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Research Article

Endpoints of Multivalued Contraction Operators

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The existence of the endpoints and approximate endpoints are studied in a general setting for the operators satisfying various contractive conditions. Some recent results are also derived as special cases.

1. Introduction

Among several generalizations of celebrated Banach fixed point theorem, one interesting extension is Nadler's [1] fixed point theorem for multivalued contraction. He exactly proved that a multivalued contraction has a fixed point in a complete metric space. Subsequently, it received great attention in applicable mathematics and was extended and generalized on various settings. Indeed, these extensions and generalizations have been influenced by the applications of the multivalued fixed point theory in mathematical economics, game theory, differential inclusions, interval arithmetic, Hammerstein equations, convex optimization, duality theory in optimization, variational inequalities and control theory, nonlinear evolution equations and nonlinear semigroups, quasivariational inequalities, and elasticity and plasticity theory (see, for instance, [2–8] and several references thereof). The results related to existence of endpoints or strict fixed-points were first given by Rus [9] in 2003. Thereafter, a number of authors established interesting results concerning existence and uniqueness of endpoints for multivalued contractions in different settings; see, for example, [10–15]. The main purpose of this paper is to establish some existence and uniqueness results for endpoints using different multivalued contractions. Our results include some recent results.

2. Preliminaries

Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be an endpoint of T if $Tx = \{x\}$. We say that a multivalued mapping $T : X \to 2^X$ has the approximate endpoint property (AEPP) if $\inf_{x\in X}\sup_{y\in Tx} d(x, y) = 0$ (also see [3, 10]). Throughout the paper, let (X, d) be a b-metric space, and let $P(X)$ denote the family of all nonempty subsets of X and $Cl(X)$ the family of all nonempty closed subsets of X. For any $A, B \in P(X)$, the Hausdorff metric is defined as

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}, \quad (1)$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the set B.

Let $I : X \to X$ be a single-valued mapping and $T : X \to Cl(X)$ a multivalued contraction. We say that the mappings I and T have an AEPP provided $\inf_{x\in X}\sup_{y\in Tx} d(Ix, y) = 0$.

A point $x \in X$ is called an endpoint of I and T if $Tx = \{Ix\}$. For each $\varepsilon > 0$, let $E_\varepsilon(I, T) = \{x \in X : \sup_{y\in Tx} d(Ix, y) \le \varepsilon\}$ be the set of all approximate endpoints of the mappings I and T.

Example 1. Let $X = (-\infty, \infty)$ with Euclidean norm. Assume that $K = [0, c]$ and $T : K \to Cl(K)$ defined by

$$Tx = \begin{cases} [x, c - x], & x \in \left[0, \dfrac{c}{2}\right) \\ [c - x, x], & x \in \left[\dfrac{c}{2}, c\right], \end{cases} \quad (2)$$

where c is a positive constant. Clearly, $\mathrm{Fix}(T) = [0, c]$ and $\mathrm{End}(T) = \{c/2\}$.

Definition 2 (see [16]). Let X be a nonempty set and $b \geq 1$ a given real number. A function $d : X \times X \to \mathfrak{R}_+$ (set of nonnegative real numbers) is said to be a b-metric iff for all $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x, y) = 0$ iff $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space.

The class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric space is a metric space when $b = 1$ in the above condition (iii). The following example shows that a b-metric on X need not be a metric on X (see also [16, page 264]).

Example 3 (see [17]). Let $X = \{x_1, x_2, x_3, x_4\}$ and $d(x_1, x_2) = k \geq 2$, $d(x_1, x_3) = d(x_1, x_4) = d(x_2, x_3) = d(x_2, x_4) = d(x_3, x_4) = 1$, $d(x_i, x_j) = d(x_j, x_i)$ for all $i, j = 1, 2, 3, 4$ and $d(x_i, x_i) = 0$, $i = 1, 2, 3, 4$. Then,

$$d\left(x_i, x_j\right) \leq \frac{k}{2}\left[d\left(x_i, x_n\right) + d\left(x_n, x_j\right)\right] \quad \text{for } n, i, j = 1, 2, 3, 4, \tag{3}$$

and if $k > 2$, the ordinary triangle inequality does not hold.

