Research Article

The Radon Transforms on the Generalized Heisenberg Group

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Let $\mathbf{H}_n^a$ be the generalized Heisenberg group. In this paper, we study the inversion of the Radon transforms on $\mathbf{H}_n^a$. Several kinds of inversion Radon transform formulas are established. One is obtained from the Euclidean Fourier transform; the other is derived from the differential operator with respect to the center variable t. Also by using sub-Laplacian and generalized sub-Laplacian we deduce an inversion formula of the Radon transform on $\mathbf{H}_n^a$.

1. Introduction

In the past decade the research of Radon transform on the Euclidean space $\mathbb{R}^n$ has made considerable progress due to its wide applications to partial differential equations, X-ray technology, radio astronomy, and so on. The basic theory and some new developments can be found in [1] by Helgason and the references therein. The combination of Radon transform and wavelet transform has proved to be very useful both on pure mathematics and its applications. Therefore, it is very meaningful to give the inversion formula of the Radon transforms by using various ways. The first result in the area is due to Holschneider who considered the classical Radon transform on the two-dimensional plane (see [2]). Rubin in [3, 4] extended the results in [2] to the k-dimensional Radon transform on $\mathbb{R}^n$ and totally geodesic Radon transforms on the sphere and hyperbolic space. Heisenberg group $\mathbf{H}^n$ is a vital Lie group with the underlying $\mathbb{C}^n \times \mathbb{R}$. Strichartz [5] discussed the Radon transform on the Heisenberg group. Nessibi and Trimèche [6] obtained an inversion formula of the Radon transform on the Laguerre hypergroup $\mathbf{K} = [0, \infty) \times \mathbb{R}$ by using the generalized wavelet transform. Afterwards, He and Liu studied the analogous problems on the Heisenberg group and Siegel type Lie group (see [7, 8]), and Rubin [9] achieved some new progress of the Radon transform on $\mathbf{H}^n$. In [10] the authors gave the definition of generalized Heisenberg group denoted by $\mathbf{H}_n^a$ and dealt with

some problems related to geometric analysis. In this paper, we investigate the inversion formulas of the Radon transform on the generalized Heisenberg group. From the Euclidean Fourier transform and group Fourier transform, we deduce inversion formulas of the Radon transform on $\mathbf{H}_n^a$ associated with differential operators and generalized sub-Laplacian.

Let $a = (a_1, a_2, \ldots, a_n)$ be an n-dimensional vector, where a_i are positive real constants for $i = 1, 2, \ldots, n$. We can turn $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ into a non-Abelian group by defining the group operation as

$$
\begin{aligned}
&(x, y, t)\left(x', y', t'\right) \\
&= \left(x + x', y + y', t + t' + \frac{1}{2}\sum_{j=1}^{n} a_j \left(x'_j y_j - y'_j x_j\right)\right).
\end{aligned}
\tag{1}
$$

This group is called the generalized Heisenberg group and is denoted by $\mathbf{H}_n^a$. It is obvious that the generalized Heisenberg group $\mathbf{H}_n^a$ becomes ordinary Heisenberg group $\mathbf{H}^n$ if all $a_i = 1$ for $i = 1, 2, \ldots, n$. For any n-dimensional vectors $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$, we define

$$
a * b = (a_1 b_1, a_2 b_2, \ldots, a_n b_n),
$$

$$
ab = a \cdot b = \sum_{j=1}^{n} a_j b_j.
\tag{2}
$$

For $c = (c_1, c_2, \ldots, c_n)$, $\lambda \in \mathbb{R}$, the following equalities are valid:

(i) $a * (b + c) = a * b + a * c$;

(ii) $(a * b)c = a(b * c)$;

(iii) $(\lambda a) * b = \lambda (a * b)$.

Therefore, (1) can be rewritten as

$$(x, y, t)\left(x', y', t'\right)$$
$$= \left(x + x', y + y', t + t' + \frac{1}{2}\left(\left(a * x'\right)y - \left(a * y'\right)x\right)\right). \tag{3}$$

Identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$; the symplectic form $(x'y - y'x)$ can be expressed by $\mathrm{Im}\, z\overline{z'}$, where $z = x + iy$ and $z' = x' + iy'$. We can write (1) by

$$(z, t)\left(z', t'\right) = \left(z + z', t + t' + \frac{1}{2}\,\mathrm{Im}\sum_{j=1}^{n} a_j z_j \overline{z_j'}\right), \tag{4}$$

or

$$(z, t)\left(z', t'\right) = \left(z + z', t + t' + \frac{1}{2}\,\mathrm{Im}\,(a * z)\,\overline{z'}\right). \tag{5}$$

In next section, we will introduce some facts of Fourier analysis on $\mathbf{H}_n^a$, which is useful to get our result.

