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Research Article

Norm of a Volterra Integral Operator on Some Analytic Function Spaces

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Let f be an analytic function in the unit disc $\mathbb{D}$. The Volterra integral operator I_f is defined as follows: $I_f(h)(z) = \int_0^z f(w)h'(w)dw, h \in H(\mathbb{D}), z \in \mathbb{D}$. In this paper, we compute the norm of I_f on some analytic function spaces.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. Denote by $d\sigma$ the normalized Lebesgue area measure in $\mathbb{D}$ and $g(a, z)$ the Green function with logarithmic singularity at a; that is, $g(a, z) = -\log|\varphi_a(z)|$, where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ is the Möbius transformation of $\mathbb{D}$.

Let $0 < p < \infty$. The Q_p is the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{Q_p}^2 = |f(0)|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1 - |\varphi_a(z)|^2\right)^p d\sigma(z) < \infty. \tag{1}$$

From [1, 2], we see that $Q_1 = \mathrm{BMOA}$, the space of all analytic functions of bounded mean oscillation. When $p > 1$, the space Q_p is the same and equal to the Bloch space $\mathfrak{B}$, which consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathfrak{B}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)| \left(1 - |z|^2\right) < \infty. \tag{2}$$

See [3, 4] for the theory of Bloch functions.

For $\alpha > 0$, the α-Bloch space, denoted by $\mathfrak{B}^\alpha$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathfrak{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)| \left(1 - |z|^2\right)^\alpha < \infty. \tag{3}$$

It is clear that $\mathfrak{B}^{\alpha_1} \subsetneqq \mathfrak{B} \subsetneqq \mathfrak{B}^{\alpha_2}$ for $0 < \alpha_1 < 1 < \alpha_2 < \infty$.

Let $1 \leqslant q \leqslant \infty$ and let $0 \leqslant \alpha \leqslant 1$. The mean Lipschitz space $\Lambda(q, \alpha)$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\Lambda(q,\alpha)} = |f(0)| + \sup_{0\leqslant r<1} \left(1 - r^2\right)^{1-\alpha} \times \left(\frac{1}{2\pi} \int_0^{2\pi} \left|f'\left(re^{i\varphi}\right)\right|^q d\varphi\right)^{1/q} < \infty. \tag{4}$$

It is obvious that $\Lambda(\infty, 0)$ is just the Bloch space $\mathfrak{B}$, which is contained in $\Lambda(q, 0)$ for every $1 < q < \infty$. Note that $\Lambda(q, 1/q)$ increases with $q \in (1, \infty)$. We refer to [5] for more information of mean Lipschitz spaces.

For $0 \leqslant s < \infty$, we say that an $f \in H(\mathbb{D})$ belongs to the growth space H_s^∞ if

$$\|f\|_{H_s^\infty} = \sup_{0\leqslant r<1} \left|f\left(re^{i\theta}\right)\right| \left(1 - r^2\right)^s < \infty. \tag{5}$$

It is easy to see that $H_0^\infty = H^\infty$.

For $-1 < \alpha < \infty$, an $f \in H(\mathbb{D})$ is said to belong to the $\mathscr{D}^{\alpha}$ space if

$$\|f\|_{\mathscr{D}^{\alpha}}^{2} = |f(0)|^{2} + \int_{\mathbb{D}} |f'(z)|^{2} (1-|z|^{2})^{\alpha} d\sigma(z) < \infty. \quad (6)$$

For $1 < p < \infty$, the Besov space $\mathscr{B}_p$ is defined to be the space of all analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{\mathscr{B}_p}^{p} = |f(0)|^{p} + \int_{\mathbb{D}} |f'(z)|^{p} (1-|z|^{2})^{p-2} d\sigma(z) < \infty. \quad (7)$$

Let $f \in H(\mathbb{D})$. The Volterra integral operators I_f and J_f are defined as follows:

$$I_f(h)(z) = \int_0^z h'(w) f(w)\, dw,$$
$$J_f(h)(z) = \int_0^z h(w) f'(w)\, dw, \quad (8)$$
$$(z \in \mathbb{D}).$$

It is easy to see that

$$(I_f + J_f) h + f(0) h(0) = M_f(h), \quad (9)$$

where M_f denotes the multiplication operator; that is, $M_f(h) = fh$. If f is a constant, then all results about I_f, J_f, or M_f are trivial. In this paper, we assume that f is a nonconstant. Both operators have been studied by many authors. See [6–23] and the references therein.

