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Research Article

Exponential Decay to the Degenerate Nonlinear Coupled Beams System with Weak Damping

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We consider a nonlinear degenerate coupled beams system with weak damping. We show using the Nakao method that the solution of this system decays exponentially when the time tends to infinity.

1. Introduction

For the last several decades, various types of equations have been employed as some mathematical models describing physical, chemical, biological, and engineering systems. Among them, the mathematical models of vibrating, flexible structures have been considerably stimulated in recent years by an increasing number of questions of practical concern. Research on stabilization of distributed parameter systems has largely focused on the stabilization of dynamic models of individual structural members such as strings, membranes, and beams.

This paper is devoted to the study of the existence, uniqueness, and uniform decay rates of the energy of solution for the nonlinear degenerate coupled beams system with weak damping given by

$$K_1(x,t)u_{tt} + \Delta^2 u - M\left(\|u\|^2 + \|v\|^2\right)\Delta u + u_t = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$K_2(x,t)v_{tt} + \Delta^2 v - M\left(\|u\|^2 + \|v\|^2\right)\Delta v + v_t = 0 \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$u = v = \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \Sigma, \tag{1.3}$$

$$(u(x,0), v(x,0)) = (u_0, v_0) \quad \text{in } \Omega, \tag{1.4}$$

$$(u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) \quad \text{in } \Omega, \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ, $T > 0$ is a real arbitrary number, and η is the unit normal at $\Sigma = \Gamma \times (0,T)$ direct towards the exterior of $\Omega \times (0,T)$. Here $K_i \in C^1([0,T]; H_0^1(\Omega) \cap L^\infty(\Omega))$, $i = 1,2$ and $M \in C^1([0,\infty[)$, see Section 2 for more details.

Problems related to the system (1.1)–(1.5) are interesting not only from the point of view of PDE general theory, but also due to its applications in mechanics. For instance, when we consider only one equation without the dissipative term, that is,

$$K(x,t)u_{tt} + \Delta^2 u - M\left(\|u\|^2\right)\Delta u = 0 \quad \text{in } \Omega \times (0,T) \tag{1.6}$$

and with $K(x,t) = 1$, it is a generalization of one-dimensional model proposed by Woinowsky-Krieger [1] as a model for the transverse deflection $u(x,t)$ of an extensible beam of natural length whose ends are held a fixed distance apart. The nonlinear term represents the change in the tension of the beam due to its extensibility. The model has also been discussed by Eisley [2], while related experimental results have been given by Burgreen [3]. Dickey [4] considered the initial-boundary value problem for one-dimensional case of (1.6) with $K(x,t) = 1$ in the case when the ends of the beam are hinged. He showed how the model affords a description of the phenomenon of "dynamic buckling." The one-dimensional case has also been studied by Ball [5]. He extended the work of Dickey [4] in several directions. In both cases he used the techniques of Lions [6] to prove that the initial boundary value problem is weakly well-posed. Menzala [7] studied the existence and uniqueness of solutions of (1.6) with $K(x,t) = 1$, $x \in \mathbb{R}^n$, and $M \in C^1[0,\infty[$ and $M(\lambda) \geq m_0 > 0$, for all $\lambda \geq 0$. The existence, uniqueness, and boundary regularity of weak solutions were considered by Ramos [8] with $K(x) \geq k_0 > 0$, $x \in \Omega$. See also Pereira et al. [9]. The abstract model

$$u_{tt} + \mathbf{A}^2 u + M\left(\left|\mathbf{A}^{1/2}\right|^2\right)\mathbf{A}u = 0 \tag{1.7}$$

of (1.6), where $\mathbf{A}$ is a nonbounded self-adjoint operator in a conveniently Hilbert space has been studied by Medeiros [10]. He proved that the abstract model is well-posed in the weak sense, since $M \in C^1[0,\infty[$ with $M(\lambda) \geq m_0 + m_1\lambda$, for all $\lambda \geq 0$, where m_0 and m_1 are positive constants. Pereira [11] considered the abstract model (1.7) with dissipative term u_t. He proved the existence, uniqueness, and exponential decay of the solutions with the following assumptions about M:

$$M \in C^0([0,\infty[) \text{ with } M(\lambda) \geq -\beta, \quad \forall \lambda \geq 0,\ 0 < \beta < \lambda_1, \tag{1.8}$$

where λ_1 is the first eigenvalue of

$$\mathbf{A}^2 u - \lambda \mathbf{A} u = 0. \tag{1.9}$$

Our main goal here is to extend the previous results for a nonlinear degenerate coupled beams system of type (1.1)–(1.5). We show the existence, uniqueness, and uniform exponential decay rates.

