

Research Article

A Brief Study of Certain Class of Harmonic Functions of Bazilevič Type

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We define and investigate a new subclass of Bazilevič type harmonic univalent functions using a linear operator. We investigated the harmonic structures in terms of its coefficient conditions, extreme points, distortion bounds, convolution, and convex combination. So, also, we discussed the subordination properties for the functions in this class.

1. Introduction

Let A denote the usual class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(0) = 0$ and $f'(0) - 1 = 0$. Also, we denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disk U by S .

Here, we recall some definitions and concepts of classes of analytic functions. Let $f \in A$. Then, $f \in S^*(\mu)$ if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \mu, \quad (0 \leq \mu < 1, z \in U). \quad (2)$$

This class is called starlike class of analytic function.

Also, let $f \in A$. Then, $f \in C(\mu)$ if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \mu, \quad (0 \leq \mu < 1, z \in U). \quad (3)$$

This class is called convex class of analytic function. The above two classes have been repeatedly investigated by various authors like [1–4] just to mention but few, as the literatures littered everywhere.

The theory of analytic functions has wide application in many physical problem: problems as in heat conduction, electrostatic potential and fluid flows, and theory of fractals constitute practical examples. The concern of this work is the study of a particular family of analytic functions defined in a given domain by certain geometric conditions which are useful in the above problems.

Let $\gamma : C^2 \rightarrow C$, and let ϕ be univalent in U . If p is analytic in U and satisfies the differential subordination $\Phi(p(z), zp'(z)) < \phi(z)$, then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, $p < q$. If p and $\Phi(p(z), zp'(z))$ are univalent in U and satisfy the differential superordination $\phi(z) < \Phi(p(z), zp'(z))$, then p is called a solution of the differential superordination. An analytic function q is called subordinate of the solution of the differential superordination if $q < p$. For details (see [5–7]).

Sălăgean [8] introduced the following differential operator:

$$\begin{aligned} D^0 f(z) &= z f'(z), \\ D^1 f(z) &= Df(z) = z f'(z), \\ D^n f(z) &= D(D^{n-1} f(z)) = z(D^{n-1} f(z))'. \end{aligned} \quad (4)$$

From (1), we can write that

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k \right)^\alpha. \quad (5)$$

Using binomial expansion on (5), we have

$$\begin{aligned} f(z)^\alpha &= z^\alpha + \alpha a_2 z^{\alpha+1} + \left[\alpha a_3 + \frac{\alpha(\alpha-1)}{2!} a_2^2 \right] z^{\alpha+2} \\ &+ \left[\alpha a_4 + \frac{\alpha(\alpha-1)}{2!} 2a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} a_2^3 \right] z^{\alpha+3} + \dots \end{aligned} \quad (6)$$

We then define the class of analytic functions of fractional power A_α as

$$f(z)^\alpha = z^\alpha + \sum_{k=\alpha}^{\infty} a_k(\alpha) z^{\alpha+k-1}, \quad (7)$$

where $\alpha > 0$ (is real, and it is principal determination only).

Thus, we obtain the differential operator

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=\alpha}^{\infty} (\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1}. \quad (8)$$

Let us also define the function $\varphi_\alpha(a, c, z)$ by

$$\varphi_\alpha(a, c; z) = z^\alpha + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{c_{k-1}} z^{\alpha+k-1}, \quad (9)$$

$$(z \in U, \alpha \in \mathbb{R}^+, a \in \mathbb{R}, c \in \mathbb{R} - \{0, -1, -2, \dots\}),$$

where $(a)_k$ is the Pochhammer symbol defined by

$$\begin{aligned} (a)_k &= \frac{\gamma(a+k)}{\gamma(a)} \\ &= \begin{cases} 1, & k=0, \\ a(a+1)(a+2)\cdots(a+n-1) & n \in \mathbb{N}. \end{cases} \end{aligned} \quad (10)$$

Corresponding to the function $\varphi_\alpha(a, c; z)$, we defined a linear operator

$$J_n^\alpha(a, c) f(z)^\alpha = \varphi_\alpha(a, c; z) * D^n f(z)^\alpha, \quad f(z)^\alpha \in A_\alpha. \quad (11)$$

Or equivalently

$$\begin{aligned} J_n^\alpha(a, c) f(z)^\alpha &= \alpha^n z^\alpha + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} (\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1}. \end{aligned} \quad (12)$$

Remark 1. For $n = 0$, $\alpha = 1$ operator (11) reduces to Carlson-shaffer operator [3], and also for different values of α , it imposes the Saitoh operator [7] and recently the Mahzoon-Lotha [9]. for $a = c$, $\alpha = 1$ poses the Salagean derivative operator.

Now, let $B_n^\alpha(\gamma)$ be the class of functions containing the operator (11) and satisfying the relation

$$\frac{z(J_n^\alpha(a, c) f(z)^\alpha)'}{\alpha J_n^\alpha(a, c) f(z)^\alpha} < \left(\frac{1+\gamma z}{1-z} \right)^\alpha \quad \alpha > 0, \gamma \neq -1. \quad (13)$$

For $a = c$, $\alpha = 1$, $\gamma = 1$, and $n = 0$, we obtain the well-known subclass

$$f'(z) < \frac{1+z}{1-z}. \quad (14)$$

Also, for $n = 0$, $a = c$, and $\gamma = 1$, we have

$$\frac{\alpha z f'(z)}{f(z)} \cdot \frac{f(z)^\alpha}{z^\alpha} < \left(\frac{1+z}{1-z} \right)^\alpha. \quad (15)$$

For $n = 0$, $\alpha = 1$, and $\gamma = 1$ we have the following subclass which contain Carlson-Shaffer operator:

$$\frac{z(J(a, c) f(z))'}{z} < \frac{1+z}{1-z}. \quad (16)$$

The starting point in the study of functions defined in (13) is the discovery in 1995 by Russian Mathematician Bazilevič [10] of functions in U defined by

$$\begin{aligned} f(z) &= \left\{ \frac{\alpha}{1+\varepsilon^2} \int_0^z \frac{p(v) - i\varepsilon}{V(1 + (i\alpha\varepsilon/(1+\varepsilon^2)))} g(v)^{\alpha/(1+\varepsilon^2)} dv \right\}^{(1+i\varepsilon)/\alpha}, \end{aligned} \quad (17)$$

where $p \in P$ and $g \in S^*$. The number $\alpha > 0$ and ε are real, and all powers are meant as principal determinant only. The family of functions in (17) became known as Bazilevič functions and is, in this work, denoted by $B(\alpha, \varepsilon)$. Except that, Bazilevič showed that each function $f \in B(\alpha, \varepsilon)$ is univalent in U , very little is known regarding the family as a whole. However, with some simplifications, it may be possible to understand and investigate the family. Indeed, it is easy to verify that, with special choices of the parameters α and ε and the function $g(z)$, the family $B(\alpha, \varepsilon)$ cracks down to some well-known subclasses of univalent functions.

