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Research Article

On the Positive Operator Solutions to an Operator Equation

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The necessary conditions and the sufficient condition for the existence of positive operator solutions to the operator equation $X^s + A^* X^{-t} A = Q$ are established. An iterative method for obtaining the positive operator solutions is proposed.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the set of all bounded linear operators on the Hilbert space $\mathscr{H}$. In this paper, we consider the nonlinear operator equation

$$X^s + A^* X^{-t} A = Q, \tag{1}$$

where $A, Q \in \mathscr{B}(\mathscr{H})$, $Q > 0$, and X is an unknown operator in $\mathscr{B}(\mathscr{H})$. Both s and t are positive integers.

This type of equation often arises from many areas such as dynamic programming [1], control theory [2, 3], stochastic filtering and statistics, and so forth [4, 5]. In the recent years, for matrices, (1) has been considered by many authors (see [6–13]), and different iterative methods for computing the positive definite solutions to (1) are proposed in finite-dimensional space. The case $s = t = 1$ has been extensively studied by several authors. In this paper, we extend the study of the operator equation (1) from a finite-dimensional space to an infinite-dimensional Hilbert space. We derive some necessary conditions for the existence of positive solutions to the operator equation (1). Moreover, conditions under which the operator equation (1) has positive operator solutions are obtained. Based on Banach's fixed-point principle, we obtain the positive operator solution to the operator equation (1). First, we introduce some notations and terminologies, which are useful later. For $A \in \mathscr{B}(\mathscr{H})$, if $(Ax, x) \geq 0$ for all $x \in \mathscr{H}$, then A is said to be a positive operator and is denoted by $A \geq 0$. If A is a positive operator and invertible, then denote $A > 0$. For A and B in $\mathscr{B}(\mathscr{H})$, $A \geq B$ means that $A - B$ is a positive operator. For $A \in \mathscr{B}(\mathscr{H})$, A^*, $\omega(A)$, $\sigma(A)$, and $r(A)$ denote the adjoint, the radius of numerical range, the spectrum, and the spectral radius of A, respectively.

For positive operators in $\mathscr{B}(\mathscr{H})$, the following facts are well known.

(1) If $P \geq Q > 0$, then $P^{-1} \leq Q^{-1}$.

(2) Denote $\lambda_{\max}(P) = \max\{\lambda : \lambda \in \sigma(P), P > 0\}$, $\lambda_{\min}(P) = \min\{\lambda : \lambda \in \sigma(P), P > 0\}$. Then $\lambda_{\min}(P)I \leq P \leq \lambda_{\max}(P)I$.

(3) If the sequence $\{X_n\}_{n=1}^{+\infty}$ of positive operator is monotonically increasing and has upper bound, that is, $X_k \leq X_{k+1} \leq C_1$, or is monotonically decreasing and has lower bound, that is, $C_2 \leq X_{k+1} \leq X_k$, then this sequence is convergent to a positive limit operator, where $C_1, C_2 \in \mathscr{B}(\mathscr{H})$ are given operators.

2. Main Results and Proofs

In order to prove our main result, we begin with some lemmas as follows.

Lemma 1. *Let $A, B \in \mathscr{B}(\mathscr{H})$. If $A \geq B \geq 0$, then $\|A\| \geq \|B\|$.*

Lemma 2. *Let A and B be self-adjoint operators in $\mathscr{B}(\mathscr{H})$. If $A \leq B$, then, for every $T \in \mathscr{B}(\mathscr{H})$, one has $T^* AT \leq T^* BT$.*

In this section, we give our main results and proofs.