Definition 4 (see [16]). Let (X, d) be a b-metric space. Then, a sequence $\{x_n\}_{n\in N}$ in X is called

(a) *convergent* if and only if there exists $x \in X$ such that $d(x_n, X) \to 0$ as $n \to \infty$. In this case, one writes $\lim_{n\to\infty} x_n = x$,

(b) *Cauchy* if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Remark 5 (see [16]). In a b-metric space (X, d), the following assertions hold.

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

Definition 6 (see [16]). Let (X, d) be a b-metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all convergent sequences of points in Y, i.e.,

$$\overline{Y} = \left\{x \in X : \text{ there exists a sequence } \{x_n\}_{n\in N} \text{ such that } \lim_{n\to\infty} x_n = x\right\}. \tag{4}$$

Definition 7 (see [16]). Let (X, d) be a b-metric space. Then, a subset $Y \subset X$ is called

(a) *closed* if and only if for each sequence $\{x_n\}_{n\in N}$ in Y which converges to an element x, one has $x \in Y$,

(b) *compact* if and only if for every sequence of elements of Y, there exists a subsequence that converges to an element of Y,

(c) *bounded* if and only if $\delta(Y) = \sup\{d(a, b) : a, b \in Y\} < \infty$.

Definition 8 (see [16]). The b-metric space (X, d) is complete iff every Cauchy sequence in X converges.

Definition 9. Let X and Y be two Hausdorff topological spaces and $T : X \to P(Y)$, a multivalued mapping with nonempty values. Then, T is said to be

(i) upper semicontinuous (u.s.c.) if, for each closed set $B \subset Y$, $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$ is closed in X;

(ii) lower semicontinuous (l.s.c.) if, for each open set $B \subset Y$, $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$ is open in X;

(iii) continuous if it is both u.s.c. and l.s.c.;

(iv) closed if its graph $\mathrm{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed;

(v) compact if closure of $T(X)$ is a compact subset of Y.

Definition 10 (see [5]). Let $I : X \to X$ be a single-valued mapping and $T : X \to Cl(X)$ a multivalued mapping. Then, T is called

(i) a multivalued I-contraction if there exists a number $\alpha \in (0, 1)$:

$$H(Tx, Ty) \leq \alpha d(Ix, Iy), \quad \forall\, x, y \in X, \tag{I-mc}$$

(ii) a multivalued I-Kannan contraction if there exists a number $\beta \in (0, 1/2)$:

$$H(Tx, Ty) \leq \beta\left[d(Ix, Tx) + d(Iy, Ty)\right], \quad \forall x, y \in X, \tag{I-mkc}$$

(iii) a multivalued I-Chatterjea contraction if there exists a number $\gamma \in (0, 1/2)$:

$$H(Tx, Ty) \leq \gamma\left[d(Ix, Ty) + d(Iy, Tx)\right], \quad \forall x, y \in X, \tag{I-mcc}$$

(iv) a multivalued I-quasi-contraction if

$$H(Tx, Ty) \leq k \cdot \max\left\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\right\} \tag{I-mqc}$$

for some $0 \leq k < 1$ and all x, y in X,

(v) a multivalued I-weak or almost contraction if there exist $\alpha \in (0, 1)$ and $L \geq 0$:

$$H(Tx, Ty) \leq \alpha d(Ix, Iy) + Ld(Iy, Tx), \quad \forall\, x, y \in X, \tag{I-mac}$$

(vi) a multivalued generalized I-almost contraction if there exists a function $\alpha : [0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+}\alpha(r) < 1$ for every $t \in [0,\infty)$ such that

$$H(Tx,Ty) \leq \alpha(d(Ix,Iy))\,d(Ix,Iy) + Ld(Iy,Tx), \quad \forall\, x,y \in X. \tag{I-gmac}$$

Remark 11. A multivalued mapping $T : X \to Cl(X)$ is called a multivalued I-Zamfirescu (I-mzc) operator if it satisfies at least one of the conditions (i), (ii), and (iii).

We have used following Cantor's intersection theorem in our results.