Our paper is organized as follows. In Section 2 we give some notations and state our main result. In Section 3 we obtain the existence and uniqueness for global weak solutions. To obtain the global weak solution we use the Faedo-Galerkin method. Finally, in Section 4 we use the Nakao method (see Nakao [12]) to derive the exponential decay of the energy.

2. Assumptions and Main Result

In what follows we are going to use the standard notations established in Lions [6].

Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u,v) = \int_{\Omega} u(x)v(x)dx \tag{2.1}$$

and norm

$$|u| = \sqrt{(u,v)}. \tag{2.2}$$

We also consider the Sobolev space $H^1(\Omega)$ endowed with the scalar product

$$(u,v)_{H^1(\Omega)} = (u,v) + (\nabla u, \nabla v). \tag{2.3}$$

We define the subspace of $H^1(\Omega)$, denoted by $H_0^1(\Omega)$. This space endowed with the norm induced by the scalar product

$$((u,v))_{H_0^1(\Omega)} = (\nabla u, \nabla v) \tag{2.4}$$

is a Hilbert space.

2.1. Assumptions on the Functions K_i, $i = 1,2$, and M

To obtain the weak solution of the system (1.1)–(1.5) we consider the following hypothesis:

$$\begin{aligned} &K_i \in C^1\left([0,T]; H_0^1(\Omega) \cap L^{\infty}(\Omega)\right), \quad i = 1,2, \\ &\text{with } K_i(x,t) \geq 0, \quad \forall (x,t) \in \Omega \times (0,T), \\ &\text{and there exists } \gamma > 0 \text{ such that } K_i(x,0) \geq \gamma > 0, \end{aligned} \tag{2.5}$$

$$\left|\frac{\partial K_i}{\partial t}\right|_{\mathbb{R}} \leq \delta + C(\delta)K_i, \quad i = 1,2, \ \forall \delta > 0, \tag{2.6}$$

$$\begin{gathered}M \in C^1([0,\infty[) \text{ with } M(\lambda) \geq -\beta, \quad \forall \lambda \geq 0, \\ 0<\beta<\lambda_1, \quad \lambda_1 \text{ is the first eingenvalue of the stationary problem}, \\ \Delta^2 u - \lambda(-\Delta u) = 0.\end{gathered} \tag{2.7}$$

Remark 2.1. Let λ_1 be the first eingevalue of $\Delta^2 u - \lambda(-\Delta u) = 0$; then (see Miklin [13])

$$\lambda_1 = \inf_{w \in H_0^2(\Omega)} \frac{|\Delta w|^2}{|\nabla w|^2} > 0. \tag{2.8}$$

3. Existence and Uniqueness Results

Now, we are in a position to state our result about the existence of weak solution to the system (1.1)–(1.5).

Theorem 3.1. *Let one take* $(u_0, v_0) \in (H_0^1(\Omega) \cap H^4(\Omega))^2$ *and* $(u_1, v_1) \in (H_0^2(\Omega))^2$*, and let one suppose that assumptions* (2.5), (2.6) *and* (2.7) *hold. Then, there exist unique functions* $u, v : [0,T] \to L^2(\Omega)$ *in the class*

$$\begin{gathered}(u,v) \in \left(\left(L^\infty_{\text{loc}}(0,\infty) : H_0^2(\Omega) \cap H^4(\Omega)\right)\right)^2, \\ (u_t, v_t) \in \left(\left(L^\infty_{\text{loc}}(0,\infty) : H_0^2(\Omega)\right)\right)^2, \\ (u_{tt}, v_{tt}) \in \left(\left(L^\infty_{\text{loc}}(0,\infty) : L^2(\Omega)\right)\right)^2\end{gathered} \tag{3.1}$$

satisfying

$$\begin{gathered}K_1(x,t)u_{tt} + \Delta^2 u - M\left(\|u\|^2 + \|v\|^2\right)\Delta u + u_t = 0 \quad \textit{in } L^2_{\text{loc}}\left(0,\infty; L^2(\Omega)\right), \\ K_2(x,t)v_{tt} + \Delta^2 v - M\left(\|u\|^2 + \|v\|^2\right)\Delta v + v_t = 0 \quad \textit{in } L^2_{\text{loc}}\left(0,\infty; L^2(\Omega)\right),\end{gathered} \tag{3.2}$$

$$\begin{gathered}(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \textit{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) \quad \textit{in } \Omega.\end{gathered} \tag{3.3}$$