Theorem 3. *If the operator equation* (1) *has a positive operator solution X, then (1) $\sqrt[t]{AQ^{-1}A^*} \leq X \leq \sqrt[s]{Q}$ and (2) $r(A) \leq (bt/(s+t))^{t/2s}(bs/(s+t))^{1/2}$, where $b = \|Q\|$.*

Proof. (1) If the operator equation (1) has a positive operator solution X, then $0 \leq A^*X^{-t}A \leq Q$. Hence, we obtain $X^s \leq Q$; that is, $X \leq \sqrt[s]{Q}$. From $A^*X^{-t}A \leq Q$, it follows that

$$Q^{-1/2}A^*X^{-t/2}X^{-t/2}AQ^{-1/2} \leq I,$$
$$X^{-t/2}AQ^{-1/2}Q^{-1/2}A^*X^{-t/2} \leq I, \quad (2)$$
$$AQ^{-1}A^* \leq X^t.$$

That is, $X \geq \sqrt[t]{AQ^{-1}A^*}$.

(2) From (1), we have

$$X^{t+s} + X^{t/2}A^*X^{-t}AX^{t/2} = X^{t/2}QX^{t/2} \leq \lambda_{\max}(Q)\,X^t. \quad (3)$$

Then

$$\begin{aligned} r(A)^2 &= r^2\left(X^{t/2}AX^{-t/2}\right) \\ &\leq \left\|X^{t/2}AX^{-t/2}\right\|^2 \\ &\leq \left\|\lambda_{\max}(Q)\,X^t - X^{t+s}\right\|. \end{aligned} \quad (4)$$

According to the spectral decomposition of X, we have

$$\lambda_{\max}(Q)\,X^t - X^{t+s} = \int_{\sigma(X)} \left(\lambda_{\max}(Q)\,\lambda^t - \lambda^{t+s}\right) dE\lambda. \quad (5)$$

Denote $\lambda_{\max}(Q) = \|Q\| = b$. Then

$$\begin{aligned} \left\|\lambda_{\max}(Q)\,X^t - X^{t+s}\right\| &\leq \max_{\lambda\in(0,\sqrt[s]{b}]} \left|b\lambda^t - \lambda^{t+s}\right| \\ &= \left(\frac{bt}{s+t}\right)^{t/s}\left(\frac{bs}{s+t}\right). \end{aligned} \quad (6)$$

Therefore, $r(A) \leq (bt/(s+t))^{t/2s}(bs/(s+t))^{1/2}$. □

Theorem 4. *If A is invertible and $A^*X^{-t}A \leq Q-(AQ^{-1}A^*)^{s/t}$ for all $X \in [\sqrt[t]{AQ^{-1}A^*}, \sqrt[s]{Q}]$, then* (1) *has a positive operator solution.*

Proof. Let $\Phi(X) = (Q - A^*X^{-t}A)^{1/s}$; then Φ is continuous for $X \in [\sqrt[t]{AQ^{-1}A^*}, \sqrt[s]{Q}]$; $\Phi(X) \leq \sqrt[s]{Q}$; and $\Phi(X) \geq [Q-(Q-(AQ^{-1}A^*)^{s/t})]^{1/s} = (AQ^{-1}A^*)^{1/t}$. So Φ maps $[\sqrt[t]{AQ^{-1}A^*}, \sqrt[s]{Q}]$ into itself. Furthermore, Φ is continuous on $[\sqrt[t]{AQ^{-1}A^*}, \sqrt[s]{Q}]$ because $X > 0$; hence, Φ has a fixed point in $[\sqrt[t]{AQ^{-1}A^*}, \sqrt[s]{Q}]$; that is, there exists $X \geq 0$ such that $(Q - A^*X^{-t}A)^{1/s} = X$; this implies that (1) has a positive operator solution. □

Theorem 5. *If the operator equation* (1) *has a positive operator solution X, then $\lambda_{\min}(T^*T) \leq (t/(t+s))^{t/s}(s/(t+s))$ and $X \leq \sqrt[s]{\tilde{\alpha}Q}$ when $\lambda_{\min}(T^*T) \neq 1$, where $T = Q^{-t/2s}AQ^{-1/2}$ and $\tilde{\alpha}$ is a solution to the equation $\alpha^{t/s}(1-\alpha) = \lambda_{\min}(T^*T)$ in $[t/(t+s), 1]$.*