For instance, if we take $\varepsilon = 0$, we have

$$f(z) = \left\{ \alpha \int_0^z \frac{p(v)}{v} g(v)^\alpha dv \right\}^{1/\alpha}. \quad (18)$$

On differentiation, the expression (18) yields

$$\frac{z f'(z) f(z)^{\alpha-1}}{g(z)^\alpha} = p(z), \quad z \in U. \quad (19)$$

Or equivalently

$$\operatorname{Re} \frac{zf'(z)f(z)^{\alpha-1}}{g(z)^\alpha} > 0, \quad z \in U. \quad (20)$$

The subclasses of Bazilevič functions satisfying (19) are called Bazilevič functions of type α and are denoted by $B(\alpha)$ (see [11]). In 1973, Noonan [12] gave a plausible description of functions of the class $B(\alpha)$ as those functions in S for which each $r < 1$, and the tangent to the curve $U_\alpha(r) = \{zf(re^{i\theta})^\alpha, 0 \leq \theta < 2\pi\}$ never turns back on itself as much as π radian. If $\alpha = 1$, the class $B(\alpha)$ reduces to the family of close-to-convex functions; that is,

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U. \quad (21)$$

If we decide to choose $g(z) = f(z)$ in (21), we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U, \quad (22)$$

which implies that $f(z)$ is starlike. Furthermore, if we replace $f(z)$ by $zf'(z)$ in (22), we obtain

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in U, \quad (23)$$

which shows that $f(z)$ is convex. Moreover, if $g(z) = z$ in (20), then we have the family $B_1(\alpha)$ [11] of functions satisfying

$$\operatorname{Re} \frac{zf'(z)f(z)^{\alpha-1}}{z^\alpha} > 0, \quad z \in U. \quad (24)$$

The various subfamilies of Bazilevič functions are being studied repeatedly by many authors; the literatures in this direction littered everywhere (see Bernard's Bibliography of Schlich functions [13]).

In 1992, Abdulhalim [14] introduced a generalization of functions satisfying (24) as

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{z^\alpha} > 0, \quad z \in U, \quad (25)$$

where the parameter α and the operator D^n are defined as earlier. He denoted this class of functions by $B_n(\alpha)$. It is easily seen that his generalization has extraneously included analytic functions satisfying

$$\operatorname{Re} \frac{f(z)^\alpha}{z^\alpha} > 0, \quad z \in U, \quad (26)$$

which are largely nonunivalent in the unit disk. By proving the inclusion

$$B_{n+1}(\alpha) \subset B_n(\alpha), \quad (27)$$

Abdulhalim was able to show that for all $n \in \mathbb{N}$, each function of the class $B_n(\alpha)$ is univalent in U .

Notable contributors like MacGregor, [15, 16], Noonan [12], Singh [11], Thomas [17], Tuan and Anh [18], Yamaguchi

[19], and Opoola [20] had earlier considered various special cases of the parameters n and α of (25) and established many interesting properties of function in those particular cases.

In some general sense, it is possible to further improve work on the function defined by the geometric condition (25). Therefore, we intend to investigate this family from the viewpoints of subordination and harmonic univalent functions and determine coefficient inequalities, extreme points, distortion bounds, convolution, and convex combination.

2. Subordination Results

The objective of this section is to find the sufficient conditions of functions belonging to the class $B_n^\alpha(\gamma, a, c)$.

For this purpose, the following Lemmas will be necessary.

Lemma 2 (see [21]). *Let $\beta \neq 0$ be a complex number. Let $q(z)$ ($q(z) \neq 0$) be a univalent function in U such that*

$$\operatorname{Re} \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > \max \left\{ 0, R \left(\frac{\beta-1}{\beta} q(z) \right) \right\}. \quad (28)$$

If p , ($p(z) \neq 0$), $z \in U$ satisfies the differential subordination

$$(1-\beta)(p(z)-1) + \beta \frac{zp'(z)}{p(z)} < (1-\beta)(q(z)-1) + \beta \frac{zq'(z)}{q(z)}, \quad (29)$$

then $p < q$ and q is the best dominant.

Lemma 3 (see [22]). *Let ω be analytic in U with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| < 1$ at a point z_0 , then*

$$z_0 \omega'(z_0) = m \omega(z_0), \quad (30)$$

where m is a real number and $m \geq 1$.

Lemma 4 (see [23]). *If $f \in A$ satisfies*

$$\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] < \frac{(\mu+1)}{2(\mu-1)}, \quad (2 \leq \mu < 3, z \in U), \quad (31)$$

then

$$\frac{zf'(z)}{f(z)} < \frac{\mu(1-z)}{\mu-z}, \quad (32)$$

$$\left| \frac{zf'(z)}{f(z)} - \frac{\mu}{\mu+1} \right| < \frac{\mu}{\mu+1}.$$

Now, we begin our main results as the following.

Theorem 5. *Let $\alpha > 0$ (α is real), $n \in \mathbb{N}_0$, and $\beta \neq 0$ be a complex number such that*

$$\operatorname{Re} \left\{ 1 + \frac{z(1-\gamma+2\gamma z)}{(1+\gamma z)(1-z)} \right\} > \max \left\{ 0, R \left(\frac{\beta-1}{\beta} \right) \left(\frac{1+\gamma z}{1-z} \right)^\alpha \right\}. \quad (33)$$

If the subordination

$$\begin{aligned} & (1-\beta) \left[\frac{z(J_n^\alpha(a, c) f(z)^\alpha)' }{\alpha^n z^\alpha} - 1 \right] \\ & + \beta \left(1 - \alpha + \frac{z(J_n^\alpha(a, c) f(z)^\alpha)'' }{(J_n^\alpha(a, c) f(z)^\alpha)'} \right) \\ & < (1-\beta) \left[\left(\frac{1+\gamma z}{1-z} \right)^\alpha - 1 \right] + \frac{\alpha\beta(1+\gamma)z}{(1+\gamma z)(1-z)} \end{aligned} \quad (34)$$

holds, then $f^\alpha \in B_n^\alpha(\gamma, a, c)$.