Theorem 12. *Let X be a compact space, and let $C_1 \supset C_2 \supset C_3 \cdots$ be a nested chain of nonempty closed subsets of X. Then, $\cap C_n \neq \phi$.*

3. Main Results

Lemma 13. *Let (X,d) be a b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a single-valued mapping such that $rd(x,y) \leq d(Ix,Iy)$ for all $x,y \in X$, where $r > 0$ is a constant. If $T : X \to Cl(X)$ satisfies* (I-mc) *with $r\alpha b^2 < 1$, then*

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b)}{r(1-\alpha b^2)}, \quad \forall\varepsilon > 0. \tag{5}$$

Proof. For any $x, y \in E_\varepsilon(I,T)$, we have

$$\begin{aligned} d(Ix,Iy) &= H(\{Ix\},\{Iy\}) \\ &\leq b[H(\{Ix\},Tx) + H(Tx,\{Iy\})] \\ &\leq b\varepsilon + b^2[H(Tx,Ty) + H(Ty,\{Iy\})] \\ &\leq b\varepsilon + b^2H(Tx,Ty) + b^2\varepsilon \\ &\leq b\varepsilon + b^2\varepsilon + \alpha b^2 d(Ix,Iy). \end{aligned} \tag{6}$$

So,

$$d(Ix,Iy) \leq \frac{b\varepsilon(1+b)}{1-\alpha b^2}. \tag{7}$$

Since $rd(x,y) \leq d(Ix,Iy)$, we have

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b)}{r(1-\alpha b^2)}, \quad \forall\varepsilon > 0. \tag{8}$$

□

Lemma 14. *Let (X,d) be a b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a continuous single-valued mapping. If $T : X \to Cl(X)$ is a lower semicontinuous multivalued mapping. Then, for each $\varepsilon > 0$, $E_\varepsilon(I,T)$ is closed.*

Proof. The proof follows from Lemma 16 of Hussain et al. [11]. □

Theorem 15. *Let (X,d) be a complete b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x,y) \leq d(Ix,Iy)$, where $r > 0$ is a constant. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying* (I-mc). *Then, I and T have a unique endpoint if and only if I and T have the AEPP.*

Proof. It is clear that if I and T have an endpoint, then I and T have the AEPP.

Then,

$$C_n = \left\{x \in X : \sup_{y\in Tx} d(Ix,y) \leq \frac{1}{n}\right\} \neq \phi, \quad \forall n \in N. \tag{9}$$

Also, we have for each $n \in N$, $C_n \supseteq C_{n+1}$. By Lemma 14, C_n is closed for each $n \in N$. Since I and T satisfy AEPP, then $C_n \neq \phi$ for each $n \in N$. Now, we show that $\lim_{n\to\infty}\delta(C_n) = 0$. To show this, let $x, y \in C_n$. Then, from Lemma 13,

$$\delta(C_n) = \delta(E_{1/n}(I,T)) \leq \frac{b\ (1+b)(1/n)}{r(1-\alpha\ b^2)} \tag{10}$$

and so $\lim_{n\to\infty}\delta(C_n) = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in N} C_n = \{x_0\}. \tag{11}$$

Thus, x_0 is the unique endpoint of I and T. □

Lemma 16. *Let (X,d) be a b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a single-valued mapping such that $rd(x,y) \leq d(Ix,Iy)$ for all $x,y \in X$, where $r > 0$ is a constant. If $T : X \to Cl(X)$ satisfies* (I-mkc), *then*

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon}{r}(1+b+2\beta b), \quad \forall\varepsilon > 0. \tag{12}$$

Proof. For any $x, y \in E_\varepsilon(I,T)$, we have

$$\begin{aligned} d(Ix,Iy) &= H(\{Ix\},\{Iy\}) \\ &\leq b[H(\{Ix\},Tx) + H(Tx,\{Iy\})] \\ &\leq b\varepsilon + b^2[H(Tx,Ty) + H(Ty,\{Iy\})] \\ &\leq b\varepsilon + b^2H(Tx,Ty) + b^2\varepsilon \\ &\leq b\varepsilon + b^2\varepsilon + b^2\beta[d(Ix,Tx) + d(Iy,Ty)] \\ &\leq b\varepsilon + b^2\varepsilon + b^2 2\beta\varepsilon \\ &\leq b\varepsilon(1+b+2\beta b). \end{aligned} \tag{13}$$

So,

$$d(Ix,Iy) \leq b\varepsilon(1+b+2\beta b). \tag{14}$$

Since $rd(x,y) \leq d(Ix,Iy)$, we have

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon}{r}(1+b+2\beta b), \quad \forall\varepsilon > 0. \tag{15}$$

□

Theorem 17. *Let (X, d) be a complete b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$, where $r > 0$ is a constant. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying* (I-mkc). *Then, I and T have a unique endpoint if and only if I and T have the AEPP.*

