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Research Article

Potra-Pták Iterative Method with Memory

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The problem is to extend the method proposed by Soleymani et al. (2012) to a method with memory. Following this aim, a free parameter is calculated using Newton's interpolatory polynomial of the third degree. So the R-order of convergence is increased from 4 to 6 without any new function evaluations. Numerically the extended method is examined along with comparison to some existing methods with the similar properties.

1. Introduction

Root finding is a great task in mathematics, both historically and practically. It has attracted attention of great mathematicians like Gauss and Newton. It has real major applications and because of these real features it is still alive as a research field.

Kung and Traub's conjecture is the basic fact to construct optimal multipoint methods without memory [1]. On the other hand, multipoint methods with memory can increase efficiency index of an optimal method without memory without consuming any new functional evaluations and merely using accelerator parameter(s). This great power of methods with memory has not been well considered until very recently. So we have been motivated to extend modified Potra-Pták [2] to its with memory method.

Traub in his book [3] introduced methods with and without memory for the first time. Moreover, he constructed a Steffensen-type method with memory using secant approach. In fact, he increased the order of convergence of the Steffensen method [4] from 2 to 2.41. This is the first method with memory based on our best knowledge. In other words, Traub changed Steffensen's method slightly as follows (see [3, pages 185–187]):

$$
\begin{gathered}
x_0,\ w_0\ \gamma_0 \text{ are given suitably,}\\
x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad 0 \neq \gamma_n \in R,\ n = 0, 1, 2, \ldots,\\
N_1(x) = f(x_n) + (x - x_n) f[x_n, w_n],\\
\gamma_{n+1} = -\frac{1}{N_1'(x_n)},\\
w_{n+1} = x_{n+1} + \gamma_{n+1} f(x_{n+1}).
\end{gathered} \tag{1}
$$

The parameter γ_n is called *self-accelerator* and method (1) has convergence order of 2.41. It is still possible to increase the convergence order using better self-accelerator parameter based on better Newton interpolation. Free derivative can be considered as another virtue of (1).

We use the symbols $\rightarrow$, O, and $\sim$ according to the following conventions [3]. If $\lim_{x_n \to \infty} g(x_n) = C$, we write $g(x_n) \rightarrow C$ or $g \rightarrow C$. If $\lim_{x \to a} g(x) = C$, we write $g(x) \rightarrow C$ or $g \rightarrow C$. If $f/g \rightarrow C$, where C is a nonzero constant, we write $f = O(g)$ or $f \sim Cg$. Let $f(x)$ be a function defined on an interval I, where I is the smallest interval containing $k+1$ distinct nodes $x_1, x_2, \ldots, x_k$. The divided difference $f[x_0, x_1, \ldots, x_k]$ with kth-order is defined as follows: $f[x_0] = f(x_0)$,

$$
\begin{aligned}
f[x_0] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \ldots, f[x_0, x_1, \ldots, x_k]\\
&= \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots x_{k-1}]}{x_k - x_0}.
\end{aligned} \tag{2}
$$

Moreover, we recall the definition of efficiency index (EI) as $E = p^{1/n}$, where p is the order of convergence and n is the total number of function evaluations per iteration.

This paper is organized as follows. Section 2 reviews modified Potra-Pták's method and we try to remodify it slightly too. Error equation for our modification is provided. In Section 3, development to with memory is carried out along with the discussion of its R-order. Numerical examinations and comparisons are presented in the last section.

2. Remodified Optimal Derivative-Free Potra-Pták's Method

In this section, our primal goal is to modify Soleymani et al. method slightly so that its error equation can provide better form in the case with memory. In fact, we prove that our modified method can generate order of convergence of 6 while theirs has order of convergence of 5.2 in the case of with memory.

Derivative-free iterative methods for solving nonlinear equation $f(x) = 0$ are important in the sense that in many practical situation it is preferable to avoid calculation of derivative of f. One such scheme is

$$x_{k+1} = x_k - \frac{\gamma f(x_k)^2}{f(x_k + \gamma f(x_k)) - f(x_k)}, \quad k = 0, 1, 2, \ldots, \ \gamma \in R - \{0\} \tag{3}$$

which is obtained from Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots \tag{4}$$

by approximating the derivative $f'(x_k)$ by the quotient $(f(x_k + \gamma f(x_k)) - f(x_k))/\gamma f(x_k)$. Scheme (3) defines a one-parameter (γ) family of methods and has the same order and efficiency index as that of Newton's method [3, 4].