Proof. Suppose that

$$p(z) = \frac{z(J_n^\alpha(a, c) f(z)^\alpha)' }{\alpha^n z^\alpha}, \quad q(z) = \left(\frac{1+\gamma z}{1-z} \right)^\alpha. \quad (35)$$

Then, simple computations give

$$\begin{aligned} & \operatorname{Re} \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} \\ & = \operatorname{Re} \left\{ 1 + \frac{z(1-\gamma+2\gamma z)}{(1+\gamma z)(1-z)} \right\} \\ & > \max \left\{ 0, \operatorname{Re} \left(\frac{\beta-1}{\beta} \right) \left(\frac{1+\gamma z}{1-z} \right)^\alpha \right\} \\ & = \max \left\{ 0, \operatorname{Re} \left(\frac{\beta-1}{\beta} \right) q(z) \right\}, \\ & (1-\beta)(p(z)-1) + \beta \frac{zp'(z)}{p(z)} \\ & = (1-\beta) \left[\frac{z(J_n^\alpha(a, c) f(z)^\alpha)' }{\alpha^n z^\alpha} - 1 \right] \\ & + \beta \left(1 - \alpha + \frac{z(J_n^\alpha(a, c) f(z)^\alpha)'' }{(J_n^\alpha(a, c) f(z)^\alpha)'} \right) \\ & < (1-\beta) \left[\left(\frac{1+\gamma z}{1-z} \right)^\alpha - 1 \right] + \frac{\alpha\beta(1+\gamma)z}{(1+\gamma z)(1-z)} \\ & = (1-\beta)(q(z)-1) + \beta \frac{zq'(z)}{q(z)}. \end{aligned} \quad (36)$$

Thus, in the view of Lemma 2, we have $f(z)^\alpha \in B_n^\alpha(\gamma, a, c)$. \square

For $\alpha = 1, \gamma = 1, n = 0$. We have the following.

Corollary 6. Let $\beta \neq 0$ be a complex number such that

$$\operatorname{Re} \left\{ 1 + \frac{2z^2}{1-z^2} \right\} > \max \left\{ 0, \operatorname{Re} \left(\frac{\beta-1}{\beta} \left(\frac{1+z}{1-z} \right) \right) \right\}. \quad (38)$$

If the subordination

$$\begin{aligned} & (1-\beta) \left[(J_0^1(a, c) f(z))' - 1 \right] + \beta \left(\frac{z(J_0^1(a, c) f(z))'' }{(J_0^1(a, c) f(z))'} \right) \\ & < (1-\beta) \left[\left(\frac{1+z}{1-z} \right) - 1 \right] + \frac{2\beta z}{1-z^2} \end{aligned} \quad (39)$$

Holds, then $f(z)^\alpha \in B_0^1(1, a, c)$.

For $a = c, \alpha = 1, \gamma = 1$, and $n = 0$, we have the following

Corollary 7. Let $\beta \neq 0$ be a complex number such that

$$\operatorname{Re} \left\{ 1 + \frac{2z^2}{1-z^2} \right\} > \max \left\{ 0, \operatorname{Re} \left(\frac{\beta-1}{\beta} \left(\frac{1+z}{1-z} \right) \right) \right\}. \quad (40)$$

If the subordination

$$(1-\beta)(f'(z)-1) + \beta \frac{zf''(z)}{f'(z)} \quad (41)$$

$$< (1-\beta) \left[\frac{1+z}{1-z} - 1 \right] + \frac{2\beta z}{1-z^2}$$

holds, then $f(z)^\alpha \in B_0^1(1, 1, 1) \equiv B$.

For $\gamma = 1, a = c$, we have the following.

Corollary 8. Let $\beta \neq 0$ be a complex number such that

$$\operatorname{Re} \left\{ 1 + \frac{2z^2}{1-z^2} \right\} > \max \left\{ 0, \operatorname{Re} \left(\frac{\beta-1}{\beta} \left(\frac{1+z}{1-z} \right)^\alpha \right) \right\}. \quad (42)$$

If the subordination

$$\begin{aligned} & (1-z) \left[\frac{z(J_n^\alpha f(z)^\alpha)' }{\alpha^n z^\alpha} - 1 \right] + \beta \left(1 - \alpha + \frac{z(J_n^\alpha f(z)^\alpha)'' }{(J_n^\alpha f(z)^\alpha)'} \right) \\ & < (1-\beta) \left[\left(\frac{1+z}{1-z} \right)^\alpha - 1 \right] + \frac{2\alpha\beta z}{1-z^2} \end{aligned} \quad (43)$$

holds, then $f^\alpha \in B_n^\alpha(1, 1, 1)$.

For $\gamma = 1, n = 0, a = c$, we have the following.

Corollary 9. Let $\beta \neq 0$ be a complex number such that

$$\operatorname{Re} \left\{ 1 + \frac{2z^2}{1-z^2} \right\} > \max \left\{ 0, \operatorname{Re} \left(\frac{\beta-1}{\beta} \left(\frac{1+z}{1-z} \right)^\alpha \right) \right\}. \quad (44)$$

If the subordination

$$\begin{aligned} & (1-\beta) \left[\frac{\alpha z f'(z)}{f(z)} \cdot \frac{f(z)^\alpha}{z^\alpha} - 1 \right] + \beta \left(1 - \beta + \frac{z(f(z)^\alpha)'' }{(f(z)^\alpha)'} \right) \\ & < (1-\beta) \left[\left(\frac{1+z}{1-z} \right)^\alpha - 1 \right] + \frac{2\alpha\beta z}{1-z^2} \end{aligned} \quad (45)$$

holds, then $f^\alpha \in B_0^\alpha \equiv B(\alpha)$.

Theorem 10. Let the functions $f(z)^\alpha$ take the form (7) and satisfy

$$\operatorname{Re} \left\{ 1 - \alpha + \frac{z(J_n^\alpha(a, c) f(z)^\alpha)''}{(J_n^\alpha(a, c) f(z)^\alpha)'} \right\} < \frac{\alpha(\gamma + 1)}{2(\gamma - 1)}, \quad z \in U, \quad \alpha > 0, \quad \gamma > 1. \quad (46)$$

Then, $f \in B_n^\alpha(\gamma, a, c)$.