Norms of some special operators, such as composition operator, weighted composition operator, and some integral operators, have been studied by many authors. The interested readers can refer [13, 24–32], for example. Recently, Liu and Xiong studied the norm of integral operators I_f and J_f on the Bloch space, Dirichlet space, BMOA space, and so on in [13]. In this paper, we study the norm of integral operator I_f on some function spaces in the unit disk.

2. Main Results

In this section, we state and prove our main results. In order to formulate our main results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 1 (see [5, page 144]). *If $f \in H^p$ $(0 < p \leqslant \infty)$, then $|f(z)| \leqslant (1-|z|^2)^{-1/p}\|f\|_p$, $|z| < 1$, and the inequality is sharp for each fixed z.*

Lemma 2. *Let $-1 < \alpha < \infty$ and $0 < p < \infty$. For any $f \in H(\mathbb{D})$, the following one has:*

$$|f(a)|^{p} \leqslant (\alpha+1) \int_{\mathbb{D}} |f(z)|^{p} \frac{(1-|a|^{2})^{2+\alpha}}{|1-\bar{a}z|^{4+2\alpha}} (1-|z|^{2})^{\alpha} d\sigma(z), \quad (10)$$

where a is any point in $\mathbb{D}$.

Proof. For any $f \in H(\mathbb{D})$, taking $z = re^{i\theta}$ and the subharmonicity of $|f(z)|^p$, we get

$$|f(0)|^{p} \leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{p} d\theta, \quad (11)$$

and so

$$\begin{aligned} |f(0)|^{p} &\leqslant (\alpha+1) \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^{p} \\ &\qquad \times (1-r^{2})^{\alpha} d\theta r\, dr \\ &= (\alpha+1) \int_{\mathbb{D}} |f(z)|^{p} (1-|z|^{2})^{\alpha} d\sigma(z). \end{aligned} \quad (12)$$

For any $a \in \mathbb{D}$, let $\varphi_a(z) = (a-z)/(1-\bar{a}z)$. Replacing f by $f \circ \varphi_a(z)$ and applying the change of variable formula give the following:

$$\begin{aligned} |f(a)|^{p} &\leqslant (\alpha+1) \int_{\mathbb{D}} |f \circ \varphi_a(z)|^{p} (1-|z|^{2})^{\alpha} d\sigma(z) \\ &= (\alpha+1) \int_{\mathbb{D}} |f(z)|^{p} (1-|\varphi_a(z)|^{2})^{\alpha} \frac{(1-|a|^{2})^{2}}{|1-\bar{a}z|^{4}} d\sigma(z) \\ &= (\alpha+1) \int_{\mathbb{D}} |f(z)|^{p} (1-|a|^{2})^{2+\alpha} \frac{(1-|z|^{2})^{\alpha}}{|1-\bar{a}z|^{2\alpha+4}} d\sigma(z). \end{aligned} \quad (13)$$

The proof is complete. □

Theorem 3. *Let $f \in H(\mathbb{D})$. The integral operator I_f is bounded on $\Lambda(1,1)$ if and only if $f \in H^{\infty}$. Moreover, one has*

$$\|I_f\| = \|f\|_{H^{\infty}}. \quad (14)$$

Proof. If $f \in H^{\infty}$, by (4), we have

$$\begin{aligned} \|I_f h\|_{\Lambda(1,1)} &= \sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi}) h'(re^{i\varphi})| d\varphi \right) \\ &\leqslant \|f\|_{H^{\infty}} \sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |h'(re^{i\varphi})| d\varphi \right) \\ &\leqslant \|f\|_{H^{\infty}} \|h\|_{\Lambda(1,1)}. \end{aligned} \quad (15)$$

Thus $\| I_f \| \leqslant \| f\|_{H^{\infty}}$.