Proof. Since $K_i \geq 0$, $i = 1,2$, we first perturb the system (1.1)–(1.5) with the terms $\varepsilon u_{tt}, \varepsilon v_{tt}$, with $0 < \varepsilon < 1$, and we apply the Faedo-Galerkin method to the perturbed system. After we pass to the limit with $\varepsilon \to 0$ in the perturbed system and we obtain the solution for the system (1.1)–(1.5).

(1) *Perturbed System*

Consider the perturbed system

$$\begin{gathered}(K_1+\varepsilon)u^\varepsilon_{tt} + \Delta u^\varepsilon + M\left(\|u^\varepsilon\|^2 + \|v^\varepsilon\|^2\right)(-\Delta u^\varepsilon) + u^\varepsilon_t = 0 \quad \text{in } \Omega \times (0,T), \\ (K_2+\varepsilon)v^\varepsilon_{tt} + \Delta v^\varepsilon + M\left(\|u^\varepsilon\|^2 + \|v^\varepsilon\|^2\right)(-\Delta v^\varepsilon) + v^\varepsilon_t = 0 \quad \text{in } \Omega \times (0,T),\end{gathered}$$

$$
\begin{aligned}
&u^{\varepsilon} = v^{\varepsilon} = \frac{\partial u^{\varepsilon}}{\partial \eta} = \frac{\partial v^{\varepsilon}}{\partial \eta} = 0 \quad \text{on } \Sigma, \\
&(u^{\varepsilon}(x,0), v^{\varepsilon}(x,0)) = (u_0(x), v_0(x)) \quad \text{in } \Omega, \\
&\left(u_t^{\varepsilon}(x,0), v_t^{\varepsilon}(x,0)\right) = (u_1(x), v_1(x)) \quad \text{in } \Omega.
\end{aligned} \tag{3.4}
$$

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^2(\Omega)$ formed by the eigenvectors of the operator $-\Delta$, that is, $-\Delta w_\nu = \lambda_\nu w_\nu$, with $\lambda_\nu \to \infty$ when $\nu \to \infty$. Let $V_m = [w_1, w_2, \dots, w_m]$ be the subspace generated by the first m vectors of $(w_\nu)_{\nu\in\mathbb{N}}$.

For each fixed ε, we consider

$$
\begin{aligned}
u_m^{\varepsilon}(t) &= \sum_{j=1}^{m} g_{j\varepsilon m}(t) w_j \in V_m, \\
v_m^{\varepsilon}(t) &= \sum_{j=1}^{m} h_{j\varepsilon m}(t) w_j \in V_m
\end{aligned} \tag{3.5}
$$

as solutions of the approximated perturbed system

$$
\begin{aligned}
&\left((K_1+\varepsilon)u_{ttm}^{\varepsilon}(t), w\right) + (-\Delta u_m^{\varepsilon}(t), -\Delta w) \\
&\quad + M\left(\|u_m^{\varepsilon}(t)\|^2 + \|v_m^{\varepsilon}(t)\|^2\right)(-\Delta u_m^{\varepsilon}(t), w) + \left(u_{tm}^{\varepsilon}(t), w\right) = 0, \quad \forall w \in V_m,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&\left((K_2+\varepsilon)v_{ttm}^{\varepsilon}(t), z\right) + (-\Delta v_m^{\varepsilon}(t), -\Delta z) \\
&\quad + M\left(\|u_m^{\varepsilon}(t)\|^2 + \|v_m^{\varepsilon}(t)\|^2\right)(-\Delta v_m^{\varepsilon}(t), z) + \left(v_{tm}^{\varepsilon}(t), z\right) = 0, \quad \forall z \in V_m,
\end{aligned} \tag{3.7}
$$

$$
(u_m^{\varepsilon}(0), v_m^{\varepsilon}(0)) = (u_{0m}, v_{0m}) \longrightarrow (u_0, v_0) \quad \text{in } \left(H_0^2(\Omega) \cap H^4(\Omega)\right)^2, \tag{3.8}
$$

$$
\left(u_{tm}^{\varepsilon}(0), v_{tm}^{\varepsilon}(0)\right) = (u_{1m}, v_{1m}) \longrightarrow (u_1, v_1) \quad \text{in } \left(H_0^2(\Omega)\right)^2. \tag{3.9}
$$

The local existence of the approximated solutions $(u_m^{\varepsilon}, v_m^{\varepsilon})$ is guaranteed by the standard results of ordinary differential equations. The extension of the solutions $(u_m^{\varepsilon}, v_m^{\varepsilon})$ to the whole interval $[0, T]$ is a consequence of the first estimate below.

