Hindawi Publishing Corporation
ISRN Mathematical Analysis
Volume 2013, Article ID 271324, 4 pages
http://dx.doi.org/10.1155/2013/271324

Research Article

A Regularity Criterion for Compressible Nematic Liquid Crystal Flows

Jishan Fan[1] and Tohru Ozawa[2]

[1] Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China
[2] Department of Applied Physics, Waseda University, Tokyo 169-8555, Japan

Correspondence should be addressed to Tohru Ozawa; txozawa@waseda.jp

Received 2 September 2013; Accepted 26 September 2013

Academic Editors: Y. Liu and X. B. Pan

We prove a blow-up criterion for local strong solutions to a simplified hydrodynamic flow modeling the compressible, nematic liquid crystal materials in a bounded domain.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following simplified version of Ericksen-Leslie system modeling the hydrodynamic flow of compressible nematic liquid crystals:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$\begin{gathered}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u \\ -(\lambda+\mu)\nabla \operatorname{div} u = -\Delta d \cdot \nabla d,\end{gathered} \tag{2}$$

$$\begin{gathered}\partial_t d + u\cdot\nabla d = \Delta d + |\nabla d|^2 d, \\ |d| = 1 \quad \text{in } \Omega\times(0,\infty),\end{gathered} \tag{3}$$

$$u = 0, \quad d = d_0(x) \quad \text{on } \partial\Omega\times(0,\infty), \tag{4}$$

$$\begin{gathered}(\rho, u, d)(x,0) = (\rho_0, u_0, d_0)(x), \\ |d_0| = 1, \quad x\in\Omega\subset\mathbb{R}^3.\end{gathered} \tag{5}$$

Here ρ is the density of the fluid, u is the fluid velocity, d represents the macroscopic average of the nematic liquid crystal orientation field, and $p(\rho) := a\rho^\gamma$ is the pressure with positive constants $a > 0$ and $\gamma > 1$. Two real constants μ and λ are the shear viscosity and the bulk viscosity coefficients of the fluid, respectively, which are assumed to satisfy the following physical condition:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0. \tag{6}$$

Equations (1) and (2) are the well-known compressible Navier-Stokes system with the external force $-\Delta d\cdot\nabla d$. Equation (3) is the well-known heat flow of harmonic map when $u = 0$.

Recently, Huang et al. [1] prove the following local-in-time well-posedness.

Proposition 1. *Let $\rho_0 \in W^{1,q}$ for some $q\in(3,6]$ and $\rho_0 \geq 0$ in Ω, $u_0\in H^2$, $d_0\in H^3$ and $|d_0| = 1$ in Ω. If, in addition, the compatibility condition*

$$-\mu\Delta u_0 - (\lambda+\mu)\nabla\operatorname{div} u_0 - \nabla p(\rho_0) - \Delta d_0\cdot\nabla d_0 = \sqrt{\rho_0}g \quad \text{for some } g\in L^2(\Omega) \tag{7}$$

holds, then there exist $T_0 > 0$ and a unique strong solution (ρ, u, d) to the problem (1)–(5).

Based on the above proposition, Huang et al. [2] prove the regularity criterion

$$\int_0^T \left(\|\mathscr{D}(u)\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right) dt < \infty \tag{8}$$

to the problem (1)–(3), (5) with the boundary condition

$$u = 0 = \frac{\partial d}{\partial\nu} \quad \text{on } \partial\Omega\times(0,\infty) \tag{9}$$

or

$$u\cdot\nu = \operatorname{curl} u\times\nu = \frac{\partial d}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty). \tag{10}$$

Here,

$$\mathscr{D}(u) := \frac{1}{2}\left(\nabla u + {}^t\nabla u\right), \tag{11}$$

where ν is the unit outward normal vector to $\partial\Omega$.

When $\Omega = \mathbb{R}^3$, Huang and Wang [3] show the following regularity criterion:

$$\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^{s_1}(0,T;L^{r_1})} + \|\nabla d\|_{L^{s_2}(0,T;L^{r_2})} < \infty, \tag{12}$$

with r_i and s_i satisfying

$$\frac{2}{s_i} + \frac{3}{r_i} = 1, \quad 3 < r_i \le \infty,\ i = 1, 2. \tag{13}$$

When the term $|\nabla d|^2 d$ in (3) is replaced by $d - |d|^2 d$, the problem (1)–(5) has been studied by L. M. Liu and X. G. Liu [4]; they proved the following regularity criterion:

$$\int_0^T \left(\|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^\infty}\right) dt < \infty. \tag{14}$$

The aim of this paper is to study the regularity criterion of local strong solutions to the problem (1)-(5). We will prove

Theorem 2. *Let the assumptions in Proposition 1 hold true. If* (12) *holds true with* $0 < T < \infty$*, then the solution* (ρ, u, d) *can be extended beyond* $T > 0$.