Proof. Let ω be defined by

$$\frac{z(J_n^\alpha(a, c) f(z)^\alpha)'}{\alpha^n z^\alpha} = \left(\frac{1 + \gamma\omega(z)}{1 - \omega(z)} \right)^\alpha, \quad \omega(z) \neq 1. \quad (47)$$

Then, ω is analytic in U , and since $\gamma \neq -1$, then $\omega(0) = 0$. Also, it follows that

$$\begin{aligned} \operatorname{Re} \left\{ 1 - \alpha + \frac{z(J_n^\alpha(a, c) f(z)^\alpha)''}{(J_n^\alpha(a, c) f(z)^\alpha)'} \right\} &= \operatorname{Re} \left\{ \frac{\alpha(\gamma + 1)z\omega'(z)}{(1 + \gamma\omega(z))(1 - \omega(z))} \right\} \\ &< \frac{\alpha(\gamma + 1)}{2(\gamma - 1)}, \quad \gamma \neq 1. \end{aligned} \quad (48)$$

Now, let us proceed to prove that $|\omega(z)| < 1$. Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1. \quad (49)$$

Then, using Lemma 3 and letting $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = me^{i\theta}$, $m \geq 1$ yields that

$$\begin{aligned} \operatorname{Re} \left\{ 1 - \alpha + \frac{z(J_n^\alpha(a, c) f(z)^\alpha)''}{(J_n^\alpha(a, c) f(z)^\alpha)'} \right\} &= \operatorname{Re} \left\{ \frac{\alpha(\gamma + 1)z_0\omega'(z_0)}{(1 + \gamma\omega(z_0))(1 - \omega(z_0))} \right\} \\ &= \operatorname{Re} \left\{ \frac{\alpha me^{i\theta}(\gamma + 1)}{(1 + \gamma\omega(e^{i\theta}))(1 - e^{i\theta})} \right\} \\ &= \frac{\alpha m(1 + \gamma)}{2(\gamma - 1)} \geq \frac{\alpha(\gamma + 1)}{2(\gamma - 1)}, \quad \gamma > 1, \quad \alpha > 0 \quad (\alpha \text{ is real}). \end{aligned} \quad (50)$$

Thus, we have

$$\operatorname{Re} \left\{ 1 - \alpha + \frac{z(J_n^\alpha(a, c) f(z)^\alpha)''}{(J_n^\alpha(a, c) f(z)^\alpha)'} \right\} \geq \frac{\alpha(\gamma + 1)}{2(\gamma - 1)}, \quad z \in U, \quad (51)$$

which contradicts the hypothesis (46). Therefore, we conclude that $|\omega(z)| < 1$ for all $z \in U$ and

$$\frac{z(J_n^\alpha(a, c) f(z)^\alpha)'}{\alpha^n z^\alpha} < \left(\frac{1 + \gamma(z)}{1 - (z)} \right)^\alpha, \quad (52)$$

$$z \in U, \quad \gamma \neq -1, \quad \alpha > 0,$$

where (α is real). This completes the proof of the theorem. \square

Letting $\alpha = 1$, $n = 0$, and $a = c$ in Theorem 10, we have the following.

Corollary 11. Let the function f take the form (1) and satisfy

$$\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} < \frac{\gamma + 1}{2(\gamma - 1)}, \quad z \in U, \quad \gamma > 1. \quad (53)$$

Then,

$$f'(z) < \frac{1 + \gamma z}{1 - z}. \quad (54)$$

Corollary 12. Let the function f take the form (1) and satisfy

$$\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} < \frac{\gamma + 1}{2(\gamma - 1)}, \quad (z \in U), \quad \gamma > 1. \quad (55)$$

Then,

$$\left| f'(z) - \frac{\gamma}{2} \right| < \frac{\gamma}{2}, \quad (z \in U). \quad (56)$$

With various special choices of the parameters involved, many existing and new subclasses of Bazilevič functions could be derived.

3. Harmonic Structure of Bazilevič Type

In this section, the authors wish to have a look into the Bazilevič type harmonic univalent functions.

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D . In any simply connected domain, we can write

$$f = h + \bar{g}, \quad (57)$$

where h and g are analytic in D . We call h the analytic part and g the coanalytic part of A . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that

$$|h'(z)| > |g'(z)|, \quad (z \in U). \quad (58)$$

Denote by S_H the class of functions f of the form (57) that are harmonic univalent and sense-preserving in the disk U . The subclasses of harmonic functions have been studied by some authors for different purposes with different properties (see [24–26]). But unfortunately, it is becoming very difficult to see the literatures on Bazilevič-type harmonic univalent function, and this may be likely associated with the problem

index α always poses. This paper is designed to address this issue.

In this work, we may express the analytic functions h and g as

$$\begin{aligned} h(z)^\alpha &= z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}, \\ g(z)^\alpha &= \sum_{k=1}^{\infty} b_k(\alpha) z^{\alpha+k-1}, \quad |b_1(\alpha)| < 1. \end{aligned} \quad (59)$$

Therefore,

$$f(z)^\alpha = h(z)^\alpha + \overline{g(z)^\alpha}. \quad (60)$$

We define our linear operator as given in (11) such that

$$J_n^\alpha(a, c) f(z)^\alpha = J_n^\alpha(a, c) h(z)^\alpha + (-1)^n \overline{J_n^\alpha(a, c) g(z)^\alpha}, \quad (61)$$

where

$$\begin{aligned} J_n^\alpha(a, c) h(z)^\alpha &= \alpha^n z^\alpha + \sum_{k=2}^{\infty} \frac{a_{k-1}}{c_{k-1}} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1}, \\ J_n^\alpha(a, c) g(z)^\alpha &= \sum_{k=1}^{\infty} \frac{a_{k-1}}{c_{k-1}} (\alpha + k - 1)^n b_k(\alpha) z^{\alpha+k-1}. \end{aligned} \quad (62)$$

We let $B_n^\alpha(\beta, a, c)$ be the family of harmonic functions f of the form (57) such that

$$\operatorname{Re} \left\{ \frac{J_n^\alpha(a, c) f(z)^\alpha}{\alpha^n z^\alpha} \right\} > \beta, \quad 0 \leq \beta < 1, \quad (63)$$

where $\alpha > 0$, (α is real), $n \in N_0$, and $J_n^\alpha(a, c)$ is earlier defined in (11).