Proof. It is clear that if I and T have an endpoint, then I and T have AEPP. Then,

$$C_n = \left\{x \in X : \sup_{y\in Tx} d(Ix, y) \leq \frac{1}{n}\right\} \neq \phi, \quad \forall n \in N. \tag{16}$$

Also, we have for each $n \in N$, $C_n \supseteq C_{n+1}$. By Lemma 14, C_n is closed for each $n \in N$. Since I and T satisfy AEPP, then $C_n \neq \phi$ for each $n \in N$. Now, we show that $\lim_{n\to\infty}\delta(C_n) = 0$. To show this, let $x, y \in C_n$. Then, from Lemma 16,

$$\delta(C_n) = \delta(E_{1/n}(I, T)) \leq \frac{b(1/n)}{r}(1 + b + 2\beta b) \tag{17}$$

and so $\lim_{n\to\infty}\delta(C_n) = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in N} C_n = \{x_0\}. \tag{18}$$

Thus, x_0 is the unique endpoint of I and T. □

Lemma 18. *Let (X, d) be a b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$ for all $x, y \in X$, where $r > 0$ is a constant. If $T : X \to Cl(X)$ satisfies* (I-mcc) *with $2b^2 r\gamma < 1$, then*

$$\delta(E_\varepsilon(I, T)) \leq \frac{b\varepsilon(1 + b + 2\gamma b)}{r(1 - 2b^2\gamma)}, \quad \forall \varepsilon > 0. \tag{19}$$

Proof. For any $x, y \in E_\varepsilon(I, T)$, we have

$$\begin{aligned}
d(Ix, Iy) &= H(\{Ix\}, \{Iy\}) \\
&\leq b[H(\{Ix\}, Tx) + H(Tx, \{Iy\})] \\
&\leq b\varepsilon + b^2[H(Tx, Ty) + H(Ty, \{Iy\})] \\
&\leq b\varepsilon + b^2 H(Tx, Ty) + b^2\varepsilon \\
&\leq b\varepsilon + b^2\varepsilon + b^2\gamma[d(Ix, Ty) + d(Iy, Tx)] \\
&\leq b\varepsilon + b^2\varepsilon + b^2\gamma[d(Ix, Iy) + d(Iy, Ty)] \\
&\quad + b^2\gamma[d(Iy, Ix) + d(Ix, Tx)] \\
&= b\varepsilon + b^2\varepsilon + b^2\gamma[H(\{Ix\}, \{Iy\}) + H(\{Iy\}, Ty)] \\
&\quad + b^2\gamma[H(\{Iy\}, \{Ix\}) + H(\{Ix\}, Tx)] \\
&\leq b\varepsilon + b^2\varepsilon + 2b^2\gamma H(\{Ix\}, \{Iy\}) + 2b^2\gamma\varepsilon \\
&= b\varepsilon + b^2\varepsilon + 2b^2\gamma d(Ix, Iy) + 2b^2\gamma\varepsilon \\
&\leq \frac{b\varepsilon(1 + b + 2\gamma\, b)}{1 - 2b^2\,\gamma}.
\end{aligned} \tag{20}$$

So,

$$d(Ix, Iy) \leq \frac{b\varepsilon(1 + b + 2\gamma b)}{1 - 2b^2\gamma}. \tag{21}$$

Since $rd(x, y) \leq d(Ix, Iy)$, we have

$$\delta(E_\varepsilon(I, T)) \leq \frac{b\varepsilon(1 + b + 2\gamma b)}{r(1 - 2b^2\gamma)}, \quad \forall \varepsilon > 0. \tag{22}$$

□

Theorem 19. *Let (X, d) be a complete b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$, where $r > 0$ is a constant. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying* (I-mcc). *Then, I and T have a unique endpoint if and only if I and T have the AEPP.*

Proof. It is clear that if I and T have an endpoint, then I and T have the AEPP.