On the other hand, denote $c = \sup_{z \in \mathbb{D}} |f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)| > c - \epsilon$. Let

$$h(z) = \frac{z_1 - z}{1 - \bar{z}_1 z} - z_1. \quad (16)$$

Then we have $\|h\|_{\Lambda(1,1)} = 1$. In fact, taking $z_1 = r_1 e^{i\varphi_1}$ and $z = re^{i\varphi}$ and using Poisson integral, we get

$$\begin{aligned}\|h\|_{\Lambda(1,1)} &= \sup_{0\leqslant r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|h'\left(re^{i\varphi}\right)\right|d\varphi\right)\\ &= \sup_{0\leqslant r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r_1^2}{\left|1-rr_1e^{i(\varphi-\varphi_1)}\right|^2}d\varphi\right)\\ &= \sup_{0\leqslant r<1}\frac{1-r_1^2}{1-r^2r_1^2} = 1.\end{aligned} \tag{17}$$

Taking $z_1 = r_1 e^{i\varphi_1}$, we obtain

$$\begin{aligned}\infty > \left\|I_f\right\| &\geqslant \left\|I_f h\right\|_{\Lambda(1,1)}\\ &= \sup_{0\leqslant r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\varphi}\right)h'\left(re^{i\varphi}\right)\right|d\varphi\right)\\ &= \sup_{0\leqslant r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\varphi}\right)\right|\right.\\ &\qquad\left.\times\frac{1-r_1^2}{\left|1-r_1re^{i(\varphi-\varphi_1)}\right|^2}d\varphi\right).\end{aligned} \tag{18}$$

So $fh' \in H^1$ and $f \in H^1$. Thus Theorem 2.6 in [5] yields

$$\left\|I_f\right\| \geqslant \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{i\varphi}\right)\right|\frac{1-r_1^2}{\left|1-r_1e^{i(\varphi-\varphi_1)}\right|^2}d\varphi\right). \tag{19}$$

By Theorem 2.12 in [5], we have

$$\left\|I_f\right\| \geqslant \left|f\left(z_1\right)\right| > c-\epsilon, \tag{20}$$

so the arbitrariness of ϵ gives $\|I_f\| \geqslant \|f\|_{H^\infty}$ and the proof is complete. □

Lemma 3 in [13] gives the norm of I_f on Dirichlet space. Here, we consider the norm of I_f on α-Dirichlet space $\mathscr{D}^\alpha$.

Theorem 4. *Let $f \in H(\mathbb{D})$ and $-1 < \alpha < \infty$. Then I_f is bounded on $\mathscr{D}^\alpha$ if and only if $f \in H^\infty$. Moreover, one has*

$$\left\|I_f\right\| = \|f\|_{H^\infty}. \tag{21}$$

Proof. First, we assume that $f \in H^\infty$. Let $h \in \mathscr{D}^\alpha$. Then (6) gives

$$\begin{aligned}\left\|I_f h\right\|_{\mathscr{D}^\alpha}^2 &= \int_{\mathbb{D}}\left|f(z)h'(z)\right|^2\left(1-|z|^2\right)^\alpha d\sigma(z)\\ &\leqslant \|f\|_{H^\infty}^2\int_{\mathbb{D}}\left|h'(z)\right|^2\left(1-|z|^2\right)^\alpha d\sigma(z)\\ &\leqslant \|f\|_{H^\infty}^2\|h\|_{\mathscr{D}^\alpha}^2,\end{aligned} \tag{22}$$

and so we have $\|I_f\| \leqslant \|f\|_{H^\infty}$.

Now we need only to show the reverse inequality. Denote $c = \sup_{z\in\mathbb{D}}|f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)| > c-\epsilon$. Let

$$h_1(z) = \int_{\Gamma(z)}\frac{\left(1-\left|z_1\right|^2\right)^{1+(\alpha/2)}}{\left(1-\overline{z}_1\zeta\right)^{2+\alpha}}d\zeta, \tag{23}$$

where $\Gamma(z)$ is any path in $\mathbb{D}$ from 0 to z. By Theorem 13.11 in [33, page 274], we know h_1 is an analytic function in $\mathbb{D}$ and $h_1'(z) = (1-|z_1|^2)^{1+(\alpha/2)}/(1-\overline{z}_1 z)^{2+\alpha}$. Also it is easy to check that $\|h_1\|_{\mathscr{D}^\alpha}^2 = 1/(\alpha+1)$. Indeed, by using the method of the proof of Lemma 4.2.2 in [4], we have