Recently, based on scheme (3), Soleymani et al. [2] have extended the idea of this family and presented Potra-Pták's derivative free families of two-point methods without memory as follows

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \qquad w_n = x_n + \gamma f(x_n), \quad \gamma \in R - \{0\}, \\ x_{n+1} &= x_n - \frac{f(x_n) + f(y_n)}{f[x_n, w_n]} - \left(\frac{2f(x_n) + af(y_n)}{f[x_n, w_n]} \left(\frac{f(y_n)}{f(x_n)} \right)^2 \right) \times \left(1 - \frac{\gamma f[x_n, w_n]}{2 + 2\gamma f[x_n, w_n]} \right). \end{aligned} \tag{5}$$

Moreover, they have proved.

Theorem 1 (see [2]). *Let α be a simple root of the sufficiently differentiable function f in an open interval D. If x_0 is sufficiently close to α, then* (5) *is of local forth order and satisfies the error equation below,*

$$e_{n+1} = \left(1 + \gamma f'(\alpha)\right) A_1 e_n^4 + O\left(e_n^5\right), \tag{6}$$

where $e_n = x_n - \alpha$, $a \in R$, $c_k = f^{(k)}(\alpha)/(k! f'(\alpha))$, $k = 2, 3, \ldots$, and

$$\begin{aligned} A_1 = -\frac{c_2}{2} \Big[& 2\left(c_3 + c_3 f'(\alpha)\gamma\right) \\ & + c_2^2 \Big(a\left(1 + f'(\alpha)\gamma\right)\left(2 + f'(\alpha)\gamma\right) \\ & - 2\left(5 + f'(\alpha)\gamma\left(5 + f'(\alpha)\gamma\right)\right)\Big)\Big]. \end{aligned} \tag{7}$$

As you can see, the order of convergence is 4. It is clear that error equation (6) has linear factor $(1 + f'(\alpha)\gamma)$; it is better to correct approach (5) in such a way that its error equation has the quadratic factor $(1 + f'(\alpha)\gamma)^2$. So, as we can prove later, this factor increases convergence order up to 6. To this end, it is just enough to correct second step in (5) as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \qquad w_n = x_n + \gamma f(x_n), \quad \gamma \in R - \{0\}, \\ x_{n+1} &= x_n - \frac{f(x_n) + f(y_n)}{f[x_n, w_n]} - \left(\frac{2f(x_n) + af(y_n)}{f[y_n, w_n]} \left(\frac{f(y_n)}{f(x_n)} \right)^2 \right) \times \left(1 - \frac{\gamma f[x_n, w_n]}{2 + 2\gamma f[x_n, w_n]} \right). \end{aligned} \tag{8}$$

Hence, method without memory (8) is still optimal and in the following theorem we establish its error equation.

Theorem 2. *Let α be a simple root of the sufficiently differentiable function f in an open interval D. If x_0 is sufficiently close to α, then* (8) *is of local forth order and satisfies the error equation below*

$$e_{n+1} = \left(1 + \gamma f'(\alpha)\right)^2 A_2 e_n^4 + O\left(e_n^5\right), \tag{9}$$

where $e_n = x_n - \alpha$, $a \in R$, $c_k = f^{(k)}(\alpha)/(k! f'(\alpha))$, $k = 2, 3, \ldots$, and

$$A_2 = -\frac{1}{2} c_2 \left(\left(2(-3 + a) + (-2 + a) f'(\alpha)\gamma \right) c_2^2 + 2c_3 \right). \tag{10}$$

Proof. We provide the Taylor expansion of any term involved in (8). By Taylor expanding around the simple root in the nth iterate, we have

$$\begin{aligned}\frac{f(x_n)}{f[x_n,w_n]} &= e_n^1 - c_2\left(1+f'(\alpha)\gamma\right)e_n^2\\ &\quad+\left(-c_3\left(1+f'(\alpha)\gamma\right)\left(2+f'(\alpha)\gamma\right)\right.\\ &\quad\left.+c_2^2\left(2+f'(\alpha)\gamma\left(2+f'(\alpha)\gamma\right)\right)\right)e_n^3\\ &\quad+O\left(e_n^4\right).\end{aligned} \tag{11}$$