2. Fourier Analysis on $\mathbf{H}_n^a$

We first state the definition of the Hermite polynomials. Let $\mathbb{Z}^+ = \{0, 1, 2, \ldots, n, \ldots\}$, $k \in \mathbb{Z}^+$, and $t \in \mathbb{R}$; the Hermite polynomials are defined by

$$H_k(t) = (-1)^k \left(\frac{d^k}{dt^k}\left\{e^{-t^2}\right\}e^{t^2}\right). \tag{6}$$

The normalized Hermite functions are then defined by

$$h_k(t) = \left(2^k \sqrt{\pi}k!\right)^{-(1/2)} H_k(t)\, e^{-(1/2)t^2}. \tag{7}$$

These functions form an orthonormal basis for $L^2(\mathbb{R})$. For any fixed $\lambda \in \mathbb{R}^+$, it is easy to calculate from (6) that

$$H_k\left(\sqrt{\lambda}t\right) = (-1)^k \frac{1}{\left(\sqrt{\lambda}\right)^k}\left(\frac{d^k}{dt^k}\left\{e^{-\lambda t^2}\right\}e^{\lambda t^2}\right); \tag{8}$$

then

$$h_k\left(\sqrt{\lambda}t\right) = \left(2^k \sqrt{\pi}k!\right)^{-(1/2)}(-1)^k \frac{1}{\left(\sqrt{\lambda}\right)^k}$$
$$\times \left(\frac{d^k}{dt^k}\left\{e^{-\lambda t^2}\right\}e^{\lambda t^2}\right)e^{-(1/2)\lambda t^2}. \tag{9}$$

Therefore, we define

$$\widetilde{h_{k,\lambda}^{a_i}}(t) = (\lambda a_i)^{1/2} h_k\left(\sqrt{a_i \lambda}t\right)$$
$$= (\lambda a_i)^{1/2}\left(2^k \sqrt{\pi}k!\right)^{-(1/2)}(-1)^k \tag{10}$$
$$\times \frac{1}{\left(\sqrt{a_i \lambda}\right)^k}\frac{d^k}{dt^k}\left\{e^{-a_i \lambda t^2}\right\}e^{(a_i \lambda / 2)t^2},$$

which also form the orthonormal basis for $L^2(\mathbb{R})$ for any positive constant a_i $(i = 1, 2, \ldots, n)$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{Z}^+)^n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $a = (a_1, a_2, \ldots, a_n)$, and $a_i > 0$ for $i = 1, 2, \ldots, n$. The higher dimensional Hermite functions denoted by Φ_α can be obtained by taking tensor products:

$$\Phi_\alpha(x) = \prod_{j=1}^{n} h_{a_j}\left(x_j\right). \tag{11}$$

Then the family $\{\Phi_\alpha : \alpha \in (\mathbb{Z}^+)^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ (see [11]).

We define

$$\Phi_{\alpha,\lambda}^a(x) = \prod_{j=1}^{n} \widetilde{h_{\alpha_j,\lambda}^{a_j}}\left(x_j\right); \tag{12}$$

then $\{\Phi_{\alpha,\lambda}^a : \alpha \in (\mathbb{Z}^+)^n\}$ for any fixed a and λ is also the orthonormal basis for $L^2(\mathbb{R}^n)$. If $\lambda = 1$ and $a_i = 1$ for $i = 1, 2, \ldots, n$, then $\Phi_\alpha(x) = \Phi_{\alpha,\lambda}^a(x)$. And $\{\Phi_\alpha : \alpha \in (\mathbb{Z}^+)^n\}$ are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$; that is, $H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha$, where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is Laplacian on $\mathbb{R}^n$, and $|\alpha| = \sum_{j=1}^{n} \alpha_j$.

For $\lambda \in \mathbb{R} \setminus \{0\}$, let $\pi_\lambda(x, y, t)$ be the Schrödinger representation of $\mathbf{H}_n^a$, which acts on $L^2(\mathbb{R}^n)$ by

$$\pi_\lambda(x, y, t)\Phi(\xi) = e^{i\lambda t + i\lambda(a*x)\cdot\xi + (i/2)\lambda(a*x)\cdot y}\Phi(\xi + y). \tag{13}$$

By a direct computation and the law of $\mathbf{H}_n^a$; we then obtain

$$\pi_\lambda(z, t)\pi_\lambda\left(z', t'\right) = \pi_\lambda\left((z, t)\left(z', t'\right)\right), \tag{14}$$

which indicates that π_λ is unitary. In addition, we deduce that π_λ is irreducible (see [12]).