The First Estimate

Setting $w = u_{tm}^{\varepsilon}$ and $z = v_{tm}^{\varepsilon}$ in (3.6) and (3.7), respectively, integrating over $(0, t)$, and taking the convergences (3.8) and (3.9) in consideration, we arrive at

$$
\begin{aligned}
&\left(K_1, \left(u_{tm}^{\varepsilon}\right)^2(t)\right) + \varepsilon\left|u_{tm}^{\varepsilon}(t)\right|^2 + |\Delta u_m^{\varepsilon}(t)|^2 + \left(K_2, \left(v_{tm}^{\varepsilon}\right)^2(t)\right) + \varepsilon\left|v_{tm}^{\varepsilon}\right|^2 \\
&\quad + |\Delta v_m^{\varepsilon}(t)|^2 + \widehat{M}\left(\|u_m^{\varepsilon}(t)\|^2 + \|v_m^{\varepsilon}(t)\|^2\right) + 2\int_0^t \left[\left|u_{tm}^{\varepsilon}(s)\right|^2 + \left|v_{tm}^{\varepsilon}(s)\right|^2\right] ds
\end{aligned}
$$

$$\leq \int_0^t \left[\left| \left(\frac{\partial K_1}{\partial t}, (u_{tm}^{\varepsilon})^2(s) \right) \right|_{\mathbb{R}} + \left| \left(\frac{\partial K_2}{\partial t}, (v_{tm}^{\varepsilon})^2(s) \right) \right|_{\mathbb{R}} \right] ds + \left(K_1(0), u_{1m}^2 \right) + \varepsilon |u_{1m}|^2$$
$$+ |\Delta u_{0m}|^2 + \left(K_2(0), v_{1m}^2 \right) + \varepsilon |v_{1m}|^2 + |\Delta v_{0m}|^2 + \widehat{M}\left(\|u_{0m}\|^2 + \|v_{0m}\|^2 \right), \tag{3.10}$$

where

$$\widehat{M}(s) = \int_0^s M(\tau) d\tau. \tag{3.11}$$

From (2.7) and (2.8), we have

$$\widehat{M}\left(\|u_m^{\varepsilon}(t)\|^2 + \|v_m^{\varepsilon}(t)\|^2 \right) \geq -\frac{\beta}{\lambda_1} \left(|\Delta u_m^{\varepsilon}(t)|^2 + |\Delta v_m^{\varepsilon}(t)|^2 \right). \tag{3.12}$$

Since $\beta < \lambda_1$ and so by (2.5)–(2.7) and convergences (3.8), (3.9), and (3.12), we obtain

$$\begin{aligned} &\left(K_1, (u_{tm}^{\varepsilon})^2(t) \right) + \left(K_2, (u_{tm}^{\varepsilon})^2(t) \right) + \varepsilon \left(\left| u_{tm}^{\varepsilon}(t) \right|^2 + \left| v_{tm}^{\varepsilon} \right|^2(t) \right) \\ &\quad + \left(1 - \frac{\beta}{\lambda_1} \right) \left(|\Delta u_m^{\varepsilon}(t)|^2 + |\Delta v_m^{\varepsilon}(t)|^2 \right) + (2 - \delta) \int_0^t \left[\left| u_{tm}^{\varepsilon}(s) \right|^2 + \left| u_{tm}^{\varepsilon}(s) \right|^2 \right] ds \\ &\leq C_0 + C(\delta) \int_0^t \left[\left(K_1, (u_{tm}^{\varepsilon})^2(s) \right) + \left(K_2, (v_{tm}^{\varepsilon})^2(s) \right) \right] ds \end{aligned} \tag{3.13}$$

with $0 < \delta < 1$ and C_0 being a positive constant independent of ε, m, and t.