Furthermore, let the subclass $VB_n^\alpha H(\beta, a, c)$ consist of harmonic functions

$$f_n^\alpha = h^\alpha + \overline{g_n^\alpha}, \quad (64)$$

so that h^α and g_n^α are of the form

$$h(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} |a_k(\alpha)| z^{\alpha+k-1}, \quad (65)$$

$$g(z)^\alpha = (-1)^n + \sum_{k=1}^{\infty} |b_k(\alpha)| z^{\alpha+k-1}, \quad |b_1(\alpha)| < 1.$$

The authors in this work wish to study the Bazilevič-type harmonic univalent functions defined by linear operator in which h^α has positive coefficients. We claim that our results are quite new and not explored in the literatures.

Assigning specific values to n, β, a, c, α in the subclass $B_n^\alpha H(\beta, a, c)$, we obtain the following subclasses which may be the expected results by using definition of earlier authors of subclasses of Bazilevič functions such as classes studied by Abduhalim [14], Yamaguchi [19], Macgregor [15], and Singh [11], just to mention but few.

We first prove a sufficient condition for the function in $B_n^\alpha H(\beta, a, c)$.

Theorem 13. Let $f^\alpha = h^\alpha + \overline{g^\alpha}$, where f^α and g^α are as earlier defined if

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha + k - 1}{\alpha} \right)^n |a_k(\alpha)| \\ &+ \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha + k - 1}{\alpha} \right)^n |b_k(\alpha)| \leq 1 - \beta, \end{aligned} \quad (66)$$

where $n \in N_0, 0 \leq \beta < 1, a \in R, c \in R - \{-1, -2, \dots\}$ and $\alpha > 0$; is real then $f(z)^\alpha$ is sense-preserving, harmonic univalent in U , and $f \in B_n^\alpha H(\beta, a, c)$.

Proof. If $z_1^\alpha \neq z_2^\alpha$, then

$$\begin{aligned} &\left| \frac{f(z_1)^\alpha - f(z_2)^\alpha}{h(z_1)^\alpha - h(z_2)^\alpha} \right| \\ &\geq 1 - \left| \frac{g(z_1)^\alpha - g(z_2)^\alpha}{h(z_1)^\alpha - h(z_2)^\alpha} \right| \\ &= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k(\alpha) (z_1^{\alpha+k-1} - z_2^{\alpha+k-1})}{(z_1^\alpha - z_2^\alpha) + \sum_{k=2}^{\infty} a_k(\alpha) (z_1^{\alpha+k-1} - z_2^{\alpha+k-1})} \right| \\ &> 1 - \frac{\sum_{k=1}^{\infty} (\alpha + k - 1) b_k(\alpha)}{\alpha + \sum_{k=2}^{\infty} (\alpha + k - 1) a_k(\alpha)} \\ &\geq 1 - \left(\left(\sum_{k=1}^{\infty} ((a)_{k-1}/(c)_{k-1}) ((\alpha + k - 1)/\alpha)^n \right. \right. \\ &\quad \times (1/(1 - \beta)) |b_k(\alpha)| \Big) \\ &\quad \times \left(1 + \sum_{k=2}^{\infty} ((a)_{k-1}/(c)_{k-1}) ((\alpha + k - 1)/\alpha)^n \right. \\ &\quad \times (1/(1 - \beta)) |a_k(\alpha)| \Big)^{-1} \Big) \\ &\geq 0, \end{aligned} \quad (67)$$

which proves the univalence. Note that f is sense-preserving in U . This is because

$$\begin{aligned} |h(z)^\alpha| &\geq \alpha - \sum_{k=2}^{\infty} (\alpha + k - 1) |a_k(\alpha)| |z|^{\alpha+k-1} \\ &> \alpha - \sum_{k=2}^{\infty} (\alpha + k - 1) |a_k(\alpha)| \\ &\geq \alpha^n - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \frac{(\alpha + k - 1)^n}{1 - \beta} |a_k(\alpha)| \\ &\geq \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \frac{(\alpha + k - 1)^n}{1 - \beta} |b_k(\alpha)| \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{k=1}^{\infty} (\alpha + k - 1) |b_k(\alpha)| \\
&> \sum_{k=2}^{\infty} (\alpha + k - 1) |b_k(\alpha)| |z|^{\alpha+k-2}| \\
&\geq |g(z)|^\alpha.
\end{aligned} \tag{68}$$

By (63) and (64), we have

$$\begin{aligned}
&\operatorname{Re} \left\{ \frac{J_n^\alpha(a, c) f(z)^\alpha}{\alpha^n z^n} \right\} \\
&= \operatorname{Re} \left\{ \frac{J_n^\alpha(a, c) h(z)^\alpha + (-1)^n \overline{J_n^\alpha(a, c) g(z)^\alpha}}{\alpha^n z^n} \right\} > \beta.
\end{aligned} \tag{69}$$

Using the fact that $\operatorname{Re}(\omega) > \beta$ if and only if $|1 - \beta + \omega| \geq |1 + \beta - \omega|$, it suffices to show that

$$\left| 1 - \beta + \frac{J_n^\alpha(a, c) f(z)^\alpha}{\alpha^n z^n} \right| - \left| 1 - \beta - \frac{J_n^\alpha(a, c) f(z)^\alpha}{\alpha^n z^n} \right| \geq 0. \tag{70}$$

That is,

$$\begin{aligned}
&|(1 - \beta) \alpha^n z^n + J_n^\alpha(a, c) f(z)^\alpha| \\
&\quad - |(1 - \beta) \alpha^n z^n - J_n^\alpha(a, c) f(z)^\alpha| \geq 0, \\
&|(1 - \beta) \alpha^n z^\alpha + J_n^\alpha(a, c) h(z)^\alpha + (-1)^n \overline{J_n^\alpha(a, c) g(z)^\alpha}| \\
&\quad - |(1 - \beta) \alpha^n z^\alpha - J_n^\alpha(a, c) h(z)^\alpha - (-1)^n \overline{J_n^\alpha(a, c) g(z)^\alpha}| \\
&= \left| \alpha^n z^\alpha - \beta \alpha^n z^\alpha + \alpha^n z^\alpha \right. \\
&\quad + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-2}} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1} \\
&\quad \left. + (-1)^n \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} (\alpha + k - 1)^n \overline{b_k(\alpha) z^{\alpha+k-1}} \right| \\
&\quad - \left| \alpha^n z^\alpha + \beta \alpha^n z^\alpha - \alpha^n z^\alpha \right. \\
&\quad - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-2}} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1} \\
&\quad \left. - (-1)^n \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} (\alpha + k - 1)^n \overline{b_k(\alpha) z^{\alpha+k-1}} \right|, \\
&= \left| \alpha^n (2 - \beta) z^\alpha + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-2}} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1} \right. \\
&\quad \left. + (-1)^n \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} (\alpha + k - 1)^n b_k(\alpha) z^{\alpha+k-1} \right|
\end{aligned}$$