Then,

$$C_n = \left\{x \in X : \sup_{y\in Tx} d(Ix, y) \leq \frac{1}{n}\right\} \neq \phi, \quad \forall n \in N. \tag{23}$$

Also, we have for each $n \in N$, $C_n \supseteq C_{n+1}$. By Lemma 14, C_n is closed for each $n \in N$. Since I and T satisfy AEPP, then $C_n \neq \phi$ for each $n \in N$. Now, we show that $\lim_{n\to\infty}\delta(C_n) = 0$. To show this, let $x, y \in C_n$. Then, from Lemma 18,

$$\delta(C_n) = \delta(E_{1/n}(I, T)) \leq \frac{b(1/n)(1 + b + 2\gamma b)}{r(1 - 2b^2\gamma)} \tag{24}$$

and so $\lim_{n\to\infty}\delta(C_n) = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in N} C_n = \{x_0\}. \tag{25}$$

Thus, x_0 is the unique endpoint of I and T. □

Lemma 20. *Let (X, d) be a b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$ for all $x, y \in X$, where*

$r > 0$ is a constant. If $T : X \to Cl(X)$ satisfies (I-mzc) with $\delta = \max\{\alpha, \beta b/(1-\beta b^2), \beta(1+b^2)/2(1-\beta b^2), \gamma b/(1-\gamma b)\}$, $r\delta < 1$, then

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b+2\delta b)}{r(1-\delta)}, \quad \forall \varepsilon > 0. \tag{26}$$

Proof. Suppose that T satisfies (I-mkc), then we have

$$\begin{aligned} H(Tx,Ty) &\leq \beta[d(Ix,Tx)+d(Iy,Ty)] \\ &\leq \beta[H(Ix,Tx)+H(Iy,Ty)] \\ &\leq \beta H(Ix,Tx)+\beta bH(Iy,Ix)+\beta bH(Ix,Ty) \\ &\leq \beta H(Ix,Tx)+\beta bH(Iy,Ix) \\ &\quad + \beta b^2 H(Ix,Tx)+\beta b^2 H(Tx,Ty) \\ &\leq (\beta+\beta b^2) H(Ix,Tx) \\ &\quad + \beta b d(Iy,Ix) + \beta b^2 H(Tx,Ty). \end{aligned} \tag{27}$$

So, we have

$$H(Tx,Ty) \leq \frac{(\beta+\beta b^2)}{1-\beta b^2} H(Ix,Tx) + \frac{\beta b}{1-\beta b^2} d(Ix,Iy). \tag{28}$$

If T satisfies (I-mcc), then we have

$$\begin{aligned} H(Tx,Ty) &\leq \gamma[d(Ix,Ty)+d(Iy,Tx)] \\ &\leq \gamma[H(Ix,Ty)+H(Iy,Tx)] \\ &\leq \gamma b[H(Ix,Tx)+H(Tx,Ty)] \\ &\quad + \gamma b[H(Iy,Ix)+H(Ix,Tx)] \\ &\leq 2\gamma bH(Ix,Tx)+\gamma bd(Ix,Iy) \\ &\quad + \gamma bH(Tx,Ty)H(Tx,Ty) \\ &\leq \frac{2\gamma b}{1-\gamma b} H(Ix,Tx) + \frac{\gamma b}{1-\gamma b} d(Ix,Iy). \end{aligned} \tag{29}$$

Let $\delta = \max\{\alpha, (\beta b)/(1-\beta b^2), (\beta(1+b^2))/(2(1-\beta b^2)), (\gamma b)/(1-\gamma b)\}$. Then,

$$H(Tx,Ty) \leq 2\delta H(Ix,Tx) + \delta d(Ix,Iy). \tag{30}$$

Thus, for any $x, y \in E_\varepsilon(I,T)$, we have

$$\begin{aligned} d(Ix,Iy) &= H(\{Ix\},\{Iy\}) \\ &\leq b[H(\{Ix\},Tx)+H(Tx,\{Iy\})] \\ &\leq b\varepsilon + b^2[H(Tx,Ty)+H(Ty,\{Iy\})] \\ &\leq b\varepsilon + b^2 H(Tx,Ty) + b^2\varepsilon \\ &\leq b\varepsilon + b^2\varepsilon + b^2 2\delta d(Ix,Tx) + \delta d(Ix,Iy) \\ &\leq b\varepsilon + b^2\varepsilon + b^2 2\delta\varepsilon + \delta d(Ix,Iy). \end{aligned} \tag{31}$$

So,

$$d(Ix,Iy) \leq \frac{b\varepsilon(1+b+2\delta b)}{1-\delta}. \tag{32}$$

Since $rd(x,y) \leq d(Ix,Iy)$, we have

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b+2\delta b)}{r(1-\delta)}, \quad \forall \varepsilon > 0. \tag{33}$$

□

Theorem 21. *Let (X,d) be a complete b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x,y) \leq d(Ix,Iy)$, where $r > 0$ is a constant. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying (I-mzc). Then, I and T have a unique endpoint if and only if I and T have the AEPP.*