Employing Gronwall's lemma in (3.13), we obtain the first estimate

$$\begin{aligned} &\left(K_1, (u_{tm}^{\varepsilon})^2(t) \right) + \left(K_2, (u_{tm}^{\varepsilon})^2(t) \right) + \varepsilon \left(\left| u_{tm}^{\varepsilon}(t) \right|^2 + \left| v_{tm}^{\varepsilon} \right|^2(t) \right) \\ &\quad + \left(1 - \frac{\beta}{\lambda_1} \right) \left(|\Delta u_m^{\varepsilon}(t)|^2 + |\Delta v_m^{\varepsilon}(t)|^2 \right) + (2 - \delta) \int_0^t \left[\left| u_{tm}^{\varepsilon}(s) \right|^2 + \left| u_{tm}^{\varepsilon}(s) \right|^2 \right] ds \leq C_1, \end{aligned} \tag{3.14}$$

where C_1 is a positive constant independent of ε, m, and t. Then, we can conclude that

$$\begin{gathered} \left(K_1^{1/2} u_{tm}^{\varepsilon} \right), \left(K_2^{1/2} v_{tm}^{\varepsilon} \right) \text{ are bounded in } L^{\infty}\left(0, T; L^2(\Omega) \right), \\ (\sqrt{\varepsilon} u_{tm}^{\varepsilon}), (\sqrt{\varepsilon} v_{tm}^{\varepsilon}) \text{ are bounded in } L^{\infty}\left(0, T; L^2(\Omega) \right), \\ (u_m^{\varepsilon}), (v_m^{\varepsilon}) \text{ are bounded in } L^{\infty}\left(0, T; H_0^2(\Omega) \right), \\ (u_{tm}^{\varepsilon}), (v_{tm}^{\varepsilon}) \text{ are bounded in } L^2\left(0, T; L^2(\Omega) \right). \end{gathered} \tag{3.15}$$

The Second Estimate

Substituting $w = -\Delta u^{\varepsilon}_{tm}(t)$ and $z = -\Delta v^{\varepsilon}_{tm}(t)$ in (3.6) and (3.7), respectively, it holds that

$$\begin{aligned}
&\frac{d}{dt}\Big[\Big(\Big(K_1, (u^{\varepsilon}_{tm})^2(t)\Big)\Big) + \Big(\Big(K_2, (v^{\varepsilon}_{tm})^2(t)\Big)\Big) + \varepsilon\Big(\|u^{\varepsilon}_{tm}(t)\|^2 + \|v^{\varepsilon}_{tm}(t)\|^2\Big) \\
&\qquad + \|\Delta u^{\varepsilon}_{m}(t)\|^2 + \|\Delta v^{\varepsilon}_{m}(t)\|^2\Big] + 2\Big(\|u^{\varepsilon}_{tm}(t)\|^2 + \|v^{\varepsilon}_{tm}(t)\|^2\Big) \\
&\qquad = \Big(\Big(\frac{\partial K_1}{\partial t}, (u^{\varepsilon}_{tm})^2(t)\Big)\Big) + \Big(\Big(\frac{\partial K_2}{\partial t}, (v^{\varepsilon}_{tm})^2(t)\Big)\Big) \\
&\qquad\quad - 2M\Big(\|u^{\varepsilon}_{m}(t)\|^2 + \|v^{\varepsilon}_{m}(t)\|^2\Big) \cdot \big[((-\Delta u^{\varepsilon}_{m}(t), u^{\varepsilon}_{tm}(t))) + ((-\Delta v^{\varepsilon}_{m}(t), v^{\varepsilon}_{tm}(t)))\big].
\end{aligned} \tag{3.16}$$

Integrating (3.16) over $(0, t)$, $0 < t < T$, and taking (2.5)–(2.7) and (3.8), (3.9), and first estimate into account, we infer

$$\begin{aligned}
&\Big(\Big(K_1, (u^{\varepsilon}_{tm})^2(t)\Big)\Big) + \Big(\Big(K_2, (v^{\varepsilon}_{tm})^2(t)\Big)\Big) + \varepsilon\Big(\|u^{\varepsilon}_{tm}(t)\|^2 + \|v^{\varepsilon}_{tm}(t)\|^2\Big) \\
&\qquad + \|\Delta u^{\varepsilon}_{m}(t)\|^2 + \|\Delta v^{\varepsilon}_{m}(t)\|^2 + 2\Big(\|u^{\varepsilon}_{tm}(t)\|^2 + \|v^{\varepsilon}_{tm}(t)\|^2\Big) \\
&\qquad + (2 - 2\delta)\int_0^t \Big(\|u^{\varepsilon}_{tm}(t)\|^2 + \|v^{\varepsilon}_{tm}(t)\|^2\Big) ds \le C_2,
\end{aligned} \tag{3.17}$$