$$\begin{aligned}\left\|h_1\right\|_{\mathscr{D}^\alpha}^2 &= \int_{\mathbb{D}}\frac{\left(1-\left|z_1\right|^2\right)^{2+\alpha}}{\left|1-\overline{z}_1 z\right|^{4+2\alpha}}\left(1-|z|^2\right)^\alpha d\sigma(z)\\ &= \left(1-\left|z_1\right|^2\right)^{2+\alpha}\int_{\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha}{\left|1-\overline{z}_1 z\right|^{4+2\alpha}}d\sigma(z)\\ &= \left(1-\left|z_1\right|^2\right)^{2+\alpha}\frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha+2)}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+2+\alpha)}{n!\Gamma(n+2+\alpha)}\left|z_1\right|^{2n}\\ &= \frac{\left(1-\left|z_1\right|^2\right)^{2+\alpha}}{\alpha+1}\sum_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(\alpha+2)}\left|z_1\right|^{2n}\\ &= \frac{\left(1-\left|z_1\right|^2\right)^{2+\alpha}}{\alpha+1}\frac{1}{\left(1-\left|z_1\right|^2\right)^{2+\alpha}} = \frac{1}{\alpha+1}.\end{aligned} \tag{24}$$

Let $h(z) = h_1(z)/\|h_1\|_{\mathscr{D}^\alpha}$, and so $\|h\|_{\mathscr{D}^\alpha} = 1$. Thus by Lemma 2 we have

$$\begin{aligned}\left\|I_f\right\|^2 &\geq \left\|I_f h(z)\right\|_{\mathscr{D}^\alpha}^2\\ &= \int_{\mathbb{D}}\left|f(z)h'(z)\right|^2\left(1-|z|^2\right)^\alpha d\sigma(z)\\ &= (\alpha+1)\int_{\mathbb{D}}|f(z)|^2\frac{\left(1-\left|z_1\right|^2\right)^{2+\alpha}}{\left|1-\overline{z}_1 z\right|^{4+2\alpha}}\left(1-|z|^2\right)^\alpha d\sigma(z)\\ &\geqslant \left|f\left(z_1\right)\right|^2 > (c-\epsilon)^2.\end{aligned} \tag{25}$$

Since ϵ is arbitrary, we get

$$\left\|I_f\right\| \geqslant \|f\|_{H^\infty}, \tag{26}$$

which implies the desired result. □

Theorem 5. *Let $f \in H(\mathbb{D})$ and let $1 < p < \infty$. The integral operator I_f is bounded on $\mathscr{B}_p$ if and only if $f \in H^\infty$. Moreover, one has*

$$\left\|I_f\right\| = \|f\|_{H^\infty}. \tag{27}$$

Proof. If $f \in H^\infty$, then by (7), we have

$$\begin{aligned}\left\|I_f h\right\|_{\mathscr{B}_p}^p &= \int_{\mathbb{D}} \left|f(z) h'(z)\right|^p \left(1-|z|^2\right)^{p-2} d\sigma(z) \\ &\leqslant \|f\|_{H^\infty}^p \int_{\mathbb{D}} \left|h'(z)\right|^p \left(1-|z|^2\right)^{p-2} d\sigma(z) \\ &\leqslant \|f\|_{H^\infty}^p \|h\|_{\mathscr{B}_p}^p,\end{aligned} \tag{28}$$

and so $\|I_f\| \leqslant \|f\|_{H^\infty}$.

Now we need only to show the reverse inequality. Denote $c = \sup_{z\in\mathbb{D}}|f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)| > c - \epsilon$. Let

$$h_1(z) = \frac{z_1 - z}{1 - z_1 z} - z_1, \quad z \in \mathbb{D}. \tag{29}$$

We see that $\|h_1\|_{\mathscr{B}_p}^p = 1/(p-1)$. Indeed,

$$\begin{aligned}\|h_1\|_{\mathscr{B}_p}^p &= \int_{\mathbb{D}} \frac{\left(1-|z_1|^2\right)^p}{\left|1-\bar{z}_1 z\right|^{2p}} \left(1-|z|^2\right)^{p-2} d\sigma(z) \\ &= \left(1-|z_1|^2\right)^p \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{p-2}}{\left|1-\bar{z}_1 z\right|^{2p}} d\sigma(z) \\ &= \left(1-|z_1|^2\right)^p \frac{\Gamma(p-1)}{\Gamma^2(p)} \sum_{n=0}^{\infty} \frac{\Gamma(n+p)^2}{n!\Gamma(n+p)} |z_1|^{2n} \\ &= \frac{\left(1-|z_1|^2\right)^p}{p-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+p)}{n!\Gamma(p)} |z_1|^{2n} \\ &= \frac{\left(1-|z_1|^2\right)^p}{p-1} \frac{1}{\left(1-|z_1|^2\right)^p} = \frac{1}{p-1}.\end{aligned} \tag{30}$$