Proof. We consider the sequence

$$\begin{aligned} \alpha_0 &= 1, \alpha_{k+1} \\ &= 1 - \frac{\lambda_{\min}\left(\left(Q^{-t/2s}AQ^{-1/2}\right)^*Q^{-t/2s}AQ^{-1/2}\right)}{\alpha_k^{t/s}}, \\ &\qquad k = 0, 1, 2, \ldots. \end{aligned} \quad (7)$$

Let X be a positive solution to (1). Then $X^s = Q - A^*X^{-t}A \leq Q = \alpha_0 Q$; that is, $X \leq \sqrt[s]{\alpha_0 Q}$. Assuming that $X \leq \sqrt[s]{\alpha_k Q}$, then

$$\begin{aligned} X^s &= Q - A^*X^{-t}A \leq Q - \frac{A^*Q^{-t/s}A}{\alpha_k^{t/s}} \\ &= Q^{1/2}\left[I - \frac{Q^{-1/2}A^*Q^{-t/2s}Q^{-t/2s}AQ^{-1/2}}{\alpha_k^{t/s}}\right]Q^{1/2} \leq Q^{1/2} \\ &\quad \times \left[1 - \frac{\lambda_{\min}\left(\left(Q^{-t/2s}AQ^{-1/2}\right)^*Q^{-t/2s}AQ^{-1/2}\right)}{\alpha_k^{t/s}}\right] \\ &\quad \times Q^{1/2} = \alpha_{k+1}Q. \end{aligned} \quad (8)$$

Hence, $X \leq \sqrt[s]{\alpha_n Q}$ for all $n = 0, 1, 2, \ldots$. It is straightforward to check that the sequence $\{\alpha_n\}$ is monotonically decreasing; also, $\{\alpha_n\}$ is bounded below; hence, $\{\alpha_n\}$ is convergent. Denote $T = Q^{-t/2s}AQ^{-1/2}$ and let $\lim_{n\to\infty}\alpha_n = \tilde{\alpha}$; then $\tilde{\alpha} = 1 - \lambda_{\min}(T^*T)/\tilde{\alpha}^{t/s}$; that is, $\tilde{\alpha}$ is a solution to the equation $\alpha^{t/s}(1-\alpha) = \lambda_{\min}(T^*T)$. Let $f(x) = x^{t/s}(1-x)$. Then

$$\max_{x\in[0,1]} f(x) = f\left(\frac{t}{t+s}\right) = \left(\frac{t}{t+s}\right)^{t/s}\left(\frac{s}{t+s}\right). \quad (9)$$

Thus, it follows that $\lambda_{\min}(T^*T) \leq (t/(t+s))^{t/s}(s/(t+s))$.

Since

$$\lambda_{\min}(T^*T) \leq \left(\frac{t}{t+s}\right)^{t/s}\left(\frac{s}{t+s}\right). \quad (10)$$

Then the equation $\alpha^t(1-\alpha) = \lambda_{\min}(T^*T)$ may have two solutions; one of these solutions is in the interval $[t/(t+s), 1]$.

In order to prove that the limit $\tilde{\alpha}$ of the sequence $\{\alpha_n\}$ is in $[t/(t+s), 1]$, we assume that $\alpha_i \geq t/(t+s)$ (obviously $\alpha_0 = 1 > t/(t+s)$); then

$$\begin{aligned} \alpha_{i+1} &= 1 - \frac{\lambda_{\min}(T^*T)}{\alpha_i^{t/s}} \\ &\geq 1 - \lambda_{\min}(T^*T)\left(\frac{t+s}{t}\right)^{t/s} \\ &\geq 1 - \left(\frac{t}{t+s}\right)^{t/s}\left(\frac{s}{t+s}\right)\left(\frac{t+s}{t}\right)^{t/s} \\ &= \frac{t}{t+s}. \end{aligned} \quad (11)$$

Therefore, $\alpha_i \geq t/(t+s)$ for each $i = 1, 2, \ldots$. Since $X \leq \sqrt[s]{\alpha_n Q}$, then $X \leq \sqrt[s]{\tilde{\alpha} Q}$ and $\tilde{\alpha} \in [t/(t+s), 1]$. The proof is completed. □

Consider the following iterative sequence:

$$X_0^s = \gamma Q, \qquad X_{k+1}^s = Q - A^* X_k^{-t} A, \quad k = 0, 1, \ldots, \gamma > 0. \tag{12}$$

We will prove the following theorem.