$$\begin{aligned}
&- \left| \beta \alpha^n z^\alpha - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-2}} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1} \right. \\
&\quad \left. - (-1)^n \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} (\alpha + k - 1)^n b_k(\alpha) z^{\alpha+k-1} \right| \\
&\geq 2\alpha^n (1 - \beta) |z|^\alpha \\
&\quad + \sum_{k=2}^{\infty} 2 \frac{(a)_{k-1}}{(c)_{k-1}} (\alpha + k - 1)^n |a_k(\alpha)| |z|^{\alpha+k-1} \\
&\quad - (-1)^n \sum_{k=1}^{\infty} 2 \frac{(a)_{k-1}}{(c)_{k-1}} (\alpha + k - 1)^n |b_k(\alpha)| |z|^{\alpha+k-1} \\
&\quad \times 2\alpha^n (1 - \beta) \left[1 + \sum_{k=2}^{\infty} \frac{1}{1 - \beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha + k - 1}{\alpha} \right)^n \right. \\
&\quad \times |a_k(\alpha)| \\
&\quad \left. + (-1)^n \sum_{k=1}^{\infty} \frac{1}{1 - \beta} \frac{(a)_{k-1}}{(c)_{k-1}} \right. \\
&\quad \left. \times \left(\frac{\alpha + k - 1}{\alpha} \right)^n |b_k(\alpha)| \right].
\end{aligned} \tag{71}$$

This last expression is nonnegative by (66), and so the proof is complete. \square

The harmonic function

$$\begin{aligned}
f(z)^\alpha &= z^\alpha + \sum_{k=2}^{\infty} \frac{(c)_{k-1} (1 - \beta)}{(a)_{k-1}} \left(\frac{\alpha}{\alpha + k - 1} \right)^n x_k z^{\alpha+k-1} \\
&\quad + \sum_{k=1}^{\infty} \frac{(c)_{k-1} (1 - \beta)}{(a)_{k-1}} \left(\frac{\alpha}{\alpha + k - 1} \right)^n y_k z^{\alpha+k-1},
\end{aligned} \tag{72}$$

where $n \in N_0$, $\alpha > 0$ (α is real), $0 \leq \beta < 1$, $a \in R$, $c \in R - \{-1, 0, 1, \dots\}$ and

$$\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1, \tag{73}$$

shows that the coefficient bound given by (66) is sharp. The functions of the form (72) are in $B_n^\alpha H(\beta, a, c)$ because

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{1}{1 - \beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha + k - 1}{\alpha} \right)^n |a_k(\alpha)| \\
&\quad + \sum_{k=1}^{\infty} \frac{1}{1 - \beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha + k - 1}{\alpha} \right)^n |b_k(\alpha)| \\
&= \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1.
\end{aligned} \tag{74}$$

In the following theorem, it is shown that the condition (66) is also necessary for functions $f_n^\alpha = h^\alpha + g_n^\alpha$, where h^α and g_n^α are as earlier defined.

Theorem 14. Let $f_n^\alpha = h^\alpha + \overline{g_n^\alpha}$. Then, $f_n \in VB_n^\alpha H(\beta, a, c)$ if and only if

$$\sum_{k=2}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |a_k(\alpha)| + \sum_{k=1}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |b_k(\alpha)| \leq 1. \quad (75)$$

Proof. Since $VB_n^\alpha H(\beta, a, c) \subset B_n^\alpha H(\beta, a, c)$, we only need to prove the “only if” part of the theorem. To this end, for functions f_n^α of the form (64), we notice that the condition

$$\operatorname{Re} \left\{ \frac{J_n^\alpha(a, c) f_n(z)^\alpha}{\alpha^n z^\alpha} \right\} > \beta \quad (76)$$

is equivalent to

$$\operatorname{Re} \left\{ \left((1-\beta) z^\alpha - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n a_k(\alpha) z^{\alpha+k-1} - (-1)^{2n} \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n b_k(\alpha) z^{\alpha+k-1} \right) (z^{-\alpha}) \right\} \geq 0. \quad (77)$$

The above required condition (75) must hold for all values of z in U . Upon clearing the values of z on the positive real axis, where $0 \leq z = r < 1$, we must have

$$(1-\beta) z^\alpha - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n a_k(\alpha) r^{k-1} - (-1)^n \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n b_k(\alpha) r^{k-1} \geq 0, \quad (78)$$

and the proof is complete. \square

Theorem 15. Let $f_n^\alpha = h^\alpha + \overline{g^\alpha}$, where h^α and g^α are as given earlier. Then, $f_n \in VB_n^\alpha H(\beta, a, c)$ if and only if

$$f_n(z)^\alpha = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{nk}(z)), \quad (79)$$

where $h_1(z)^\alpha = z^\alpha$, $h_k(z)^\alpha = z^\alpha + ((c)_{k-1}(1-\beta)(\alpha)^n / (a)_{k-1}(\alpha+k-1)^n) z^{\alpha+k-1}$ ($k = 2, 3, \dots$),

$$g_{mk}(z)^\alpha = z^\alpha + (-1)^n \frac{\alpha^n (1-\beta)(c)_{k-1}}{(\alpha+k-1)^n (a)_{k-1}} \overline{z^{\alpha+k-1}} \quad (k = 2, 3, \dots), \quad (80)$$

where $x_k \geq 0$, $y_k \geq 0$, $x_1 = 1 - (\sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k) \geq 0$.

In particular, the extreme points of $VB_n^\alpha H(\beta, a, c)$ are $\{b_k\}$ and $\{g_{nk}\}$.