Let $h(z) = h_1(z)/\|h_1\|_{\mathscr{B}_p}$. Then $\|h\|_{\mathscr{B}_p} = 1$. Thus by Lemma 2, we have

$$\begin{aligned}\left\|I_f\right\|^p \geq \left\|I_f h(z)\right\|_{\mathscr{B}_p}^p &= \int_{\mathbb{D}} \left|f(z) h'(z)\right|^p \left(1-|z|^2\right)^{p-2} d\sigma(z) \\ &= (p-1) \int_{\mathbb{D}} |f(z)|^p \frac{\left(1-|z_1|^2\right)^p}{\left|1-\bar{z}_1 z\right|^{2p}} \\ &\quad \times \left(1-|z|^2\right)^{p-2} d\sigma(z) \\ &\geqslant \left|f(z_1)\right|^p > (c-\epsilon)^p.\end{aligned} \tag{31}$$

Since ϵ is arbitrary, we get $\|I_f\| \geqslant \|f\|_{H^\infty}$. The proof is complete. □

Theorem 6. *Let $f \in H(\mathbb{D})$ and let $0 < \alpha \leqslant \beta < \infty$. The integral operator I_f is bounded from $\mathfrak{B}^\alpha$ to $\mathfrak{B}^\beta$ if and only if $f \in H_{\beta-\alpha}^\infty$. Moreover, one has*

$$\left\|I_f\right\| = \sup_{z\in\mathbb{D}} |f(z)| \left(1-|z|^2\right)^{\beta-\alpha}. \tag{32}$$

Proof. If $f \in H_{\beta-\alpha}^\infty$, then by (3), we have

$$\begin{aligned}\left\|I_f h\right\|_{\mathfrak{B}^\beta} &= \sup_{z\in\mathbb{D}} \left|f(z) h'(z)\right| \left(1-|z|^2\right)^\beta \\ &\leqslant \|f\|_{H_{\beta-\alpha}^\infty} \sup_{z\in\mathbb{D}} \left|h'(z)\right| \left(1-|z|^2\right)^\alpha \\ &\leqslant \|f\|_{H_{\beta-\alpha}^\infty} \|h\|_{\mathfrak{B}^\alpha}.\end{aligned} \tag{33}$$

Hence $\| I_f \| \leqslant \| f\|_{H_{\beta-\alpha}^\infty}$.

For the converse, denote $c = \sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^{\beta-\alpha}$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)|(1 - |z_1|^2)^{\beta-\alpha} > c - \epsilon$. Set

$$h(z) = \int_{\Gamma(z)} \frac{\left(1-|z_1|^2\right)^\alpha}{\left(1-\bar{z}_1\zeta\right)^{2\alpha}} d\zeta, \tag{34}$$

where $\Gamma(z)$ is any path in $\mathbb{D}$ from 0 to z. By Theorem 13.11 in [33, page 274], we know that h is an analytic function in $\mathbb{D}$ and $h\prime(z) = (1-|z_1|^2)^\alpha/(1-\bar{z}_1 z)^{2\alpha}$, and it is easy to check that $\|h\|_{\mathfrak{B}^\alpha} = 1$. Thus

$$\begin{aligned}\left\|I_f\right\| &\geqslant \left\|I_f h\right\|_{\mathfrak{B}^\beta} \\ &= \sup_{z\in\mathbb{D}} \left|f(z) h'(z)\right| \left(1-|z|^2\right)^\beta \\ &\geqslant \left|f(z_1) h'(z_1)\right| \left(1-|z_1|^2\right)^\beta \\ &\geqslant \left|f(z_1)\right| \frac{\left(1-|z_1|^2\right)^\alpha}{\left(1-|z_1|^2\right)^{2\alpha}} \left(1-|z_1|^2\right)^\beta \\ &= \left|f(z_1)\right| \left(1-|z_1|^2\right)^{\beta-\alpha} > c-\epsilon.\end{aligned} \tag{35}$$