Theorem 6. *If the operator equation* (1) *has a positive operator solution X, then it has a maximal one X_L. Moreover, the sequence $\{X_i\}$ in* (12) *for $\gamma \in [\tilde{\alpha}, 1]$ is monotonically decreasing and converges to X_L, where $\tilde{\alpha}$ is defined in Theorem 5.*

Proof. Consider the iterative sequence (12) with $\gamma \in [\tilde{\alpha}, 1]$. According to Theorem 4, we have $X^s \leq \tilde{\alpha} Q \leq \gamma Q = X_0^s$ for any positive operator solution X to (1).

Suppose that $X_i^s \geq X^s$. Then

$$X_{i+1}^s = Q - A^* X_i^{-t} A \geq Q - A^* X^{-t} A = X^s. \tag{13}$$

Hence, $X_i \geq X$ hold for each i. According to the definition of $\tilde{\alpha}$ and the monotonicity of the function $f(x) = x^{t/s}(1-x)$ on $[t/(t+s), 1]$, we have

$$\gamma^{t/s}(1-\gamma) I \leq \tilde{\alpha}^{t/s}(1-\tilde{\alpha}) I = \lambda_{\min}(T^*T) I \leq T^*T \tag{14}$$

for all $\gamma \in [\tilde{\alpha}, 1]$, where $T = Q^{-t/2s} A Q^{-1/2}$. We compute

$$\begin{aligned} X_1^s &= Q - A^*(\gamma Q)^{-t/s} A \\ &= Q^{1/2}\left[I - Q^{-1/2} A^* (\gamma Q)^{-t/s} A Q^{-1/2}\right] Q^{1/2} \\ &= Q^{1/2}\left(I - \gamma^{-t/s} T^*T\right) Q^{1/2}. \end{aligned} \tag{15}$$

Using inequality (14) and equality (15), we obtain $X_1^s \leq \gamma Q = X_0^s$.

Assuming that $X_i^s \leq X_{i-1}^s$, then

$$X_{i+1}^s = Q - A^* X_i^{-t} A \leq Q - A^* X_{i-1}^{-t} A = X_i^s. \tag{16}$$

Therefore, $X_{i+1}^s \leq X_i^s$ for all $i = 0, 1, \ldots$; that is, the sequence $\{X_n^s\}$ is monotonically decreasing. Hence, $\{X_n^s\}$ converges to the positive operator solution X_L to (1). Since $X_L^s \geq X^s$ for any positive operator solution X, it follows that X_L is the maximal solution. □

Theorem 7. *Let $\lambda_{\min}(T^*T) < \lambda_{\max}(T^*T) \leq (t/(t+s))^{t/s}(s/(t+s))$, and β_1, β_2 are solutions to the equation $\beta^{t/s}(1-\beta) = \lambda_{\max}(T^*T)$ in $[0, t/(t+s)]$ and $[t/(t+s), 1]$, respectively. Then there are the following conclusions.*

(i) *If $\gamma \in [\beta_1, \beta_2]$, then the sequence $\{X_n\}$ in* (12) *is monotonically increasing and converges to a positive operator solution $X_\gamma \in [\sqrt[s]{\gamma Q}, \sqrt[s]{\tilde{\alpha} Q}]$ to* (1).

