z^n \in S$.*

Proof. Let S be relatively compact, thus there exist $g_1, \dots, g_k \in H^p(\beta)$ such that

$$S \subseteq \bigcup_{i=1}^k B(g_i, 1). \quad (2.42)$$

For every $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in S$, there is g_i such that $f \in B(g_i, 1)$. By Minkowski inequality we get

$$\begin{aligned} \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p &\leq \left[\left(\sum_{n=0}^{\infty} |\hat{f}(n) - \hat{g}_i(n)|^p \beta(n)^p \right)^{1/p} + \left(\sum_{n=0}^{\infty} |\hat{g}_i(n)|^p \beta(n)^p \right)^{1/p} \right]^p \\ &\leq (\|f - g_i\| + \|g_i\|)^p \\ &\leq (1 + \|g_i\|)^p \\ &\leq (1 + \max\{\|g_i\| : i = 1, \dots, k\})^p. \end{aligned} \quad (2.43)$$

Thus for every $f \in S$ and $n \in \mathbb{N} \cup \{0\}$, we get

$$|\hat{f}(n)\beta(n)| \leq 1 + \max\{\|g_i\| : i = 1, \dots, k\}. \quad (2.44)$$

So (i) holds. Now suppose that ϵ is an arbitrary positive number. Since S is relatively compact, thus there exist $h_1, \dots, h_k \in H^p(\beta)$ such that

$$S \subseteq \bigcup_{i=1}^k B(h_i, \frac{\epsilon}{2}). \quad (2.45)$$

Since $h_i \in H^p(\beta)$, there exists $N_i \in \mathbb{N}$ such that

$$\sum_{n=N_i}^{\infty} |\hat{h}_i(n)|^p \beta(n)^p < \frac{\epsilon^p}{2^p} \quad (2.46)$$

for $i = 1, \dots, k$. Put

$$N_0 = \max\{N_i : i = 1, \dots, k\}, \quad (2.47)$$

and consider $f \in S$. Then there exists $i \in \{1, \dots, k\}$, such that $f \in B(h_i, \epsilon/2)$. Hence we get

$$\begin{aligned} \sum_{n=N_0}^{\infty} |\hat{f}(n)|^p \beta(n)^p &\leq \left[\left(\sum_{n=N_0}^{\infty} |\hat{f}(n) - \hat{h}_i(n)|^p \beta(n)^p \right)^{1/p} + \left(\sum_{n=N_0}^{\infty} |\hat{h}_i(n)|^p \beta(n)^p \right)^{1/p} \right]^p \\ &\leq \left(\|f - h_i\| + \frac{\epsilon}{2} \right)^p \\ &\leq \epsilon^p. \end{aligned} \quad (2.48)$$

So (ii) holds.

Conversely, assume that $\epsilon > 0$ be given and let (i) and (ii) hold. By condition (ii), there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \frac{\epsilon^p}{2}, \quad (2.49)$$

for all $f \in S$. Let M_{n_0} be the closed linear span of the set $\{1, z, \dots, z^{n_0-1}\}$ in $H^p(\beta)$. Consider \mathbb{C}^{n_0} and M_{n_0} with norms

$$\|(z_1, \dots, z_{n_0})\| = \left(\sum_{n=1}^{n_0} |z_n|^p \beta(n)^p \right)^{1/p}, \quad (2.50)$$

for all $(z_i)_{i=1}^{n_0} \in \mathbb{C}^{n_0}$, and

$$\left\| \sum_{i=0}^{n_0-1} a_i z^i \right\| = \left(\sum_{i=0}^{n_0-1} |a_i|^p \beta(i)^p \right)^{1/p} \quad (2.51)$$

for all $\sum_{i=0}^{n_0-1} a_i z^i \in M_{n_0}$. Define $L : M_{n_0} \rightarrow \mathbb{C}^{n_0}$, by

$$L \left(\sum_{i=0}^{n_0-1} a_i z^i \right) = (a_0, \dots, a_{n_0-1}). \quad (2.52)$$

Clearly, we can see that L is a bounded linear operator. Now, consider the compact subset

$$\left\{ (z_i)_{i=1}^{n_0} : \sum_{i=1}^{n_0} |z_i|^p \beta(i)^p \leq n_0 M^p \right\} \quad (2.53)$$

in \mathbb{C}^{n_0} . Then we have

$$\left\{ \sum_{i=0}^{n_0-1} \hat{f}(i) z^i : \sum_{n=0}^{\infty} \hat{f}(n) z^n \in S \right\} \subseteq L^{-1} \left\{ (z_i)_{i=1}^{n_0} : \sum_{i=1}^{n_0} |z_i|^p \beta(i)^p \leq n_0 M^p \right\}. \quad (2.54)$$