Proof. For functions $f_n^\alpha = h^\alpha + \overline{g^\alpha}$, where h^α and g^α are as earlier defined, we have

$$\begin{aligned} f_n(z)^\alpha &= \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{mk}(z)) \\ &= \sum_{k=1}^{\infty} (x_k + y_k) z^\alpha + \sum_{k=2}^{\infty} \frac{\alpha^n (1-\beta)(c)_{k-1}}{(\alpha+k-1)^n (a)_{k-1}} x_k z^{\alpha+k-1} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{\alpha^n (1-\beta)(c)_{k-1}}{(\alpha+k-1)^n (a)_{k-1}} y_k z^{\alpha+k-1}. \end{aligned} \quad (81)$$

Then,

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n \frac{1}{1-\beta} |a_k(\alpha)| \\ + \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n \frac{1}{1-\beta} |b_k(\alpha)| \\ = \sum_{k=2}^{\infty} x_k + \sum_{k=2}^{\infty} x_k = 1 - x_1 \leq 1, \end{aligned} \quad (82)$$

and so $f_n^\alpha \in B_n^\alpha H(\beta, a, c)$.

Conversely, suppose that $f_n^\alpha \in B_n^\alpha H(\beta, a, c)$. Setting

$$\begin{aligned} x_k &= \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |a_k(\alpha)|, \\ 0 &\leq x_k \leq 1 \quad (k = 2, 3, \dots), \\ y_k &= \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |b_k(\alpha)|, \\ 0 &\leq y_k \leq 1 \quad (k = 1, 2, 3, \dots), \end{aligned} \quad (83)$$

and $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$, therefore, f_n^α can be written as

$$\begin{aligned} f_n(z)^\alpha &= z^\alpha + \sum_{k=2}^{\infty} |a_k(\alpha)| z^{\alpha+k-1} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} |b_k(\alpha)| \overline{z^{\alpha+k-1}} \\ &= z^\alpha + \sum_{k=2}^{\infty} \frac{\alpha^n (1-\beta)(c)_{k-1}}{(\alpha+k-1)^n (a)_{k-1}} x_k z^{\alpha+k-1} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{\alpha^n (1-\beta)(c)_{k-1}}{(\alpha+k-1)^n (a)_{k-1}} y_k \overline{z^{\alpha+k-1}} \\ &= \sum_{k=2}^{\infty} h_k(z)^\alpha x_k + \sum_{k=1}^{\infty} g_{nk}(z)^\alpha y_k \\ &\quad + z^\alpha \left(1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \right) \\ &= \sum_{k=1}^{\infty} (h_k(z)^\alpha x_k + g_{nk}(z)^\alpha y_k), \end{aligned} \quad (84)$$

as required. \square

Our next result is on distortion bounds for the functions in the class $VB_n^\alpha H(\beta, a, c)$.

Theorem 16. Let $f_n^\alpha \in VB_n^\alpha H(\beta, a, c)$. Then, for $|z| = r < 1$, one has

$$\begin{aligned} |f_n(z)^\alpha| &\leq (1 + |b_1(\alpha)|) r^\alpha \\ &\quad + \left(\frac{\alpha}{\alpha+1} \right)^n \left(\frac{c(1-\beta)}{a} - \frac{1+\beta}{a} |b_1(\alpha)| \right) r^{\alpha+1}, \\ |f_n(z)^\alpha| &\geq (1 - |b_1(\alpha)|) r^\alpha \\ &\quad - \left(\frac{\alpha}{\alpha+1} \right)^n \left(\frac{c(1-\beta)}{a} - \frac{1+\beta}{a} |b_1(\alpha)| \right) r^{\alpha+1}. \end{aligned} \quad (85)$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted.

Let $f_n^\alpha \in VB_n^\alpha H(\beta, a, c)$. Taking the absolute value of f_n^α , we obtain

$$\begin{aligned} |f_n(z)^\alpha| &= \left| z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1} + (-1)^n \sum_{k=1}^{\infty} b_k(\alpha) \overline{z^{\alpha+k-1}} \right| \\ &\leq (1 + |b_1(\alpha)|) r^\alpha + \sum_{k=2}^{\infty} (|a_k(\alpha)| + |b_k(\alpha)|) r^{\alpha+k-1} \\ &\leq (1 + |b_1(\alpha)|) r^\alpha + r^{\alpha+1} \sum_{k=2}^{\infty} (|a_k(\alpha)| + |b_k(\alpha)|) \\ &\leq (1 + |b_1(\alpha)|) r^\alpha + \frac{\alpha^n (1-\beta) c}{(\alpha+1)^n a} \\ &\quad \times \left[\sum_{k=2}^{\infty} \frac{1}{1-\beta} \left(\frac{\alpha+1}{\alpha} \right)^n \frac{a}{c} |a_k(\alpha)| \right. \\ &\quad \left. + \frac{a}{c(1+\beta)} \left(\frac{\alpha+1}{\alpha} \right)^n |b_k(\alpha)| \right] r^{\alpha+1} \\ &\leq (1 + |b_1(\alpha)|) r^\alpha + \frac{\alpha^n (1-\beta) (c)}{(\alpha+1)^n (a)} \\ &\quad \times \left[\sum_{k=2}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |a_k(\alpha)| \right. \\ &\quad \left. + \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |b_k(\alpha)| \right] r^{\alpha+1} \\ &\leq (1 + |b_1(\alpha)|) r^\alpha \left(\frac{\alpha}{\alpha+1} \right)^n \\ &\quad \times \left[\frac{c(1-\beta)}{a} - \frac{c(1+\beta)}{a} |b_1(\alpha)| \right] r^{\alpha+1}, \end{aligned} \quad (86)$$

for $b_1(\alpha) < 1$. This shows that the bound given in Theorem 16 is sharp. \square

The following covering results follow from the left-hand inequality in Theorem 18.