Suppose that $f \in L^1(\mathbf{H}_n^a)$, the Fourier transform of f is an operator-valued function acting on $L^2(\mathbb{R}^n)$ by

$$\widehat{f}(\lambda) = \int_{\mathbf{H}_n^a} f(z, t)\pi_\lambda(z, t)\, dz\, dt. \tag{15}$$

If we write $\pi_\lambda(z, t) = e^{i\lambda t}\pi_\lambda(z)$, where $\pi_\lambda(z) = \pi_\lambda(z, 0)$, and define

$$f^\lambda(z) = \int_{-\infty}^{+\infty} e^{i\lambda t} f(z, t)\, dt, \tag{16}$$

then (15) can be rewritten as

$$\widehat{f}(\lambda) = \int_{\mathbb{R}^{2n}} f^\lambda(z)\pi_\lambda(z)\, dz = \mathbf{W}_\lambda\left(f^\lambda\right), \tag{17}$$

where $\mathbf{W}_\lambda$ is the Weyl transform. By the same argument of the theory of Weyl transforms on the Heisenberg group we have

$$\left\|f^\lambda\right\|^2_{L^2(\mathbb{R}^{2n})} = \prod_{i=1}^n a_i \left\|\widehat{f}(\lambda)\right\|^2_{\mathrm{HS}}. \tag{18}$$

From this identity we obtain the Plancherel formula

$$\|f\|_{L^2(\mathbf{H}_n^a)} = \left\{\int_{-\infty}^{+\infty} \left\|\widehat{f}(\lambda)\right\|^2_{\mathrm{HS}} d\mu(a,\lambda)\right\}^{1/2}, \tag{19}$$
$$f \in L^1(\mathbf{H}_n^a) \cap L^2(\mathbf{H}_n^a),$$

where $d\mu(a,\lambda) = (2\pi)^{-n-1}|\lambda|^n \prod_{i=1}^n a_i \, d\lambda$ and $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of operators. For $f, g \in L^1(\mathbf{H}_n^a) \cap L^2(\mathbf{H}_n^a)$, the Parseval formula is

$$\langle f, g\rangle_{L^2(\mathbf{H}_n^a)} = \int_{-\infty}^{+\infty} \mathrm{tr}\left(\widehat{g}(\lambda)^* \widehat{f}(\lambda)\right) d\mu(a,\lambda), \tag{20}$$

where $\widehat{g}(\lambda)^*$ denotes the adjoint of $\widehat{g}(\lambda)$. For any $(z,t) \in \mathbf{H}_n^a$, $z = x + iy$, where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Thus, (z,t) can be written as $(x, y, t) \in \mathbf{H}_n^a$. We define the left-invariant vector fields X_j, Y_j $(j = 1, 2, \ldots, n)$, and T on $\mathbf{H}_n^a$. These are given by

$$X_j = \left(\frac{\partial}{\partial x_j} - \frac{1}{2} a_j y_j \frac{\partial}{\partial t}\right), \quad j = 1, 2, \ldots, n,$$
$$Y_j = \left(\frac{\partial}{\partial y_j} + \frac{1}{2} a_j x_j \frac{\partial}{\partial t}\right), \quad j = 1, 2, \ldots, n, \tag{21}$$

and $T = \partial/\partial t$ (see [10]). The $2n + 1$ vectors fields generate the Lie algebra of the generalized Heisenberg group.

The generalized Heisenberg sub-Laplacian is explicitly given by

$$\ell = -\sum_{j=1}^n \left(X_j^2 + Y_j^2\right). \tag{22}$$

Also, a direct computation shows that

$$\ell = -\sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) - \sum_{j=1}^n a_j \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial t}$$
$$- \frac{1}{4} \frac{\partial^2}{\partial t^2} \sum_{j=1}^n a_j^2 \left(x_j^2 + y_j^2\right). \tag{23}$$

We know that $(-\Delta + |x|^2)\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha$ (see [11]). In dimension one, in fact, we have

$$\left(-\frac{d^2}{dx_1^2} + x_1^2\right) h_{\alpha_1}(x_1) = (2\alpha_1 + 1) h_{\alpha_1}(x_1). \tag{24}$$

In n dimensions, this equation together with (11) shows that Φ_α is an eigenfunction of the Hermite operators in each variable:

$$\left(-\frac{d^2}{dx_1^2} + x_1^2\right) \Phi_\alpha(x) = (2\alpha_1 + 1) \Phi_\alpha(x). \tag{25}$$

Therefore, for each λ, we have

$$\left(-\frac{d^2}{dx_1^2} + \lambda^2 x_1^2\right) \Phi_\alpha\left(|\lambda|^{1/2} x\right) = |\lambda|(2\alpha_1 + 1) \Phi_\alpha\left(|\lambda|^{1/2} x\right). \tag{26}$$

Let $a_i > 0$ for $i = 1, 2, \ldots, n$; then

$$\sum_{i=1}^n \left(-\frac{d^2}{dx_i^2} + (a_i\lambda)^2 x_i^2\right) \Phi_{\alpha,\lambda}^a(x)$$
$$= \sum_{i=1}^n \left(-\frac{d^2}{dx_i^2} + (a_i\lambda)^2 x_i^2\right)$$
$$\cdot \left(\lambda^{n/2} \prod_{i=1}^n a_i^{1/2} \prod_{j=1}^n h_{\alpha_j}\left(\left|a_j\lambda\right|^{1/2} x_j\right)\right) \tag{27}$$
$$= |\lambda| \sum_{i=1}^n a_i(2\alpha_i + 1) \cdot \prod_{i=1}^n \left((\lambda a_i)^{1/2} h_{\alpha_i}\left(\left|a_j\lambda\right|^{1/2} x_i\right)\right)$$
$$= \left(2\sum_{i=1}^n a_i\alpha_i + \sum_{i=1}^n a_i\right)|\lambda| \cdot \prod_{j=1}^n \widetilde{h_{\alpha_j,\lambda}^{a_j}}(x_j)$$
$$= (2a\alpha + |a|)|\lambda| \Phi_{\alpha,\lambda}^a(x),$$

where $a\alpha = \sum_{i=1}^n a_i\alpha_i$ and $|a| = \sum_{i=1}^n a_i$.