Since ϵ is arbitrary, we obtain the desired result. The proof is complete. □

Theorem 7. *Let $f \in H(\mathbb{D})$. The integral operator I_f is bounded from $\Lambda(1,1)$ to $\mathfrak{B}$ if and only if $f \in H^\infty$. Moreover, one has*

$$\left\|I_f\right\| = \|f\|_{H^\infty}. \tag{36}$$

Proof. If $f \in H^\infty$, then by Lemma 1, we have

$$\begin{aligned}\left\|I_f h\right\|_{\mathfrak{B}} &= \sup_{z\in\mathbb{D}} \left|f(z) h'(z)\right| \left(1-|z|^2\right) \\ &\leqslant \|h\|_{\Lambda(1,1)} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{-1} \left(1-|z|^2\right) |f(z)| \\ &= \|h\|_{\Lambda(1,1)} \sup_{z\in\mathbb{D}} |f(z)|,\end{aligned} \tag{37}$$

and hence $\|I_f\| \leqslant \|f\|_{H^\infty}$.

For the converse, denote $c = \sup_{z\in\mathbb{D}}|f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)| > c - \epsilon$. Let

$$h(z) = \frac{z_1 - z}{1 - z_1 z} - z_1, \quad z \in \mathbb{D}. \tag{38}$$

Then by the proof of Theorem 3, we see that $\|h\|_{\Lambda(1,1)} = 1$. In the meantime, we know that $|h'(z_1)|(1 - |z_1|^2) = 1$, which gives

$$\begin{aligned}\|I_f\| &\geqslant \|I_f h\|_{\mathfrak{B}} \\ &= \sup_{z\in\mathbb{D}} |f(z)h'(z)|(1-|z|^2) \\ &\geqslant |f(z_1)h'(z_1)|(1-|z_1|^2) \\ &= |f(z_1)| > c - \epsilon.\end{aligned} \tag{39}$$

Since ϵ is arbitrary, we get the desired result. The proof is complete. □

Finally, we consider the norm of I_f from $\Lambda(\infty, 1)$ to some Banach spaces.

Theorem 8. *If $f \in H(\mathbb{D})$, then the following assertions hold.*

(1) *Let $0 < p < \infty$. The integral operator I_f is bounded from $\Lambda(\infty, 1)$ space to Q_p space if and only if f satisfies*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^p d\sigma(z) < \infty. \tag{40}$$

Moreover, one has

$$\|I_f\| = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^p d\sigma(z). \tag{41}$$

(2) *Let $0 \leqslant \alpha \leqslant 1$ and let $0 \leqslant q \leqslant \infty$. The integral operator I_f is bounded from $\Lambda(\infty, 1)$ space to $\Lambda(q, \alpha)$ space if and only if f satisfies*

$$\sup_{0\leqslant r<1} (1 - r^2)^{1-\alpha} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^q d\varphi \right)^{1/q} < \infty. \tag{42}$$

Moreover, one has

$$\|I_f\| = \sup_{0\leqslant r<1} (1 - r^2)^{1-\alpha} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^q d\varphi \right)^{1/q}. \tag{43}$$

(3) *Let $0 < \alpha < \infty$. The integral operator I_f is bounded from $\Lambda(\infty, 1)$ space to $\mathfrak{B}^\alpha$ space if and only if f satisfies $\sup_{z\in\mathbb{D}} |f(z)|(1 - |z|^2)^\alpha < \infty$. Moreover, one has*

$$\|I_f\| = \sup_{z\in\mathbb{D}} |f(z)| (1 - |z|^2)^\alpha. \tag{44}$$

(4) *Let $-1 < \alpha < \infty$. The integral operator I_f is bounded from $\Lambda(\infty, 1)$ space to $\mathcal{D}^\alpha$ space if and only if f satisfies $\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha d\sigma(z) < \infty$. Moreover, one has*

$$\|I_f\| = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha d\sigma(z). \tag{45}$$

Proof. The assertion (1) will be proved only here, and the conclusions of (2), (3), and (4) follow by using the similar arguments to that used in proving (1), and so the proofs are omitted.