(ii) *If $\gamma \in [\tilde{\alpha}, 1]$, then the sequence $\{X_n\}$ in* (12) *is monotonically decreasing and converges to the maximal positive operator solution $X_L \in [\sqrt[s]{\beta_2 Q}, \sqrt[s]{\tilde{\alpha} Q}]$ to* (1).

(iii) *If $\gamma \in (\beta_2, \tilde{\alpha})$ and $(t/s)\|A\|^2 \leq (\beta_2 \lambda_{\min}(Q))^{(s+t)/s}$, then the sequence $\{X_n\}$ in* (12) *converges to the unique solution $X_L \in [\sqrt[s]{\beta_2 Q}, \sqrt[s]{\tilde{\alpha} Q}]$, where $\tilde{\alpha}$ and T are defined in Theorem 5.*

Proof. Consider the function $f(x) = x^{t/s}(1-x)$, $x \in [0, 1]$. It is monotonically increasing on $[0, t/(t+s)]$ and monotonically decreasing on $[t/(t+s), 1]$, and

$$\max_{x \in [0,1]} f(x) = f\left(\frac{t}{t+s}\right) = \left(\frac{t}{t+s}\right)^{t/s}\left(\frac{s}{t+s}\right). \tag{17}$$

Since $\lambda_{\max}(T^*T) \leq (t/(t+s))^{t/s}(s/(t+s))$, then, for each γ_1, γ_2 such that $0 < \beta_1 \leq \gamma_1 \leq \beta_2 < \tilde{\alpha} \leq \gamma_2 \leq 1$, the inequalities

$$\begin{aligned} \gamma_2^{t/s}(1-\gamma_2) I &\leq \lambda_{\min}(T^*T) \leq T^*T \\ &\leq \lambda_{\max}(T^*T) \leq \gamma_1^{t/s}(1-\gamma_1) I \end{aligned} \tag{18}$$

are satisfied.

(i) Let $\gamma \in [\beta_1, \beta_2]$. We will prove that the operator sequence $\{X_n\}$ in (12) is monotonically increasing and is bounded above.

According to the fourth inequality in (18), we compute

$$\begin{aligned} X_1^s &= Q - A^*(\gamma Q)^{-t/s} A \\ &= Q^{1/2}\left(I - \gamma^{-t/s} Q^{-1/2} A^* Q^{-t/s} A Q^{-1/2}\right) Q^{1/2} \\ &= Q^{1/2}\left(I - \gamma^{-t/s} T^*T\right) Q^{1/2} \geq \gamma Q = X_0^s. \end{aligned} \tag{19}$$

Assuming that $X_i^s \geq X_{i-1}^s$, then

$$X_{i+1}^s = Q - A^* X_i^{-t} A \geq Q - A^* X_{i-1}^{-t} A = X_i^s. \tag{20}$$

Therefore, $X_{i+1}^s \geq X_i^s$ for all $i = 0, 1, \ldots$; that is, the sequence $\{X_n^s\}$ is monotonically increasing. Obviously $X_0^s = \gamma Q \leq \beta_2 Q \leq \tilde{\alpha} Q$. We suppose that $X_i^s \leq \tilde{\alpha} Q$; from the first inequality in (18), we obtain

$$X_{i+1}^s = Q - A^* X_i^{-t} A \leq Q - A^*(\tilde{\alpha} Q)^{-t/s} A \leq \tilde{\alpha} Q. \tag{21}$$

Hence, $\{X_n^s\}$ converges to a positive operator solution X_γ^s to (1). Since $X_i \in [\sqrt[s]{\gamma Q}, \sqrt[s]{\tilde{\alpha} Q}]$ for all $i = 0, 1, \ldots$, we have $X_\gamma \in [\sqrt[s]{\gamma Q}, \sqrt[s]{\tilde{\alpha} Q}]$.