Remark 3. Theorem 2 is also true for the boundary condition (9). But it is an open problem to prove (12) when the homogeneous Dirichlet boundary condition $u = 0$ is replaced by

$$u \cdot \nu = 0, \quad \operatorname{curl} u \times \nu = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{15}$$

2. Proof of Theorem 2

Since (ρ, u, d) is the local strong solution, we only need to prove a priori estimates.

First, testing (2) and (3) by u and $\Delta d + |\nabla d|^2 d$, respectively, and adding the resulting equations together, we see that

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|\nabla d|^2 + \frac{a\rho^\gamma}{\gamma - 1}\right)dx \\ &\quad + \int\left(\mu|\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 + \left|\Delta d + |\nabla d|^2 d\right|^2\right)dx = 0,\end{aligned} \tag{16}$$

which gives

$$\begin{aligned}&\int\left(\rho|u|^2 + |\nabla d|^2\right)dx \\ &\quad + \int_0^T\int\left(|\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|^2\right)dx\,dt \le C.\end{aligned} \tag{17}$$

We decompose the velocity u into two parts: $u = v + w$, where $v(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies

$$\mu\Delta v + (\lambda + \mu)\nabla \operatorname{div} v = \nabla p(\rho), \tag{18}$$

and thus $w(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies

$$\mu\Delta w + (\lambda + \mu)\nabla \operatorname{div} w = \rho\dot{u} + \Delta d \cdot \nabla d, \tag{19}$$

where we used $\dot{u} := \partial_t u + u \cdot \nabla u$ to denote the material derivative of u. Then, together with the standard $W^{1,p}$ theory and H^2 theory for elliptic systems, we obtain

$$\begin{aligned}&\|\nabla v\|_{L^6} \le C\|p(\rho)\|_{L^6}, \\ &\|\nabla w\|_{L^6} + \left\|\nabla^2 w\right\|_{L^2} \le C\|\rho\dot{u}\|_{L^2} + C\|\Delta d\nabla d\|_{L^2}.\end{aligned} \tag{20}$$

Testing (3) by $\Delta\partial_t d$ and using (4), (20), (3), and the identity $0 = \Delta(d\partial_t d) = d\Delta\partial_t d + \partial_t d\Delta d + 2\nabla d\partial_t d$, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\Delta d|^2 dx + \int|\nabla\partial_t d|^2 dx \\
&= \int\left(u\cdot\nabla d - |\nabla d|^2 d\right)\Delta\partial_t d\,dx \\
&= -\int\nabla(u\cdot\nabla d)\nabla\partial_t d\,dx - \int|\nabla d|^2(d\Delta\partial_t d)\,dx \\
&= -\int\left(u\nabla^2 d + \nabla u\cdot\nabla d\right)\nabla\partial_t d\,dx \\
&\quad + \int|\nabla d|^2(\partial_t d\Delta d + 2\nabla d\nabla\partial_t d)\,dx \\
&= -\int\left(u\cdot\nabla^2 d + \nabla u\cdot\nabla d\right)\nabla\partial_t d\,dx \\
&\quad + \int|\nabla d|^2\left(\Delta d + |\nabla d|^2 d - u\cdot\nabla d\right)\Delta d\,dx \\
&\quad + 2\int|\nabla d|^2\nabla d\nabla\partial_t d\,dx \\
&\le \|u\|_{L^{r_1}}\left\|\nabla^2 d\right\|_{L^{2r_1/(r_1-2)}}\|\nabla\partial_t d\|_{L^2} \\
&\quad + \|\nabla d\|_{L^{r_2}}\|\nabla u\|_{L^{2r_2/(r_2-2)}}\|\nabla\partial_t d\|_{L^2} \\
&\quad + C\|\nabla d\|_{L^{r_2}}^2\|\Delta d\|_{L^{2r_2/(r_2-2)}} + C\|u\|_{L^{r_1}}^2\|\Delta d\|_{L^{2r_1/(r_1-2)}}^2 \\
&\quad + \epsilon\|\nabla\partial_t d\|_{L^2}^2 \\
&\le C\epsilon\|\nabla\partial_t d\|_{L^2}^2 + C\|u\|_{L^{r_1}}^{s_1}\left\|\nabla^2 d\right\|_{L^2}^2 \\
&\quad + C\|\nabla d\|_{L^{r_2}}^{s_2}\|\nabla u\|_{L^2}^2 + \epsilon\|\nabla u\|_{L^6}^2 \\
&\quad + C\|\nabla d\|_{L^{r_2}}^{s_2}\|\Delta d\|_{L^2}^2 + \epsilon\|d\|_{H^3}^2 + C \\
&\le C\epsilon\|\nabla\partial_t d\|_{L^2}^2 + C\epsilon\|d\|_{H^3}^2 + C\epsilon\|\rho\dot{u}\|_{L^2}^2 \\
&\quad + C\|\nabla d\|_{L^{r_2}}^{s_2}\left(\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) \\
&\quad + C\|u\|_{L^{r_1}}^{s_1}\left\|\nabla^2 d\right\|_{L^2}^2 + C
\end{aligned} \tag{21}$$