If $h \in \Lambda(\infty, 1)$, then by (1), we have

$$\begin{aligned}\|I_f h\|_{Q_p}^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)h'(z)|^2 (1 - |\varphi_a(z)|^2)^p d\sigma(z) \\ &\leqslant \|h\|_{\Lambda(\infty,1)}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^p d\sigma(z),\end{aligned} \tag{46}$$

and so

$$\|I_f\| \leqslant \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^p d\sigma(z). \tag{47}$$

For the converse, let $h(z) = z$. It is easy to see that $\|h\|_{\Lambda(\infty,1)} = 1$. Thus

$$\begin{aligned}\|I_f\| &\geqslant \|I_f h\|_{Q_p}^2 \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)h'(z)|^2 (1 - |\varphi_a(z)|^2)^p d\sigma(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^p d\sigma(z).\end{aligned} \tag{48}$$

The desired result follows by (47) and (48). The proof is complete. □

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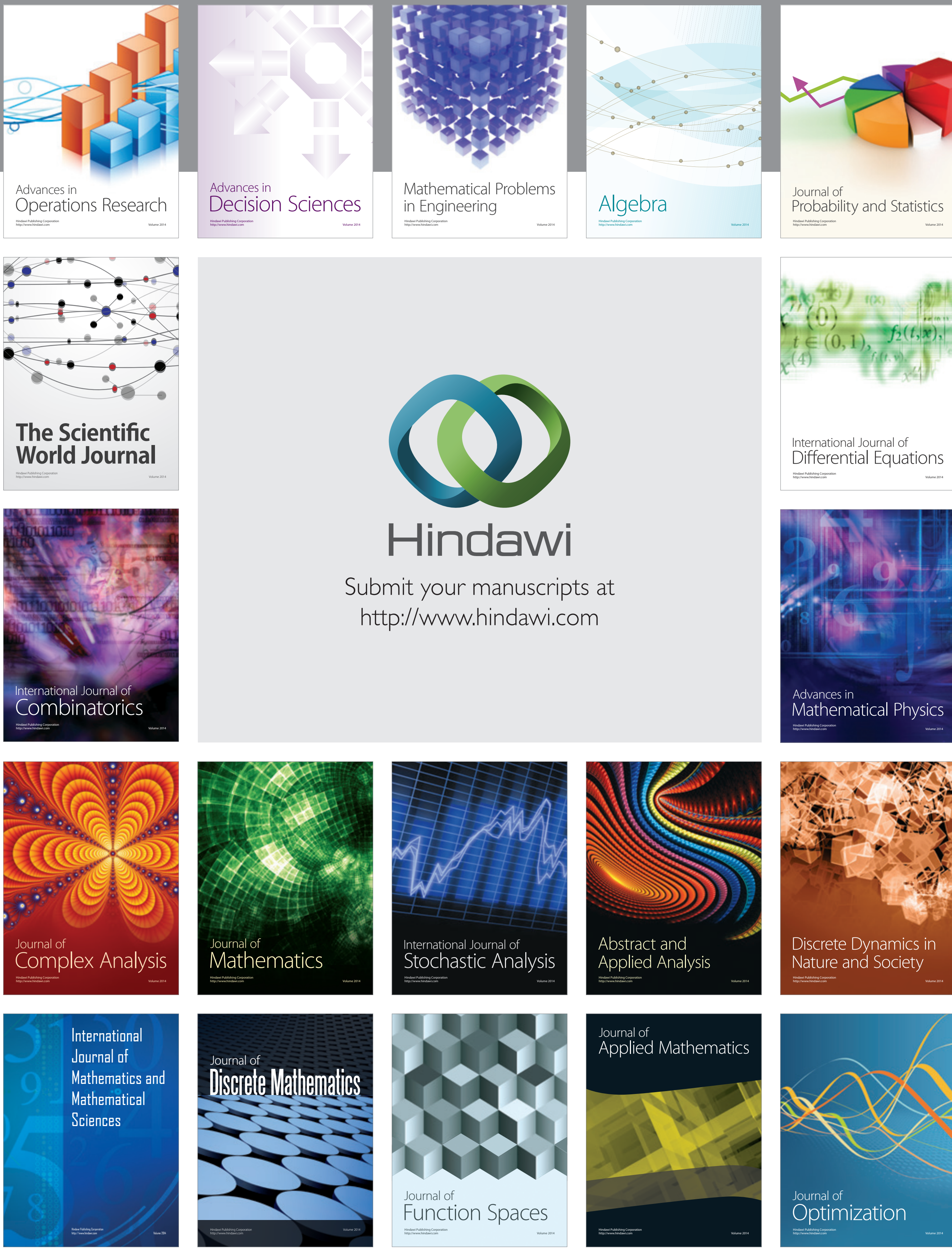
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