Therefore, $\{\Phi_{\alpha,\lambda}^a : \alpha \in (\mathbb{Z}^+)^n\}$ are eigenfunctions of operator $\sum_{i=1}^n (-(d^2/dx_i^2) + (a_i\lambda)^2 x_i^2)$ with eigenvalues $(2a\alpha + |a|)|\lambda|$.

Now we consider the space $\mathbf{H}_k$ spanned by $\{\Phi_{\alpha,\lambda}^a(x) : |\alpha| = k\}$. It is clear that the dimension of $\mathbf{H}_k$ is $d_k = (n + k - 1)!/(n - 1)!k!$ (see [13]). Then we have

$$L^2(\mathbb{R}^n) = \bigoplus_{k=0}^\infty \mathbf{H}_k. \tag{28}$$

Let $\mathscr{P}_k$ denote the orthogonal projection operator from $L^2(\mathbb{R}^n)$ to $\mathbf{H}_k$. For $k \in \mathbf{Z}^+, \sigma = +\text{or}-$. Let $\widetilde{\mathbf{H}_k^\sigma}$ be the subspace of $L^2(\mathbf{H}_n^a)$ such that

$$\widetilde{\mathbf{H}_k^+} = \left\{f \in L^2(\mathbf{H}_n^a) : \widehat{f}(\lambda) = \widehat{f}(\lambda)\mathscr{P}_k,\right.$$
$$\left.\mathscr{P}_k = 0 \text{ if } \lambda < 0\right\},$$
$$\widetilde{\mathbf{H}_k^-} = \left\{f \in L^2(\mathbf{H}_n^a) : \widehat{f}(\lambda) = \widehat{f}(\lambda)\mathscr{P}_k,\right.$$
$$\left.\mathscr{P}_k = 0 \text{ if } \lambda > 0\right\}. \tag{29}$$

Then we have

$$L^2(\mathbf{H}_n^a) = \bigoplus_{k=0}^\infty \left(\widetilde{\mathbf{H}_k^+} \oplus \widetilde{\mathbf{H}_k^-}\right). \tag{30}$$

Set $\widetilde{\mathbf{H}_k} = \widetilde{\mathbf{H}_k^+} \oplus \widetilde{\mathbf{H}_k^-} = \{f \in L^2(\mathbf{H}_n^a) : \widehat{f}(\lambda) = \widehat{f}(\lambda)\mathscr{P}_k\}$; then we can write the above decomposition as

$$L^2(\mathbf{H}_n^a) = \bigoplus_{k=0}^\infty \widetilde{\mathbf{H}_k}. \tag{31}$$

More details can be found in [14].

3. Inversion of the Radon Transform on $\mathbf{H}_n^a$

The Radon transform R for a function f on the generalized Heisenberg group $\mathbf{H}_n^a$ is defined by

$$
R(f)(z,t) = \int_{\mathbb{C}^n} f\left((z,t)(z',0)\right) dz'
$$

$$
= \int_{\mathbb{C}^n} f\left(z', t + \frac{1}{2}\ \mathrm{Im}\sum_{j=1}^n a_j z_j \overline{z_j'}\right) dz'
$$

$$
= \int_{\mathbb{R}^n \times \mathbb{R}^n} f\left(u, v, t\right.
$$

$$
\left. + \frac{1}{2}\left((a*y)u - (a*x)v\right)\right) du\, dv,
$$

(32)

where $z' = (z'_1, z'_2, \ldots, z'_n)$, $z' = u + iv$, $z'_j = u_j + iv_j$, $z = x + iy$, and $dz' = du\, dv$. Clearly, when $a \equiv 1$, that is, $a_i = 1$ for $i = 1, 2, \ldots, n$, the above formula is just the Radon transform on the Heisenberg group $\mathbf{H}^n$.

Next, we will obtain some inversion formulas for the inverse Radon transform by means of the Euclidean Fourier transform, differential operators, and sub-Laplacian. We first consider the way of the Euclidean Fourier transform. In fact, Strichartz [5] had obtained the inverse Radon transform on $\mathbf{H}^n$ by using Euclidean Fourier transform. However, he did not show on which space the formula holds. In this section, we not only find a subspace of $\mathcal{S}(\mathbf{H}_n^a)$ on which the Radon transform is a bijection, but also give the inversion Radon transform on $\mathbf{H}_n^a$, where $\mathcal{S}(\mathbf{H}_n^a)$ is the Schwartz space on $\mathbf{H}_n^a$.