where C_2 is a positive constant independent of ε, m, and t. From the above estimate we conclude that

$$\begin{gathered}
\Big(K_1^{1/2} u^{\varepsilon}_{tm}\Big), \Big(K_2^{1/2} v^{\varepsilon}_{tm}\Big) \text{ are bounded in } L^{\infty}\Big(0, T; H^1_0(\Omega)\Big), \\
(\sqrt{\varepsilon} u^{\varepsilon}_{tm}), (\sqrt{\varepsilon} v^{\varepsilon}_{tm}) \text{ are bounded in } L^{\infty}\Big(0, T; H^1_0(\Omega)\Big), \\
(u^{\varepsilon}_{m}), (v^{\varepsilon}_{m}) \text{ are bounded in } L^{\infty}\Big(0, T; H^2_0(\Omega) \cap H^4(\Omega)\Big), \\
(u^{\varepsilon}_{tm}), (v^{\varepsilon}_{tm}) \text{ are bounded in } L^{2}\Big(0, T; H^1_0(\Omega)\Big).
\end{gathered} \tag{3.18}$$

The Third Estimate

Differentiating (3.6) and (3.7) with respect to t and setting $w = u^{\varepsilon}_{ttm}$ and v^{ε}_{ttm}, respectively, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\Big[K_1, (u^{\varepsilon}_{ttm}(t))^2 + \Big(K_2, (v^{\varepsilon}_{ttm}(t))^2\Big) + \varepsilon\Big(|u^{\varepsilon}_{ttm}(t)|^2 + |v^{\varepsilon}_{ttm}|^2\Big) + |\Delta u^{\varepsilon}_{tm}(t)|^2 + |\Delta v^{\varepsilon}_{tm}(t)|^2\Big] \\
&\qquad + 2\Big(|u^{\varepsilon}_{tm}(s)|^2 + |v^{\varepsilon}_{tm}(s)|^2\Big)
\end{aligned}$$

$$= -2M\left(\|u_m^\varepsilon(t)\|^2 + \|v_m^\varepsilon(t)\|^2\right) \cdot \left[\left(-\Delta u_{tm}^\varepsilon(t), u_{ttm}^\varepsilon(t)\right) + \left(-\Delta v_{tm}^\varepsilon(t), v_{ttm}^\varepsilon(t)\right)\right]$$
$$- 4\left[\left(u_m^\varepsilon(\mathrm{t}), -\Delta u_{tm}^\varepsilon(t)\right) + \left(v_m^\varepsilon(t), -\Delta v_{tm}^\varepsilon(t)\right)\right] \cdot M'\left(\|u_m^\varepsilon(t)\|^2 + \|v_m^\varepsilon(t)\|^2\right)$$
$$\cdot \left[\left(-\Delta u_{tm}^\varepsilon(t), u_{ttm}^\varepsilon(t)\right) + \left(-\Delta v_{tm}^\varepsilon(t), v_{ttm}^\varepsilon(t)\right)\right] + \left(\frac{\partial K_1}{\partial t}, \left(u_{ttm}^\varepsilon\right)^2(t)\right) + \left(\frac{\partial K_2}{\partial t}, \left(v_{ttm}^\varepsilon\right)^2(t)\right). \tag{3.19}$$

Integrating (3.19) over $(0,t)$, and using (2.5), (3.8), (3.9), and the norms $|u_{ttm}^\varepsilon(0)|^2 \leq C_3$ and $|v_{ttm}^\varepsilon(0)|^2 \leq C_4$ after employing Gronwall's lemma, we obtain the third estimate