By considering this relation and the first step of (8), we obtain

$$\begin{aligned}x_n-\frac{f(x_n)}{f[x_n,w_n]} &= \alpha + c_2\left(1+f'(\alpha)\gamma\right)e_n^2\\ &\quad-\left(-c_3\left(1+f'(\alpha)\gamma\right)\left(2+f'(\alpha)\gamma\right)\right.\\ &\quad\left.+c_2^2\left(2+f'(\alpha)\gamma\left(2+f'(\alpha)\gamma\right)\right)\right)e_n^3\\ &\quad+O\left(e_n^4\right).\end{aligned} \tag{12}$$

At this time, we should expand $f(y_n)$ around the root by taking into consideration (12). Accordingly, we have

$$\begin{aligned}&f(y_n)\\ &= c_2\left(1+f'(\alpha)\gamma\right)f'(\alpha)e_n^2\\ &\quad-\left(-c_3\left(1+f'(\alpha)\gamma\right)\left(2+f'(\alpha)\gamma\right)\right.\\ &\quad\left.+c_2^2\left(2+f'(\alpha)\gamma\left(2+f'(\alpha)\gamma\right)\right)\right)f'(\alpha)e_n^3+f'(\alpha)\\ &\quad\times\left(c_4\left(1+f'(\alpha)\gamma\right)\left(3+f'(\alpha)\gamma\left(3+f'(\alpha)\gamma\right)\right)\right.\\ &\quad+c_2^3\left(5+f'(\alpha)\gamma\left(7+f'(\alpha)\gamma\left(4+f'(\alpha)\gamma\right)\right)\right)-c_2c_3\\ &\quad\left.\times\left(7+f'(\alpha)\gamma\left(10+f'(\alpha)\gamma\left(7+2f'(\alpha)\gamma\right)\right)\right)\right)e_n^4\\ &\quad+O\left(e_n^5\right).\end{aligned} \tag{13}$$

Additionally, we obtain

$$\begin{aligned}x_n&-\frac{f(x_n)+f(y_n)}{f[x_n,w_n]}\\ &= \alpha + c_2^2\left(1+f'(\alpha)\gamma\right)\left(2+f'(\alpha)\gamma\right)e_n^3+O\left(e_n^4\right).\end{aligned} \tag{14}$$

Similarly, the same Taylor expansion results in

$$\begin{aligned}&\left(\frac{2f(x_n)+af(y_n)}{f[y_n,w_n]}\left(\frac{f(y_n)}{f(x_n)}\right)^2\right)\left(1-\frac{\gamma f[x_n,w_n]}{2+2\gamma f[x_n,w_n]}\right)\\ &\quad= c_2^2\left(1+f'(\alpha)\gamma\right)\left(2+f'(\alpha)\gamma\right)e_n^3\\ &\quad+\frac{1}{2}c_2\left(1+f'(\alpha)\gamma\right)^2\\ &\quad\times\left(\left(2(-3+a)+(-2+a)f'(\alpha)\gamma\right)c_2^2+2c_3\right)e_n^4\\ &\quad+O\left(e_n^5\right).\end{aligned} \tag{15}$$

Using (12)–(15) in the last step of (8) provides finally

$$\begin{aligned}&x_{n+1}-\alpha\\ &\quad= x_n-\frac{f(x_n)+f(y_n)}{f[x_n,w_n]}\\ &\quad-\left(\frac{2f(x_n)+af(y_n)}{f[y_n,w_n]}\left(\frac{f(y_n)}{f(x_n)}\right)^2\right)\\ &\quad\times\left(1-\frac{\gamma f[x_n,w_n]}{2+2\gamma f[x_n,w_n]}\right)-\alpha\\ &\quad= -\frac{1}{2}c_2\left(1+f'(\alpha)\gamma\right)^2\\ &\quad\times\left(\left(2(-3+a)+(-2+a)f'(\alpha)\gamma\right)c_2^2+2c_3\right)e_n^4\\ &\quad+O\left(e_n^5\right),\end{aligned} \tag{16}$$

which shows that (8) is a derivative-free family of two-step methods with optimal convergence rate of 4. This completes the proof. □