Let $\mathcal{F}_3$ denote the Euclidean Fourier transform with respect to the central variable t alone and let $\mathcal{F}$ denote the full Euclidean Fourier transform; that is,

$$
\mathcal{F}_3(f)(u, v, \lambda) = \int_{\mathbb{R}} f(u, v, t) e^{i\lambda t} dt,
$$

$$
\mathcal{F}(f)(x, y, \lambda) = \iint_{\mathbb{R}^n} \int_{\mathbb{R}} f(u, v, t) e^{i\lambda t + iux + ivy} dt\, du\, dv.
$$

(33)

Because

$$
\mathcal{F}_3(R(f))(x, y, \lambda)
$$

$$
= \int_{\mathbb{R}} R(f)(x, y, t) e^{i\lambda t} dt
$$

$$
= \int_{\mathbb{R}} \int_{\mathbb{R}^n \times \mathbb{R}^n} f\left(u, v, t + \frac{1}{2}\left((a*y)u - (a*x)v\right)\right)
$$

$$
\times e^{i\lambda t} du\, dv\, dt
$$

$$
= \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_{\mathbb{R}} f(u, v, t) e^{i\lambda t} e^{-(i/2)\lambda((a*y)u - (a*x)v)} dt\, du\, dv
$$

$$
= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{F}_3(f)(u, v, \lambda) e^{iu(-(\lambda/2)(a*y)) + iv((\lambda/2)(a*x))} du\, dv,
$$

(34)

we have

$$
\mathcal{F}_3(R(f))(x, y, \lambda) = \mathcal{F}(f)\left(-\frac{\lambda}{2}(a*y), \frac{\lambda}{2}(a*x), \lambda\right).
$$

(35)

We define

$$
\mathcal{S}_*(\mathbf{H}_n^a)
$$

$$
= \left\{ f(x, y, t) \in \mathcal{S}(\mathbf{H}_n^a) : \int_{\mathbb{R}} f(x, y, t) t^k dt = 0 \right.
$$

$$
\left. \forall x, y \in \mathbb{R}^n, \ k \in \mathbb{Z}^+ \right\}
$$

$$
\mathcal{S}^*(\mathbf{H}_n^a) = \left\{ f(x, y, t) \in \mathcal{S}(\mathbf{H}_n^a) : \left. \frac{\partial^k}{\partial t^k} f(x, y, t)\right|_{t=0} \right.
$$

$$
\left. \forall x, y \in \mathbb{R}^n, \ k \in \mathbb{Z}^+ \right\}.
$$

(36)

By argument analogous to [9], we also find that $f \in \mathcal{S}_*(\mathbf{H}_n^a)$ if and only if $\mathcal{F}(f) \in \mathcal{S}^*(\mathbf{H}_n^a)$, and $\mathcal{F}_3$ is an isomorphism from $\mathcal{S}_*(\mathbf{H}_n^a)$ onto $\mathcal{S}^*(\mathbf{H}_n^a)$. The spaces $\mathcal{S}_*(\mathbf{H}_n^a)$ and $\mathcal{S}^*(\mathbf{H}_n^a)$ are regarded as Semyanistyi-Lizorkin type spaces that have many applications (see [6, 15]). We define an operator $\mathcal{J}_a$ which is given by

$$
\mathcal{J}_a(f)(x, y, t)
$$

$$
= f\left(-\frac{t}{2} a_1 y_1, -\frac{t}{2} a_2 y_2, \ldots, -\frac{t}{2} a_n y_n, \frac{t}{2} a_1 x_1, \ldots, \frac{t}{2} a_n x_n, t\right).
$$

(37)

It is easy to know that $\mathcal{J}_a$ is a bijection on $\mathcal{S}^*(\mathbf{H}_n^a)$. The inversion of $\mathcal{J}_a$ is given by

$$
\mathcal{J}_a^{-1}(f)(x, y, t)
$$

$$
= \begin{cases} f\left(-\dfrac{2y_1}{a_1 t}, -\dfrac{2y_2}{a_2 t}, \ldots, -\dfrac{2y_n}{a_n t}, \right. \\ \left. \dfrac{2x_1}{a_1 t}, \ldots, \dfrac{2x_n}{a_n t}, t\right), & \text{for } t \neq 0; \\ 0, & \text{for } t = 0. \end{cases}
$$

(38)

Now (35) reads as

$$
\mathcal{F}_3(R(f))(x, y, \lambda) = \mathcal{J}_a(\mathcal{F}(f))(x, y, \lambda).
$$

(39)

Therefore, we have an inversion formula of the Radon transform as follows.

Theorem 1. Let $f \in \mathcal{S}_*(\mathbf{H}_n^a)$. Then one has

$$
R^{-1}(f) = \mathcal{F}^{-1} \mathcal{J}_a^{-1} \mathcal{F}_3(f).
$$

(40)

We can give another inversion formula of R by using operator $L = \prod_{j=1}^n a_j((1/4\pi i)(\partial/\partial t))^n$. First of all, we give the Fourier transform of Radon transform for a function f.