$$\begin{aligned} &\left(K_1, \left(u_{ttm}^\varepsilon\right)^2(t)\right) + \left(K_2, \left(v_{ttm}^\varepsilon\right)^2(t)\right) + \varepsilon\left(\left|u_{ttm}^\varepsilon(t)\right|^2 + \left|v_{ttm}^\varepsilon(t)\right|^2\right) \\ &\quad + \left|\Delta u_{tm}^\varepsilon(t)\right|^2 + \left|\Delta v_{tm}^\varepsilon(t)\right|^2\Big] + (2-2\delta)\int_0^t \left(\left|u_{ttm}^\varepsilon(t)\right|^2 + \left|v_{ttm}^\varepsilon(t)\right|^2\right) ds \leq C_5, \end{aligned} \tag{3.20}$$

where C_5 is a positive constant independent of ε, m, and t. From the above estimate we conclude that

$$\begin{gathered} \left(K_1^{1/2} u_{ttm}^\varepsilon\right), \left(K_2^{1/2} v_{ttm}^\varepsilon\right) \text{ are bounded in } L^\infty\left(0,T; L^2(\Omega)\right), \\ \left(\sqrt{\varepsilon} u_{ttm}^\varepsilon\right), \left(\sqrt{\varepsilon} v_{ttm}^\varepsilon\right) \text{ are bounded in } L^\infty\left(0,T; L^2(\Omega)\right), \\ \left(u_{tm}^\varepsilon\right), \left(v_{tm}^\varepsilon\right) \text{ are bounded in } L^\infty\left(0,T; H_0^2(\Omega)\right), \\ \left(u_{ttm}^\varepsilon\right), \left(v_{ttm}^\varepsilon\right) \text{ are bounded in } L^2\left(0,T; L^2(\Omega)\right). \end{gathered} \tag{3.21}$$

(2) *Limits of Approximated Solutions*

From the Aubin-Lions theorem (see [6]) we deduce that there exist subsequences of $(u_m^\varepsilon)_{m\in\mathbb{N}}$ and $(v_m^\varepsilon)_{m\in\mathbb{N}}$ such that

$$\begin{aligned} u_m^\varepsilon &\longrightarrow u^\varepsilon \quad \text{strongly in } L^2\left(0,T; H_0^1(\Omega)\right), \\ v_m^\varepsilon &\longrightarrow v^\varepsilon \quad \text{strongly in } L^2\left(0,T; H_0^1(\Omega)\right), \end{aligned} \tag{3.22}$$

and since M is continuous, it follows that

$$M\left(\|u_m^\varepsilon(t)\|^2 + \|v_m^\varepsilon(t)\|^2\right) \longrightarrow M\left(\|u^\varepsilon(t)\|^2 + \|v^\varepsilon(t)\|^2\right). \tag{3.23}$$

From the above estimate we can conclude that there exist subsequences of $(u_m^\varepsilon)_{m\in\mathbb{N}}$ and $(v_m^\varepsilon)_{m\in\mathbb{N}}$, that we denote also by $(u_m^\varepsilon)_{m\in\mathbb{N}}$ and $(v_m^\varepsilon)_{m\in\mathbb{N}}$ such that as $m \to \infty$ and $\varepsilon \to 0$ we have

$$
\begin{aligned}
& u_m^\varepsilon \longrightarrow u, \qquad v_m^\varepsilon \longrightarrow v \quad \text{weak star in } L^\infty\Big(0,T;H_0^2(\Omega)\cap H^4(\Omega)\Big), \\
& u_{tm}^\varepsilon \longrightarrow u_t, \qquad v_{tm}^\varepsilon \longrightarrow v_t \quad \text{weak star } L^\infty\Big(0,T;H_0^2(\Omega)\Big), \\
& u_{ttm}^\varepsilon \longrightarrow u_{tt}, \qquad v_{ttm}^\varepsilon \longrightarrow v_{tt} \quad \text{weak star } L^2\Big(0,T;L^2(\Omega)\Big), \\
& \Delta u_m^\varepsilon \longrightarrow \Delta u, \qquad \Delta v_m^\varepsilon \longrightarrow \Delta v \quad \text{weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big), \\
& K_1 u_{ttm}^\varepsilon \longrightarrow K_1 u_{tt}, \qquad K_2 v_{ttm}^\varepsilon \longrightarrow K_2 v_{tt} \quad \text{weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big), \\
& \Delta^2 u_m^\varepsilon \longrightarrow \Delta^2 u, \qquad \Delta^2 v_m^\varepsilon \longrightarrow \Delta^2 v \quad \text{weak star in } L^2\Big(0,T;L^2(\Omega)\Big), \\
& \sqrt{\varepsilon} u_{ttm}^\varepsilon \longrightarrow \sqrt{\varepsilon} u_{tt}, \qquad \sqrt{\varepsilon} v_{ttm}^\varepsilon \longrightarrow \sqrt{\varepsilon} v_{tt} \quad \text{weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big), \\
& M\Big(\|u_m^\varepsilon(t)\|^2 + \|v_m^\varepsilon(t)\|^2\Big)(-\Delta u_m^\varepsilon - \Delta v_m^\varepsilon) \\
& \qquad \longrightarrow M\Big(\|u(t)\|^2 + \|v(t)\|^2\Big)(-\Delta u - \Delta v) \quad \text{weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big).
\end{aligned}
\tag{3.24}
$$