Since

$$L^{-1} \left\{ (z_i)_{i=1}^{n_0} : \sum_{i=1}^{n_0} |z_i|^p \beta(i)^p \leq n_0 M^p \right\} \quad (2.55)$$

is a compact subspace of M_{n_0} , so there exist $g_1, \dots, g_k \in M_{n_0}$ such that

$$L^{-1} \left\{ (z_i)_{i=1}^{n_0} : \sum_{i=1}^{n_0} |z_i|^p \beta(i)^p \leq n_0 M^p \right\} \in \bigcup_{i=1}^k B \left(g_i, \frac{\epsilon}{2^{1/p}} \right). \quad (2.56)$$

Hence for every

$$f \in \left\{ \sum_{i=0}^{n_0-1} \hat{f}(i) z^i : f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in S \right\} \quad (2.57)$$

there is $i \in \{1, \dots, k\}$ satisfying

$$\sum_{n=0}^{n_0-1} \left| \hat{f}(n) - \hat{g}_i(n) \right|^p \beta(n)^p \leq \frac{\epsilon^p}{2}. \quad (2.58)$$

Also, we have

$$\begin{aligned} \left(\|f - g_i\|_{\beta} \right)^p &\leq \sum_{n=0}^{n_0-1} \left| \hat{f}(n) - \hat{g}_i(n) \right|^p \beta(n)^p + \sum_{n=n_0}^{\infty} \left| \hat{f}(n) \right|^p \beta(n)^p \\ &\leq \frac{\epsilon^p}{2} + \frac{\epsilon^p}{2} \\ &\leq \epsilon^p. \end{aligned} \quad (2.59)$$

Thus, S is relatively compact and so the proof is complete. \square

Theorem 2.12. *Let the mean matrix operator A be bounded on $H^p(\beta)$, and*

$$\lim_{m \rightarrow \infty} \left(\sum_{n=m}^{\infty} \frac{\beta(n)^{p^2+p}}{A_n^p} \right)^{1/p} \left(\sum_{k=0}^m \left(\frac{a_k}{\beta(k)} \right)^q \right)^{1/q} = 0, \quad (2.60)$$

where $1/p + 1/q = 1$. Then A is a compact operator on $H^p(\beta)$.

Proof. Let $B_{H^p(\beta)}$ be the closed unit ball of $H^p(\beta)$. Define $S = A(B_{H^p(\beta)})$ and note that S is a bounded subset of $H^p(\beta)$. Put $r_n = |\hat{f}(n)|a_n$, $u_n = \beta(n)^{p^2+p}/A_n^p$, $v_k = (\beta(k)/a_k)^p$, and

$$E_m = \left(\sum_{n=m}^{\infty} u_n \right)^{1/p} \left(\sum_{k=0}^m v_k^{1-q} \right)^{1/q}. \quad (2.61)$$

Note that $\lim_{m \rightarrow \infty} E_m = 0$. So for every $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $E_m < \epsilon/(q^{p-1}p)^{1/p}$ for all $m \geq m_0$. Note that if

$$f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k \in B_{H^p(\beta)}, \quad (2.62)$$

then

$$Af(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{a_k \beta(n)^p \hat{f}(k)}{A_n} \right) z^n \in S. \quad (2.63)$$

Since $\|f\|_{\beta}^p \leq 1$, we have

$$\begin{aligned} \sum_{n=m}^{\infty} |\hat{A}f(n)|^n \beta(n)^p &\leq \sum_{n=m}^{\infty} \frac{\beta(n)^{p^2+p}}{A_n^p} \left(\sum_{k=0}^n a_k |\hat{f}(k)| \right)^p \\ &= \sum_{n=m}^{\infty} u_n \left(\sum_{k=0}^n r_k \right)^p \\ &\leq \epsilon^p \sum_{k=0}^{\infty} (r_k)^p v_k \\ &\leq \epsilon^p \sum_{k=0}^{\infty} |\hat{f}(k)|^p \beta(k)^p \\ &\leq \epsilon^p. \end{aligned} \quad (2.64)$$

Thus by Theorem 2.11, S is compact and so the proof is complete. \square

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