Lemma 2. *Let $f \in L^2(\mathbf{H}_n^a)$; then*

$$\left(\widehat{R(f)}(\lambda)\Phi_\alpha\right)(\eta) = \frac{(-1)^{|\alpha|}|\lambda|^{-n}2^n(2\pi)^n}{\prod_{j=1}^n a_j}\left(\widehat{f}(\lambda)\Phi_\alpha\right)(\eta),$$

(41)

where Φ_α is the orthonormal basis for $L^2(\mathbb{R}^n)$.

Proof. Because

$$\mathscr{F}_3(f)(u,v,\lambda) = \int_{\mathbb{R}} f(u,v,t)e^{i\lambda t}dt,$$

(42)

then we have

$$\left(\widehat{R(f)}(\lambda)\Phi_\alpha\right)(\eta)$$

$$= \int_{\mathbf{H}_n^a} R(f)(x,y,t)e^{i\lambda t+i\lambda(a*x)\eta+(i/2)\lambda(a*x)y}$$

$$\times \Phi_\alpha(\eta+y)\,dx\,dy\,dt$$

$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathscr{F}_3(f)(u,v,\lambda)$$

$$\cdot \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\lambda(a*x)\eta+(i/2)\lambda(a*x)y-(i/2)\lambda[(a*y)u-(a*x)v]}\right.$$

$$\left.\times \Phi_\alpha(\eta+y)\,dx\,dy\right)du\,dv.$$

(43)

Let $\widehat{\Phi_\alpha}$ denote the ordinary Fourier transform of Φ_α on $\mathbb{R}^n$; then we have

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\lambda(a*x)\eta+(i/2)\lambda(a*x)y-(i/2)\lambda[(a*y)u-(a*x)v]}$$

$$\times \Phi_\alpha(\eta+y)\,dx\,dy$$

$$= \int_{\mathbb{R}^n} e^{(i/2)\lambda(a*u)\eta+(i/2)\lambda(a*x)\eta+(i/2)\lambda(a*x)v}$$

$$\times \left(\int_{\mathbb{R}^n} \Phi_\alpha(\eta+y)e^{-(i/2)\lambda(\eta+y)(a*u-a*x)}dy\right)dx$$

$$= \int_{\mathbb{R}^n} e^{(i/2)\lambda(a*u)\eta+(i/2)\lambda(a*x)\eta+(i/2)\lambda(a*x)v}$$

$$\times \widehat{\Phi_\alpha}\left(\frac{\lambda}{2}a*(u-x)\right)dx$$

$$= e^{i\lambda(a*u)\eta+(i/2)\lambda(a*u)v}$$

$$\times \int_{\mathbb{R}^n} \widehat{\Phi_\alpha}\left(\frac{\lambda}{2}a*(u-x)\right)e^{(i/2)\lambda(a*u-a*x)(-\eta-v)}dx$$

$$= \frac{2^n|\lambda|^{-n}(2\pi)^n}{\prod_{j=1}^n a_j}e^{i\lambda(a*u)\eta+(i/2)\lambda(a*u)v}\Phi_\alpha(-v-\eta).$$

(44)

On the other hand, by the recursion formula of Hermite polynomials (see [16]) we can get

$$\Phi_\alpha(-\eta) = (-1)^{|\alpha|}\Phi_\alpha(\eta),$$

(45)

so we have

$$\left(\widehat{R(f)}(\lambda)\Phi_\alpha\right)(\eta)$$

$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathscr{F}_3 f(u,v,\lambda)\frac{2^n|\lambda|^{-n}(2\pi)^n}{\prod_{j=1}^n a_j}$$

$$\times e^{i\lambda(a*u)\eta+(i/2)\lambda(a*u)v}\Phi_\alpha(-v-\eta)\,du\,dv$$

$$= \frac{2^n|\lambda|^{-n}(2\pi)^n}{\prod_{j=1}^n a_j}$$

$$\times \int_{\mathbf{H}_n^a} f(u,v,t)e^{i\lambda t+i\lambda(a*u)\eta+(i/2)\lambda(a*u)v}$$

$$\times \Phi_\alpha(-v-\eta)\,du\,dv\,dt$$

$$= \frac{(-1)^{|\alpha|}|\lambda|^{-n}(2\pi)^n}{\prod_{j=1}^n a_j}\left(\widehat{f}(\lambda)\Phi_\alpha\right)(\eta).$$

(46)

This completes the proof. □

Let $L = \prod_{j=1}^n a_j((1/4\pi i)(\partial/\partial t))^n$, $f \in \mathscr{S}(\mathbf{H}_n^a)$; then

$$\int_{\mathbb{R}} L(f)(x,y,t)e^{i\lambda t}dt = \frac{(-\lambda)^n}{(4\pi)^n}\prod_{j=1}^n a_j\mathscr{F}_3(f)(x,y,\lambda).$$

(47)

Furthermore,

$$\widehat{L(f)}(\lambda) = \frac{(-\lambda)^n}{(4\pi)^n}\prod_{j=1}^n a_j\widehat{f}(\lambda).$$

(48)

By (41), we have

$$\left(\widehat{LR(f)}(\lambda)\Phi_\alpha\right)(\eta)$$

$$= \begin{cases} (-1)^{n+|\alpha|}\left(\widehat{f}(\lambda)\Phi_\alpha\right)(\eta), & \text{if } \lambda > 0; \\ (-1)^{|\alpha|}\left(\widehat{f}(\lambda)\Phi_\alpha\right)(\eta), & \text{if } \lambda < 0. \end{cases}$$

(49)

We can verify that $\widehat{(LR)^2(f)}(\lambda) = \widehat{f}$ and $R^{-1} = LRL$.