Proof. It is clear that if I and T have an endpoint, then I and T have the AEPP. Then,

$$C_n = \left\{x \in X : \sup_{y\in Tx} d(Ix,y) \leq \frac{1}{n}\right\} \neq \phi, \quad \forall n \in N. \tag{34}$$

Further, we have for each $n \in N$, $C_n \supseteq C_{n+1}$. By Lemma 14, C_n is closed for each $n \in N$. Since I and T satisfy AEPP, then $C_n \neq \phi$ for each $n \in N$. Now, we show that $\lim_{n\to\infty}\delta(C_n) = 0$. To show this, let $x, y \in C_n$. Then, from Lemma 20,

$$\delta(C_n) = \delta(E_{1/n}(I,T)) \leq \frac{b(1/n)(1+b+2\delta b)}{r(1-\delta)}. \tag{35}$$

and so $\lim_{n\to\infty}\delta(C_n) = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in N} C_n = \{x_0\}. \tag{36}$$

Thus, x_0 is the unique endpoint of I and T. □

Lemma 22. *Let (X,d) be a b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a single-valued mapping such that $rd(x,y) \leq d(Ix,Iy)$ for all $x, y \in X$, where $r > 0$ is a constant. If $T : X \to Cl(X)$ satisfies (I-mqc) with $k < 1/(r(b^2+b))$, then*

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b+kb)}{r(1-kb^2)}, \quad \forall \varepsilon > 0. \tag{37}$$

Proof. For any $x, y \in E_\varepsilon(I,T)$, we have

$$\begin{aligned} d(Ix,Iy) &= H(\{Ix\},\{Iy\}) \\ &\leq b[H(\{Ix\},Tx)+d(Tx,\{Iy\})] \\ &\leq b\varepsilon + b^2[H(Tx,Ty)+H(Ty,\{Iy\})] \\ &\leq b\varepsilon + b^2 H(Tx,Ty) + b^2\varepsilon \\ &\leq b\varepsilon + b^2\varepsilon + kb^2 \\ &\quad \cdot \max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), \\ &\qquad d(Ix,Ty), d(Iy,Tx)\} \\ &\leq b\varepsilon + b^2\varepsilon + kb^2\{d(Ix,Iy)+\varepsilon\}. \end{aligned} \tag{38}$$

So,

$$d(Ix, Iy) \leq \frac{b\varepsilon(1+b+kb)}{1-kb^2}. \tag{39}$$

Since $rd(x, y) \leq d(Ix, Iy)$, we have

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b+kb)}{r(1-kb^2)}, \quad \forall\varepsilon > 0. \tag{40}$$

□

Theorem 23. *Let (X, d) be a complete b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$, where $r > 0$ is a constant. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying* (I-mqc). *Then, I and T have a unique endpoint if and only if I and T have the AEPP.*

Proof. It is clear that if I and T have an endpoint, then I and T have the AEPP.

Then,

$$C_n = \left\{x \in X : \sup_{y\in Tx} d(Ix, y) \leq \frac{1}{n}\right\} \neq \phi, \quad \forall n \in N. \tag{41}$$

Also, it is clear that, for each $n \in N$, $C_n \supseteq C_{n+1}$. By the above Lemma 14, C_n is closed for each $n \in N$. Since I and T satisfy AEPP, then $C_n \neq \phi$ for each $n \in N$. Now, we show that $\lim_{n\to\infty}\delta(C_n) = 0$. To show this, let $x, y \in C_n$. Then, from Lemma 22,

$$\delta(C_n) = \delta(E_{1/n}(I,T)) \leq \frac{b(1/n)(1+b+kb)}{r(1-kb^2)} \tag{42}$$

and so $\lim_{n\to\infty}\delta(C_n) = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in N} C_n = \{x_0\}. \tag{43}$$

Thus, x_0 is the unique endpoint of I and T. □

Lemma 24. *Let (X, d) be a b-metric space and $I : X \to X$ a single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$ for all $x, y \in X$, where $r > 0$ is a constant. If $T : X \to Cl(X)$ satisfies* (I-mac) *with $rb^2(\alpha + bL) < 1$, then*

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b+Lb^2)}{r(1-b^2(\alpha+bL))}, \quad \forall\varepsilon > 0. \tag{44}$$