3. Development and Construction with Memory Family

This section concerns with extension of (8) to a method with memory since its error equation contains the parameter γ which can be approximated in such a way that increase the local order of convergence. So we set $\gamma=\gamma_k$ as the iteration proceeds by the formula $\gamma_k=-1/\overline{f}'(\alpha)$ for $k=1,2,\ldots$, where $\overline{f}'(\alpha)$ is an approximation of $f'(\alpha)$. We have a method through the following forms of γ_k:

$$\gamma_k=-\frac{1}{\overline{f}'(\alpha)}=-\frac{1}{N_3'(x_k)}, \tag{17}$$

where $N_3'(t) = N_3(t; x_k, y_{k-1}, w_{k-1}, x_{k-1})$ is Newton's interpolatory polynomial of third degree, set through four available approximations x_k, y_{k-1}, w_{k-1}, and x_{k-1} and

$$\begin{aligned} N_3'(x_k) &= \left[\frac{d}{dt}N_3(t)\right]_{t=x_k} \\ &= \left[\frac{d}{dt}\left[f(x_k) + f[x_k, y_{k-1}](t-x_k)\right.\right. \\ &\qquad + f[x_k, y_{k-1}, w_{k-1}](t-x_k)(t-y_{k-1}) \\ &\qquad + f[x_k, y_{k-1}, w_{k-1}, x_{k-1}] \\ &\qquad \left.\left.\times (t-x_k)(t-y_{k-1})(t-w_{k-1})\right]\right]_{t=x_k} \\ &= f[x_k, y_{k-1}] + f[x_k, y_{k-1}, w_{k-1}](x_k - y_{k-1}) \\ &\quad + f[x_k, y_{k-1}, w_{k-1}, x_{k-1}](x_k - y_{k-1})(x_k - w_{k-1}). \end{aligned} \tag{18}$$

By using Taylor's expansion of $f(x)$ around the root α, we have

$$f(x) = f'(\alpha)\left(e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + \cdots\right), \tag{19}$$

where $e = x - \alpha$. By using (18) and (19), we calculate

$$\begin{aligned} N_3'(x_k) = f'(\alpha)&\left[1 + 2c_2e_k + 3c_3e_k^2\right. \\ &+ c_4\left(e_{k-1}e_k^2 + e_{k-1,y}^2e_k^2\right. \\ &+ e_{k-1,w}^2e_k^2 - e_{k-1}e_{k-1,y}e_k \\ &- e_{k-1,y}e_{k-1,w}e_k - e_{k-1}e_{k-1,w}e_k \\ &\left.\left.+ e_{k-1}e_{k-1,y}e_{k-1,w} + 3e_k^3\right) + \cdots\right] \\ = f'(\alpha)&\left[1 + c_4e_{k-1}e_{k-1,y}e_{k-1,w} + O(e_k)\right]. \end{aligned} \tag{20}$$

According to this and (17) we find

$$1 + \gamma f'(\alpha) \sim c_4e_{k-1}e_{k-1,y}e_{k-1,w}. \tag{21}$$

For general case one can consult [3].

In order to obtain the order of convergence of the family of two-point methods with memory (8), where γ_k is calculated using the formula (17), we will use the concept of the R-order of convergence [3]. Now, we can state the following convergence theorem.

Theorem 3. *If an initial approximation x_0 is sufficiently close to the zero α of $f(x)$ and the parameter γ_k in the iterative scheme* (8) *is recursively calculated by the forms given in* (17), *then the R-order of convergence is at least 6.*

Proof. Let $\{x_k\}$ be a sequence of approximations generated by an iterative method with memory (IM). If this sequence converges to the zero α of f with the R-order ($\geq r$) of IM, then we write

$$e_{k+1} \sim D_{k,r}e_k^r, \qquad e_k = x_k - \alpha, \tag{22}$$

where $D_{k,r}$ tends to the asymptotic error constant D_r of IM when $k \to \infty$. Thus

$$e_{k+1} \sim D_{k,r}\left(D_{k-1,r}e_{k-1}^r\right)^r = D_{k,r}D_{k-1,r}^re_{k-1}^{r^2}. \tag{23}$$

Let $e_{k-1,y} = y_{k-1} - \alpha$, $e_{k-1,w} = w_{k-1} - \alpha$, then we have

$$e_{k,w} \sim \left(1 + \gamma_kf'(\alpha)\right)e_k + O\left(e_k^2\right), \tag{24}$$

$$e_{k,y} \sim c_2\left(1 + \gamma_kf'(\alpha)\right)e_k^2 + O\left(e_k^3\right), \tag{25}$$

$$e_{k+1} \sim A_2\left(1 + \gamma_kf'(\alpha)\right)^2e_k^4 + O\left(e_k^5\right), \tag{26}$$

where $A_2 = -(1/2)c_2[(2(-3+a) + (-2+a)f'(\alpha)\gamma)c_2^2 + 2c_3]$. In the sequel, we obtain the R-order of convergence of family (8) for approach (17) applied to the calculation of γ_k.