However, we know that $R(f)$ may not belong to $L^2(\mathbf{H}_n^a)$ for a function $f \in L^2(\mathbf{H}_n^a)$. We naturally hope to find a space on which the Radon transform is a bijection. Suppose that Ω is a subspace of $L^2(\mathbf{H}_n^a)$ such that the Radon transform R is a bijection. That is, if $f \in \Omega$, then for all $j \in \mathbb{Z}, R^j(f) \in \Omega$. From Lemma 2 and the Plancherel formula (19), we have

$$\left\|R^j(f)\right\|_{L^2(\mathbf{H}_n^a)}^2 = \frac{(2\pi)^{2jn}4^{jn}}{\prod_{i=1}^n a_i^{2j}}\int_{\mathbf{H}_n^a}\left\|\widehat{f}(\lambda)\right\|_{\text{HS}}^2|\lambda|^{-2jn}d\mu(a,\lambda).$$

(50)

Define the subspace $R(\mathbf{H}_n^a)$ of $L^2(\mathbf{H}_n^a)$ by

$$R(\mathbf{H}_n^a) = \left\{f \in \mathscr{S}(\mathbf{H}_n^a) : \int_{\mathbb{R}}\left\|\widehat{f}(\lambda)\right\|_{\text{HS}}^2|\lambda|^{2jn}d\mu(a,\lambda) < +\infty,\right.$$

$$\left.\forall j \in \mathbb{Z}\right\}.$$

(51)

Obviously, if $f_1, f_2 \in R(\mathbf{H}_n^a), f_1 \neq f_2$, then $R(f_1) \neq R(f_2)$. Furthermore, for any $g \in R(\mathbf{H}_n^a)$, we can find that $f \in R(\mathbf{H}_n^a)$, such that $R(f) = g$. In fact, we take f satisfying $\widehat{f}(\lambda) = \prod_{j=1}^n a_j \cdot (-1)^{|\alpha|} |\lambda|^n 2^{-n} (2\pi)^{-n} \widehat{g}(\lambda)$. Since

$$\int_{\mathbb{R}} \left\| \widehat{g}(\lambda) \right\|_{\mathrm{HS}}^2 |\lambda|^{2jn} d\mu(a,\lambda)$$
$$= \frac{(4\pi)^n}{\prod_{i=1}^n a_i^2} \int_{\mathbb{R}} \left\| \widehat{f}(\lambda) \right\|_{\mathrm{HS}}^2 |\lambda|^{2(j-1)n} d\mu(a,\lambda), \tag{52}$$

we can see that $f \in R(\mathbf{H}_n^a)$. This is to say that the Radon transform is a bijection from $R(\mathbf{H}_n^a)$ onto itself.

From the above discussion, we have the following.

Theorem 3. *Let $f \in R(\mathbf{H}_n^a)$; then*

$$R^{-1}(f) = LRL(f). \tag{53}$$

Next, we will give another inversion formula associated with the generalized Heisenberg group sub-Laplacian. In fact, a direct calculation shows that

$$\pi_\lambda(X_j) \varphi(\xi) = i\lambda a_j \xi_j \varphi(\xi),$$
$$\pi_\lambda(Y_j) \varphi(\xi) = \frac{\partial}{\partial \xi_j} \varphi(\xi) \tag{54}$$

and consequently

$$\pi_\lambda(\ell) = -\Delta + \lambda^2 |a * \xi|^2 \tag{55}$$

is the scaled Hermite operator, where X_j and Y_j are the left-invariant vector fields on $\mathbf{H}_n^a$.

Because

$$\sum_{i=1}^n \left(-\frac{d^2}{d\xi_i^2} + (a_i \lambda)^2 \xi_i^2 \right) = -\Delta + \lambda^2 |a * \xi|^2 \tag{56}$$

and $\{\Phi_{\alpha,\lambda}^a : \alpha \in (\mathbb{Z}^+)^n\}$ are eigenfunctions of operator $\sum_{i=1}^n (-(d^2/d\xi_i^2) + (a_i\lambda)^2 \xi_i^2)$ with eigenvalues $(2a\alpha + |a|)|\lambda|$, we have

$$\pi_\lambda(\ell) \Phi_{\alpha,\lambda}^a(\xi) = (2a\alpha + |a|) |\lambda| \Phi_{\alpha,\lambda}^a(\xi). \tag{57}$$

Set $f \in \widetilde{\mathbf{H}_k}$; then we have from [17]

$$\widehat{\ell f}(\lambda) = (2a\alpha + |a|) |\lambda| \widehat{f}(\lambda), \tag{58}$$

where $\sum_{i=1}^n \alpha_i = k$.