for any $0 < \epsilon < 1$, where we have used the Hölder inequality

$$\|\nabla u\|_{L^{2r_1/(r_1-2)}} \le C\|\nabla u\|_{L^2}^{1-(3/r_1)}\|\nabla u\|_{L^6}^{3/r_1} \tag{22}$$

and the Gagliardo-Nirenberg inequality

$$\begin{aligned} \left\|\nabla^2 d\right\|_{L^{2r_2/(r_2-2)}} &\leq C\left\|\nabla^2 d\right\|_{L^2}^{1-(3/r_2)}\|d\|_{H^3}^{3/r_2}, \\ \|\nabla u\|_{L^6} &\leq \|\nabla v\|_{L^6}+\|\nabla w\|_{L^6} \\ &\leq C+\|\nabla w\|_{L^6}. \end{aligned} \tag{23}$$

By the H^3 theory of the elliptic equations, it follows from (3) that

$$\begin{aligned} \|d\|_{H^3} &\leq C\left(1+\|\nabla \Delta d\|_{L^2}\right) \\ &\leq C\left(1+\left\|\nabla\left(\partial_t d+u\cdot\nabla d-|\nabla d|^2 d\right)\right\|_{L^2}\right) \\ &\leq C\left(1+\left\|\nabla \partial_t d\right\|_{L^2}+\|u\|_{L^{r_1}}\left\|\nabla^2 d\right\|_{L^{2r_1/(r_1-2)}}\right. \\ &\quad \left.+\|\nabla d\|_{L^{r_2}}\|\nabla u\|_{L^{2r_2/(r_2-2)}}+\|\nabla d\|_{L^{r_2}}\left\|\nabla^2 d\right\|_{L^{2r_2/(r_2-2)}}\right) \\ &\leq C\left(1+\left\|\nabla \partial_t d\right\|_{L^2}+\epsilon\|d\|_{H^3}\right. \\ &\quad +\|u\|_{L^{r_1}}^{s_1/2}\left\|\nabla^2 d\right\|_{L^2}+\epsilon\|\nabla u\|_{L^6} \\ &\quad \left.+\|\nabla d\|_{L^{r_2}}^{s_2/2}\|\nabla u\|_{L^2}+\|\nabla d\|_{L^{r_2}}^{s_2/2}\|\Delta d\|_{L^2}\right), \end{aligned} \tag{24}$$

which yields

$$\begin{aligned} \|d\|_{H^3} &\leq C\left(1+\left\|\nabla \partial_t d\right\|_{L^2}+\|u\|_{L^{r_1}}^{s_1/2}\left\|\nabla^2 d\right\|_{L^2}\right. \\ &\quad \left.+\|\nabla d\|_{L^{r_2}}^{s_2/2}\|\nabla u\|_{L^2}+\|\nabla d\|_{L^{r_2}}^{s_2/2}\left\|\nabla^2 d\right\|_{L^2}\right). \end{aligned} \tag{25}$$

Testing (2) by $\partial_t u$ and setting $M(d) := \nabla d \odot \nabla d - (1/2)|\nabla d|^2 \mathbb{I}_3$, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div} u)^2\right)dx+\int \rho|\dot{u}|^2 dx \\ &\quad -\frac{d}{dt}\int p\operatorname{div} u\,dx-\frac{d}{dt}\int M(d):\nabla u\,dx \\ &= \int \rho\dot{u}\cdot(u\cdot\nabla u)\,dx-\int \partial_t p\operatorname{div} u\,dx \\ &\quad -\int \partial_t M(d):\nabla u\,dx \\ &\leq \left\|\rho\dot{u}\right\|_{L^2}\|u\|_{L^{r_1}}\|\nabla u\|_{L^{2r_1/(r_1-2)}} \\ &\quad +C\|\nabla d\|_{L^{r_2}}\|\nabla u\|_{L^{2r_2/(r_2-2)}}\left\|\nabla\partial_t d\right\|_{L^2}-\int p_t\operatorname{div} u\,dx. \end{aligned} \tag{26}$$

Now we deal with the last term.