Proof. For any $x, y \in E_\varepsilon(I,T)$, we have

$$\begin{aligned}
d(Ix, Iy) &= H(\{Ix\}, \{Iy\}) \\
&\leq b[H(\{Ix\}, Tx) + d(Tx, \{Iy\})] \\
&\leq b\varepsilon + b^2[H(Tx, Ty) + H(Ty, \{Iy\})] \\
&\leq b\varepsilon + b^2 H(Tx, Ty) + b^2\varepsilon \\
&\leq b\varepsilon + b^2\varepsilon + b^2[\alpha d(Ix, Iy) + Ld(Iy, Tx)] \\
&\leq b\varepsilon + b^2\varepsilon + b^2\alpha d(Ix, Iy) \\
&\quad + b^3 L[d(Ix, Iy) + d(Ix, Tx)] \\
&\leq b\varepsilon + b^2\varepsilon + b^2\alpha d(Ix, Iy) \\
&\quad + b^3 Ld(Ix, Iy) + b^3 L\varepsilon \\
&\leq b\varepsilon(1+b+Lb^2) + b^2(\alpha + sL)d(Ix, Iy).
\end{aligned} \tag{45}$$

So,

$$d(Ix, Iy) \leq \frac{b\varepsilon(1+b+Lb^2)}{1-b^2(\alpha+bL)}. \tag{46}$$

Since $rd(x, y) \leq d(Ix, Iy)$, we have

$$\delta(E_\varepsilon(I,T)) \leq \frac{b\varepsilon(1+b+Lb^2)}{r(1-b^2(\alpha+bL))}, \quad \forall\varepsilon > 0. \tag{47}$$

□

Theorem 25. *Let (X, d) be a complete b-metric space with the b-metric as a continuous functional. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$, where $r > 0$ is a constant. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying* (I-mac). *Then, I and T have a unique endpoint if and only if I and T have the AEPP.*

Proof. It is clear that if I and T have an endpoint, then I and T have the AEPP. Then,

$$C_n = \left\{x \in X : \sup_{y\in Tx} d(Ix, y) \leq \frac{1}{n}\right\} \neq \phi, \quad \forall n \in N. \tag{48}$$

Also, we have for each $n \in N$, $C_n \supseteq C_{n+1}$. By Lemma 14, C_n is closed for each $n \in N$. Since I and T satisfy AEPP, then $C_n \neq \phi$ for each $n \in N$. Now, we show that $\lim_{n\to\infty}\delta(C_n) = 0$. To show this, let $x, y \in C_n$. Then, from Lemma 24,

$$\delta(C_n) = \delta(E_{1/n}(I,T)) \leq \frac{b(1/n)(1+b+Lb^2)}{r(1-b^2(\alpha+bL))} \tag{49}$$

and so $\lim_{n\to\infty}\delta(C_n) = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in N} C_n = \{x_0\}. \tag{50}$$

Thus, x_0 is the unique endpoint of I and T. □

On putting $b = 1$ in the above Theorem 25, we obtain the following result of [11].

Corollary 26 (see [11]). *Let (X, d) be a complete metric space. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x, y) \leq d(Ix, Iy)$, where $r > 0$ is a constant. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying* (I-mac). *Then, I and T have a unique endpoint if and only if I and T have the AEPP.*

If I is the identity mapping on X and $b = 1$, then the above result reduces to the following results:

Corollary 27 (see [11, Corollary 3.5]). *Let (X, d) be a metric space, and let $T : X \to Cl(X)$ satisfy* (I-mac) *with $\alpha + L < 1$. Then, for each $\varepsilon > 0$,*

$$\delta\left(E_{\varepsilon}(T)\right) \leq \frac{\varepsilon(2 + L)}{1 - (\alpha + L)}, \tag{51}$$

where $E_{\varepsilon}(T) = \{x \in X : \sup_{y \in Tx} d(x, y) \leq \varepsilon\}$.

Corollary 28 (see [11, Corollary 3.6]). *Let (X, d) be a complete metric space. Let $T : X \to Cl(X)$ be a lower semicontinuous map satisfying* (I-mac) *with $\alpha + L < 1$. Then, T has a unique endpoint if and only if T has the AEPP.*

If $L = 0$, in almost contraction, then we have following result in metric space.

Corollary 29 ([10, Corollary 2.2]). *Let (X, d) be a complete metric space. Let $T : X \to Cl(X)$ satisfy (mc). Then, T has a unique endpoint if and only if T has the AEPP.*

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