(ii) Let $\gamma \in [\tilde{\alpha}, 1]$. From the proving procession of Theorem 6, the sequence (12) is monotonically decreasing. Since $X_0 = \sqrt[s]{\gamma Q} \geq \sqrt[s]{\beta_2 Q}$ and supposing that $X_i^s \geq \beta_2 Q$, it follows that

$$\begin{aligned} X_{i+1}^s &= Q - A^* X_i^{-t} A \geq Q - A^*(\beta_2 Q)^{-t/s} A \\ &= Q^{1/2}\left(I - \beta_2^{-t/s} Q^{-1/2} A^* Q^{-t/s} A Q^{-1/2}\right) Q^{1/2} \\ &\geq Q^{1/2}\left(I - \beta_2^{-t/s} \lambda_{\max}(T^*T) I\right) Q^{1/2} \\ &= \beta_2 Q. \end{aligned} \tag{22}$$

By induction principle, $X_i^s \geq \beta_2 Q$ hold for each $i = 0, 1, 2, \ldots$. Hence, $\{X_n^s\}$ is convergent. From Theorem 6, $\{X_n^s\}$ converges to X_L^s such that $X_L \in [\sqrt[s]{\beta_2 Q}, \sqrt[s]{\gamma Q}]$.

(iii) Considering the sequence $\{X_n^s\}$ in (12) for $\gamma \in (\beta_2, \tilde{\alpha})$, that is, $X_0^s \in (\beta_2 Q, \tilde{\alpha} Q)$ and supposing that $X_i^s \in (\beta_2 Q, \tilde{\alpha} Q)$, then, for X_{i+1}^s,

$$\begin{aligned} X_{i+1}^s &= Q - A^* X_i^{-t} A \\ &< Q^{1/2} \left(I - \beta_2^{-t/s} Q^{-1/2} A^* Q^{-t/s} A Q^{-1/2} \right) Q^{1/2} \\ &\leq \tilde{\alpha} Q, \\ X_{i+1}^s &= Q - A^* X_i^{-t} A \\ &> Q^{1/2} \left(I - \tilde{\alpha}^{-t/s} Q^{-1/2} A^* Q^{-t/s} A Q^{-1/2} \right) Q^{1/2} \\ &\geq \beta_2 Q. \end{aligned} \tag{23}$$

Hence, $X_i^s \in (\beta_2 Q, \gamma Q)$, for all $i = 0, 1, 2, \ldots$. Now, we considering $\|X_{i+1}^s - X_i^s\|$,

$$\begin{aligned} \left\| X_{i+1}^s - X_i^s \right\| &= \left\| \left(Q - A^* X_i^{-t} A \right) - \left(Q - A^* X_{i-1}^{-t} A \right) \right\| \\ &\leq \frac{1}{s} \frac{1}{\left(\sqrt[s]{\beta_2 \lambda_{\min}(Q)} \right)^{s-1}} \\ &\quad \times \frac{t \|A\|^2}{\left(\sqrt[s]{\beta_2 \lambda_{\min}(Q)} \right)^{t+1}} \left\| X_i - X_{i-1} \right\| \\ &= \frac{t}{s} \|A\|^2 \frac{1}{\left(\sqrt[s]{\beta_2 \lambda_{\min}(Q)} \right)^{t+s}} \\ &\quad \times \left\| X_i - X_{i-1} \right\|, \end{aligned} \tag{24}$$

and $(t/s)\|A\|^2 \leq (\beta_2 \lambda_{\min}(Q))^{(s+t)/s}$, it follows that $\{X_n\}$ in (12) is a Cauchy sequence in the Banach space $[\sqrt[s]{\beta_2 Q}, \sqrt[s]{\tilde{\alpha} Q}]$. Hence, it has a limit X_γ in $[\sqrt[s]{\beta_2 Q}, \sqrt[s]{\tilde{\alpha} Q}]$, and X_γ is a unique solution to (1) in $[\sqrt[s]{\beta_2 Q}, \sqrt[s]{\tilde{\alpha} Q}]$. According to Theorem 6, it is the maximal solution, that is, $X_\gamma = X_L$. The proof is completed. □

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