Let $f \in \widetilde{\mathbf{H}_k} \bigcap \mathcal{S}(\mathbf{H}_n^a)$; then we can get

$$\widehat{\ell^n f}(\lambda) = (2a\alpha + |a|)^n |\lambda|^n \widehat{f}(\lambda). \tag{59}$$

Write $\Gamma = \ell^n$; by (41), we can deduce

$$\widehat{\Gamma R f}(\lambda) = \frac{\left(2\sum_{i=1}^n a_i\alpha_i + n \right)^n (-1)^k 4^n \pi^n}{\prod_{j=1}^n a_j} \widehat{f}(\lambda). \tag{60}$$

Consequently, by Plancherel formula we obtain

$$\left\| (\Gamma R)^2 f \right\|_{L^2(\mathbf{H}_n^a)}^2 = \int_{-\infty}^{+\infty} \left\| \widehat{(\Gamma R)^2 f}(\lambda) \right\|_{\mathrm{HS}}^2 d\mu(a,\lambda)$$
$$= \frac{(2a\alpha + |a|)^{4n} 4^{4n} \pi^{4n}}{\prod_{j=1}^n a_j^4} \left\| f \right\|_{L^2(\mathbf{H}_n^a)}^2. \tag{61}$$

Theorem 4. *Suppose that $f \in \widetilde{\mathbf{H}_k} \bigcap \mathcal{S}(\mathbf{H}_n^a)$; then we have*

$$R^{-1}(f) = \prod_{j=1}^n a_j^2 \cdot (2a\alpha + |a|)^{-2n} (4\pi)^{-2n} (\Gamma R \Gamma)(f), \tag{62}$$

where $\sum_{i=1}^n \alpha_i = k$.

We conclude this section by giving the inverse Radon transform with generalized sub-Laplacian.

Let u_j be positive constants for $j = 1, 2, \ldots, n$. We define generalized sub-Laplacian by

$$\widetilde{\ell} = -\sum_{j=1}^n u_j \left(X_j^2 + Y_j^2 \right), \tag{63}$$

where X_j and Y_j are the left-invariant vector fields on $\mathbf{H}_n^a$. Let $f \in \widetilde{\mathbf{H}_k}$; by an analogous computation, we have

$$\pi_\lambda(\widetilde{\ell}) \Phi_{\alpha,\lambda}^a(\xi) = |\lambda| \sum_{i=1}^n u_i (2a_i\alpha_i + a_i) \Phi_{\alpha,\lambda}^a(\xi), \tag{64}$$

$$\widehat{\widetilde{\ell} f}(\lambda) = |\lambda| \sum_{i=1}^n u_i (2a_i\alpha_i + a_i) \widehat{f}(\lambda). \tag{65}$$

If we take $u_j = 1/a_j$, then

$$\widehat{\widetilde{\ell} f}(\lambda) = |\lambda| (2k + n) \widehat{f}(\lambda). \tag{66}$$

In this case the Fourier transform of f under the action of generalized sub-Laplacian ℓ of $\mathbf{H}_n^a$ is the same as that of the sub-Laplacian on $\mathbf{H}^n$ (see [11]). Write $\widetilde{\Gamma} = \widetilde{\ell}^n$; by Lemma 2, Plancherel formula, and (65), we obtain the theorem below.

Theorem 5. *Suppose that $f \in \widetilde{\mathbf{H}_k} \bigcap \mathcal{S}(\mathbf{H}_n^a)$. Then we have*

$$R^{-1}(f) = \prod_{j=1}^n a_j^2 \cdot \left(\sum_{i=1}^n u_i a_i (2\alpha_i + 1) \right)^{-2n} (4\pi)^{-2n} \left(\widetilde{\Gamma} R \widetilde{\Gamma} \right)(f), \tag{67}$$

where $\sum_{i=1}^n \alpha_i = k$. Especially, when $u_i = 1/a_i$, we have

$$R^{-1}(f) = \prod_{j=1}^n a_j^2 \cdot (2k + n)^{-2n} (4\pi)^{-2n} \left(\widetilde{\Gamma} R \widetilde{\Gamma} \right)(f). \tag{68}$$
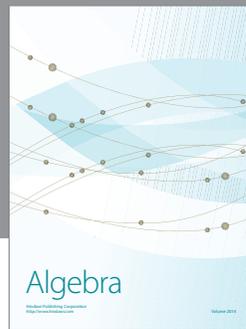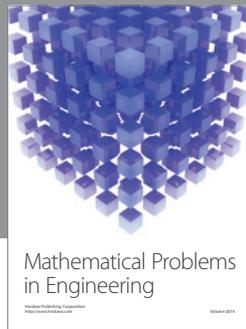
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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