First, (1) implies that

$$\partial_t p+\operatorname{div}(pu)+(\gamma-1)p\operatorname{div} u=0. \tag{27}$$

Using (27) and (20), we have

$$\begin{aligned} -\int \partial_t p\operatorname{div} u\,dx &= -\int \partial_t p\operatorname{div} v\,dx-\int \partial_t p\operatorname{div} w\,dx \\ &= \int \nabla\partial_t p v\,dx+\int \operatorname{div}(pu)\operatorname{div} w\,dx \\ &\quad +(\gamma-1)\int p\operatorname{div} u\operatorname{div} w\,dx \\ &= -\frac{d}{dt}\int\left(\frac{\mu}{2}|\nabla v|^2+\frac{\lambda+\mu}{2}(\operatorname{div} v)^2\right)dx \\ &\quad -\int pu\nabla\operatorname{div} w\,dx \\ &\quad +(\gamma-1)\int p\operatorname{div} u\operatorname{div} w\,dx \\ &\leq -\frac{d}{dt}\int\left(\frac{\mu}{2}|\nabla u|^2+\frac{\lambda+\mu}{2}(\operatorname{div} v)^2\right)dx \\ &\quad +C\left\|\sqrt{\rho}u\right\|_{L^2}\|\nabla\operatorname{div} w\|_{L^2} \\ &\quad +C\|\operatorname{div} u\|_{L^2}\|\operatorname{div} w\|_{L^2}. \end{aligned} \tag{28}$$

Inserting (28) into (26) and using (20), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div} u)^2\right)dx \\ &\quad +\frac{d}{dt}\int\left(\frac{\mu}{2}|\nabla v|^2+\frac{\lambda+\mu}{2}(\operatorname{div} v)^2\right)dx \\ &\quad -\frac{d}{dt}\int p\operatorname{div} u\,dx-\frac{d}{dt}\int M(d):\nabla u\,dx \\ &\quad +\int \rho|\dot{u}|^2 dx \\ &\leq \left\|\rho\dot{u}\right\|_{L^2}\|u\|_{L^{r_1}}\|\nabla u\|_{L^{2r_1/(r_1-2)}} \\ &\quad +C\|\nabla d\|_{L^{r_2}}\|\nabla u\|_{L^{2r_2/(r_2-2)}}\left\|\nabla\partial_t d\right\|_{L^2} \\ &\quad +C\|\nabla\operatorname{div} w\|_{L^2}+C\|\operatorname{div} u\|_{L^2}^2+C \\ &\leq \epsilon\int \rho|\dot{u}|^2 dx+C\|u\|_{L^{r_1}}^{s_1}\|\nabla u\|_{L^2}^2+\epsilon\|\nabla u\|_{L^6}^2 \\ &\quad +C\|\nabla d\|_{L^{r_2}}^{s_2}\|\nabla u\|_{L^2}^2+\epsilon\left\|\nabla\partial_t d\right\|_{L^2}^2 \\ &\quad +C\|\nabla d\|_{L^{r_2}}^{s_2}\|\Delta d\|_{L^2}^2+C\|\operatorname{div} u\|_{L^2}^2+C \\ &\leq C\epsilon\int \rho|\dot{u}|^2 dx+C\epsilon\left\|\nabla\partial_t d\right\|_{L^2}^2+C\|u\|_{L^{r_1}}^{s_1}\|\nabla u\|_{L^2}^2 \\ &\quad +C\|\nabla d\|_{L^{r_2}}^{s_2}\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+C. \end{aligned} \tag{29}$$

Combining (21), (25), and (29), taking ϵ small enough, and using the Gronwall inequality, we conclude that

$$\begin{aligned} &\|u\|_{L^\infty(0,T;H^1)}+\|d\|_{L^\infty(0,T;H^2)}+\|d\|_{L^2(0,T;H^3)} \\ &\quad +\left\|\sqrt{\rho}\dot{u}\right\|_{L^2(0,T;L^2)}\leq C. \end{aligned} \tag{30}$$

Now by the same calculations as those in [3, 5], we prove that

$$\begin{aligned}
\|\rho\|_{L^\infty(0,T;W^{1,q})} + \|\partial_t\rho\|_{L^\infty(0,T;L^q)} &\le C, \\
\|\sqrt{\rho}\partial_t u\|_{L^\infty(0,T;L^2)} + \|\partial_t u\|_{L^2(0,T;H^1)} &\le C, \\
\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;W^{2,q})} &\le C, \\
\|d\|_{L^\infty(0,T;H^3)} &\le C, \\
\|\partial_t d\|_{L^\infty(0,T;H^1)} &\le C.
\end{aligned} \tag{31}$$

This completes the proof.

Acknowledgment

This work is partially supported by NSFC (no. 11171154).

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