Corollary 17. If function $f_n^\alpha = h^\alpha + \overline{g^\alpha}$, where h^α and g^α are as given earlier in $VB_n^\alpha H(\beta, a, c)$, then

$$\begin{aligned} \left\{ \omega : |\omega| \leq \left(\frac{\alpha}{\alpha+1} \right)^n \left(\frac{\alpha+1}{\alpha} \right)^n \right. \\ \left. - \left(\frac{\alpha}{\alpha+1} \right)^n \frac{c}{a} (1-\beta) - \left(\frac{\alpha}{\alpha+1} \right)^n \left(\frac{\alpha+1}{\alpha} \right)^n \right. \\ \left. - \left(\frac{\alpha}{\alpha+1} \right)^n \frac{c}{a} (1-\beta) |b_1(\alpha)| \right\} \subset f_n(U). \end{aligned} \quad (87)$$

For harmonic functions

$$\begin{aligned} f_n(z)^\alpha &= z^\alpha + \sum_{k=2}^{\infty} |a_k(\alpha)| z^{\alpha+k-1} + (-1)^n \sum_{k=1}^{\infty} |b_k(\alpha)| \overline{z^{\alpha+k-1}}, \\ F_n(z)^\alpha &= z^\alpha + \sum_{k=2}^{\infty} |A_k(\alpha)| z^{\alpha+k-1} + (-1)^n \sum_{k=1}^{\infty} |B_k(\alpha)| \overline{z^{\alpha+k-1}}. \end{aligned} \quad (88)$$

The convolution of f_n^α and F_n^α is given by

$$\begin{aligned} (f_n^\alpha * F_n^\alpha)(z) &= f_n(z)^\alpha * F_n(z)^\alpha \\ &= z^\alpha + |a_k(\alpha)| |A_k(\alpha)| z^{\alpha+k-1} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} |b_k(\alpha)| |B_k(\alpha)| \overline{z^{\alpha+k-1}}. \end{aligned} \quad (89)$$

Using this definition, one shows that the class $VB_n^\alpha H(\beta, a, c)$ is closed under convolution.

Theorem 18. For $0 \leq \lambda \leq \beta < 1$, let $f_n^\alpha \in VB_n^\alpha H(\beta, a, c)$, and let $F_n^\alpha \in VB_n^\alpha H(\beta, a, c)$. Then, $f_n^\alpha * F_n^\alpha \in VB_n^\alpha H(\beta, a, c) \subseteq VB_n^\alpha H(\beta, a, c)$.

Proof. Let the functions

$$\begin{aligned} f_n(z)^\alpha &= z^\alpha + \sum_{k=2}^{\infty} |a_k(\alpha)| z^{\alpha+k-1} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} |b_k(\alpha)| \overline{z^{\alpha+k-1}} \text{ be in } VB_n^\alpha H(\beta, a, c) \end{aligned} \quad (90)$$

and $F_n(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} |A_k(\alpha)| z^{\alpha+k-1} + (-1)^n \sum_{k=1}^{\infty} |B_k(\alpha)| \overline{z^{\alpha+k-1}}$ be in $VB_n^\alpha H(\beta, a, c)$. Then, the convolution $f_n^\alpha * F_n^\alpha$ is given by (89).

We wish to show that the coefficients of $f_n^\alpha * F_n^\alpha$ satisfy the required condition given in Theorem 14. For $F_n^\alpha \in VB_n^\alpha H(\beta, a, c)$, $A_k(\alpha) \leq 1$, and $|B_k(\alpha)| \leq 1$. Now, for the convolution function $(f_n^\alpha * F_n^\alpha)(z)$, we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |a_k(\alpha)| |A_k(\alpha)| \\ & + \sum_{k=1}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |b_k(\alpha)| |B_k(\alpha)| \\ & \leq \sum_{k=2}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |a_k(\alpha)| \\ & + \sum_{k=1}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |b_k(\alpha)| \\ & \leq 1. \end{aligned} \quad (91)$$

Therefore $(f_n^\alpha * F_n^\alpha)(z) \in VB_n^\alpha H(\beta, a, c) \subseteq VB_n^\alpha H(\beta, a, c)$.

Next, we show that the class $VB_n^\alpha H(\beta, a, c)$ is closed under convex combinations of its members.

Let functions $f_{nj}(z)^\alpha$ be defined, for $j = 1, 2, \dots$, by

$$\begin{aligned} f_{nj}(z)^\alpha &= z^\alpha + \sum_{k=2}^{\infty} |a_{k,j}(\alpha)| z^{\alpha+k-1} + (-1)^n \sum_{k=1}^{\infty} |b_{k,j}(\alpha)| z^{\alpha+k-1}. \end{aligned} \quad (92)$$

Theorem 19. Let the functions $f_{nj}(z)^\alpha$ defined by (92) be in the class $VB_n^\alpha H(\beta, a, c)$ for every $j = 1, 2, \dots, m$. Then, the functions $t_j(z)^\alpha$ defined by

$$t_j(z)^\alpha = \sum_{j=1}^m \gamma_j f_{nj}(z) \quad (0 \leq \gamma_j \leq 1) \quad (93)$$

are also in the class $VB_n^\alpha H(\beta, a, c)$, where $\sum_{j=1}^m \gamma_j = 1$.

Proof. According to the definition of t^α , we can write

$$\begin{aligned} t^\alpha &= z^\alpha + \sum_{k=2}^{\infty} \left(\sum_{j=1}^m \gamma_j a_{k,j}(\alpha) \right) z^{\alpha+k-1} \\ &+ (-1)^n \sum_{k=1}^{\infty} \left(\sum_{j=1}^m \gamma_j b_{k,j}(\alpha) \right) z^{\alpha+k-1}. \end{aligned} \quad (94)$$

Furthermore, since $f_{nj}(z)^\alpha$ are in $VB_n^\alpha H(\beta, a, c)$ for every $(j = 1, 2, \dots)$, then by (66), we have

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n \left(\sum_{j=1}^m \gamma_j |a_{k,j}(\alpha)| \right) \\ & + \sum_{k=1}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n \left(\sum_{j=1}^m \gamma_j |b_{k,j}(\alpha)| \right) \end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^m \gamma_j \left(\sum_{k=2}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |a_{k,j}(\alpha)| \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \frac{1}{1-\beta} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha} \right)^n |b_{k,j}(\alpha)| \right) \\ &\leq \sum_{j=1}^m \gamma_j = 1. \end{aligned} \quad (95)$$

Hence, the theorem is proved. \square

Corollary 20. The class $VB_n^\alpha H(\beta, a, c)$ is closed under convex linear combination.

Proof. Let the functions $f_{nj}(z)^\alpha$ ($j = 1, 2$) defined by (92) be in the class $MVB_n^\alpha H(\beta, a, c)$. Then, the function $\Psi(z)^\alpha$ defined by

$$\Psi(z)^\alpha = \mu f_{n1}(z)^\alpha + (1-\mu) f_{n2}(z)^\alpha \quad (0 \leq \mu \leq 1) \quad (96)$$

is in the class $MVB_n^\alpha H(\lambda, a, c)$. Also, by taking $m = 2$, $t_1 = \mu$ and $t_2 = (1-\mu)$ in Theorem 19, we have the above corollary. \square

With various special choices of the parameters involved, both the existing classes and new one could be derived.

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