Assume that the iterative sequences y_k and x_k have the R-orders; p and r, respectively, then, bearing in mind (22) we obtain

$$e_{k,y} \sim D_{k,p}e_k^p \sim D_{k,p}\left(D_{k-1,r}e_{k-1}^r\right)^p \sim D_{k,p}D_{k-1,r}^pe_{k-1}^{rp}, \tag{27}$$

and then, we obtain

$$\begin{aligned} e_{k,y} &\sim c_2\left(1 + \gamma_kf'(\alpha)\right)e_k^2 \sim c_2\left(c_4e_{k-1}e_{k-1,y}e_{k-1,w}\right)e_k^2 \\ &\sim c_2c_4\left(e_{k-1}\right)\left(D_{k-1,p}e_{k-1}^p\right)\left(D_{k-1,s}e_{k-1}^s\right)\left(D_{k-1,r}e_{k-1}^r\right)^2 \\ &\sim c_2c_4D_{k-1,p}D_{k-1,s}D_{k-1,r}^2e_{k-1}^{2r+s+p+1}. \end{aligned} \tag{28}$$

Assume that the iterative sequence w_k has the R-order s; then bearing in mind (22) we obtain

$$e_{k,w} \sim D_{k,s}e_k^s \sim D_{k,s}\left(D_{k-1,r}e_{k-1}^r\right)^s \sim D_{k,s}D_{k-1,r}^se_{k-1}^{rs}. \tag{29}$$

and then, we obtain

$$\begin{aligned} e_{k,w} &\sim \left(1 + \gamma_kf'(\alpha)\right)e_k \sim \left(c_4e_{k-1}e_{k-1,y}e_{k-1,w}\right)e_k \\ &\sim c_4e_{k-1}\left(D_{k-1,p}e_{k-1}^p\right)\left(D_{k-1,s}e_{k-1}^s\right)\left(D_{k-1,r}e_{k-1}^r\right) \\ &\sim c_4D_{k-1,p}D_{k-1,s}D_{k-1,r}e_{k-1}^{r+s+p+1}, \end{aligned} \tag{30}$$

$$\begin{aligned} e_{k+1} &\sim a_{k,4}\left(1 + \gamma_kf'(\alpha)\right)^2e_k^4 \sim a_{k,4}\left(c_4e_{k-1}e_{k-1,y}e_{k-1,w}\right)^2e_k^4 \\ &\sim a_{k,4}c_4\left(e_{k-1}\right)^2\left(D_{k-1,p}e_{k-1}^p\right)^2\left(D_{k-1,s}e_{k-1}^s\right)^2\left(D_{k-1,r}e_{k-1}^r\right)^4 \\ &\sim a_{k,4}c_4D_{k-1,p}^2D_{k-1,s}^2D_{k-1,r}^4e_{k-1}^{4r+2s+2p+2}. \end{aligned} \tag{31}$$

Combining the exponents of e_{k-1} on the right-hand sides of (27)-(28), (29)-(30), and (23)–(31), we form the nonlinear system of three equations in p, s, and r:

$$\begin{aligned} rp - 2r - (p+s) - 1 &= 0, \\ rs - r - (p+s) - 1 &= 0, \\ r^2 - 4r - 2(p+s) - 2 &= 0. \end{aligned} \tag{32}$$

Table 1: $f_1(x) = \log(1+x^2) + e^{x^2-3x}\sin x$, $\alpha = 0$, $x_0 = 0.35$, $\gamma_0 = 0.01$.

Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	$r_c(20)$
Potra-Pták without memory	6.040 (−3)	5.672 (−10)	8.280 (−37)	3.822
Equations (8) and (17) with memory	6.050 (−3)	8.723 (−15)	6.425 (−84)	5.841
Kung and Traub [1] without memory	5.545 (−3)	4.864 (−9)	3.045 (−33)	3.999
Equation (34) with memory	5.546 (−3)	4.708 (−17)	1.374 (−96)	5.654
Zheng et al. [5] without memory	3.541 (−2)	2.974 (−6)	2.127 (−22)	3.930
Equation (35) with memory	5.647 (−3)	8.272 (−18)	1.022 (−101)	5.658

Table 2: $f_2(x) = \prod_{i=1}^{5}(x-i)$, $\alpha = 2$, $x_0 = 1.6$, $\gamma_0 = 0.01$.

Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	$r_c(20)$
Potra-Pták without memory	5.522 (−3)	1.063 (−10)	7.391 (−41)	3.910
Equations (8) and (17) with memory	5.623 (−3)	3.822 (−16)	1.170 (−93)	5.888
Kung and Traub [1] (without memory)	6.125 (−3)	2.402 (−9)	5.453 (−35)	4.004
Equation (34) with memory	6.125 (−3)	5.888 (−15)	1.651 (−86)	5.955
Zheng at al. [5] without memory	6.021 (−3)	1.525 (−9)	6.077 (−36)	4.003
Equation (35) with memory	6.021 (−3)	3.251 (−15)	1.274 (−88)	5.985

Nontrivial solution of this system is $s = 2$, $p = 3$, and $r = 6$, and we conclude that the lower bound of the R-order of the method with memory is 6. □

Similarly, one can prove the following.

Theorem 4. *If an initial approximation x_0 is sufficiently close to the zero α of $f(x)$ and the parameter γ_k in the iterative scheme* (5) *is recursively calculated by the forms given in* (17), *then the R-order of convergence is at least 5.2.*

4. Numerical Examples

To examine practical aspects of the proposed modified Potra-Pták's without and with memory we implement it here in action. In other words, we demonstrate the convergence behavior of the method with memory (8), where γ_k is calculated by (17). For comparison purposes, we pick up Kung and Traub [1] and Zheng et al. [5] with and without memories. We use these notations. The errors $|x_k - \alpha|$ denote approximations to the sought zeros. $A(-h)$ stands for $A \times 10^{-h}$. Moreover, r_c indicates computational order of convergence and is computed [2]

$$r_c = \frac{\log\left(\left|f(x_k)/f(x_{k-1})\right|\right)}{\log\left(\left|f(x_{k-1})/f(x_{k-2})\right|\right)}. \tag{33}$$

The software Mathematica 8, with 1000 arbitrary precision arithmetic, has been used in our computations. The results alongside the test functions are given in Tables 1 and 2, while $\gamma = \gamma_0 = 0.01$ [3]. From Tables 1 and 2, we can conclude that our methods work numerically well and are successfully competing with the existing methods. Indeed the last columns of these tables show that both numerical and theoretical aspects support each other.

For comparison purposes, we consider the following methods.

Two-Point Method by Kung and Traub [1]:

$$\begin{aligned}
&x_0,\ w_0\ \gamma_0 \text{ are given suitably},\\
&y_n = x_n - \frac{f(x_n)}{f[w_n, x_n]}, \quad n = 0,1,2,\ldots,\\
&x_{n+1} = y_n - \frac{f(y_n) f(w_n)}{[f(w_n) - f(y_n)] f[x_n, y_n]}, \quad k = 0,1,\ldots\\
&\gamma_{n+1} = -\frac{1}{N_3'(x_n)},\\
&w_{n+1} = x_{n+1} + \gamma_{n+1} f(x_{n+1}).
\end{aligned} \tag{34}$$

Two-Point Method by Zheng et al. [5]:

$$\begin{aligned}
&x_0,\ w_0\ \gamma_0 \text{ are given suitably},\\
&y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad n = 0,1,2,\ldots,\\
&x_{n+1} = y_n - \frac{f(y_n)}{f[y_n, w_n] + f[y_n, x_n, w_n](y_n - x_n)},\\
&\gamma_{n+1} = -\frac{1}{N_3'(x_n)},\\
&w_{n+1} = x_{n+1} + \gamma_{n+1} f(x_{n+1}).
\end{aligned} \tag{35}$$

From Tables 1 and 2, it can be seen that our modified method without memory works truly; moreover, its with memory competes the existing methods. To sum up, Potra and Pták [6] constructed two-point method without memory with convergence order of 3; it is not optimal in the sense of Kung and Traub. Cordero et al. [7] could make it optimal. In other words, they introduce optimal two- and three-point methods with order of convergence of 4 and 8, respectively. Though their methods are optimal, they are not derivative-free. Freshly, Soleymani et al. [2] have drawn two point methods without memory from Potra and pták method. One is derivative-free and the other is not. In addition their derivative method results two steps method by Cordero et al. [7] for $a = 0$ (See (5)). In this work, we modified their derivative-free method at first. Then, we generalized it to method with memory with efficiency index $E(p, n) = 4^{1/3} = 1.8$; see more about efficiency index in [7]. Therefore, a two-step method with memory can obtain performance even better than four-step methods without memory with efficiency index $16^{1/5} = 1.7$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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