Now, multiplying (3.6), (3.7) by $\theta \in \mathfrak{D}(0,T)$ and integrating over $(0,T)$, we arrive at

$$
\begin{aligned}
& \int_0^T ((K_1+\varepsilon)u_{ttm}^\varepsilon(t), w)\theta(t)dt + \int_0^T (-\Delta u_m^\varepsilon(t), -\Delta w)\theta(t)dt \\
& \qquad + \int_0^T M\Big(\|u_m^\varepsilon(t)\|^2 + \|v_m^\varepsilon(t)\|^2\Big)(-\Delta u_m^\varepsilon(t), w)\theta\, dt \\
& \qquad + \int_0^T (u_{tm}^\varepsilon, w)\theta dt = 0, \quad \forall w \in V_m,\ \forall \theta \in \mathfrak{D}(0,T), \\
& \int_0^T ((K_2+\varepsilon)v_{ttm}^\varepsilon(t), z)\theta(t)dt + \int_0^T (-\Delta v_m^\varepsilon(t), -\Delta z)\theta(t)dt \\
& \qquad + \int_0^T M\Big(\|u_m^\varepsilon(t)\|^2 + \|v_m^\varepsilon(t)\|^2\Big)(-\Delta v_m^\varepsilon(t), z)\theta\, dt \\
& \qquad + \int_0^T (v_{tm}^\varepsilon, z)\theta\, dt = 0, \quad \forall z \in V_m,\ \forall \theta \in \mathfrak{D}(0,T).
\end{aligned}
\tag{3.25}
$$

The convergences (3.24) are sufficient to pass to the limit in (3.25) in order to obtain

$$
\begin{aligned}
& K_1 u_{tt} + \Delta^2 u + M\Big(\|u(t)\|^2 + \|v(t)\|^2\Big)(-\Delta u) + u_t = 0 \quad \text{in } L^\infty_{\mathrm{loc}}\Big(0,\infty;L^2(\Omega)\Big), \\
& K_2 v_{tt} + \Delta^2 v + M\Big(\|u(t)\|^2 + \|v(t)\|^2\Big)(-\Delta v) + v_t = 0 \quad \text{in } L^\infty_{\mathrm{loc}}\Big(0,\infty;L^2(\Omega)\Big),
\end{aligned}
\tag{3.26}
$$

and (u,v) satisfies (3.1).

The uniqueness and initial conditions follow by using the standard arguments as in Lions [6]. The proof is now complete. □

4. Asymptotic Behavior

In this section we study the asymptotic behavior of solutions to the system (1.1)–(1.5). We show using the Nakao method that the system (1.1)–(1.5) is exponentially stable. The main result of this paper is given by the following theorem.

Theorem 4.1. *Let one take* $(u_0, v_0) \in (H_0^1(\Omega) \cap H^4(\Omega))^2$, *and* $(u_1, v_1) \in (H_0^2(\Omega))^2$ *and let one suppose that assumptions* (2.5), (2.6), *and* (2.7) *hold. Then, the solution* (u, v) *of system* (1.1)–(1.5) *satisfies*

$$\left|K_1^{1/2}u_t(t)\right|^2 + \left|K_2^{1/2}v_t(t)\right|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2 + \int_t^{t+1}\left(|u_t(s)|^2 + |v_t(s)|^2\right)ds \le \alpha_1 e^{-\alpha_2 t}, \tag{4.1}$$

for all $t \ge 1$, *where* α_1 *and* α_2 *are positive constants.*

Proof. Multiplying (3.2) by $u_t(t)$ and $v_t(t)$, respectively, and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left[\left|K_1^{1/2}u_t(t)\right|^2 + \left|K_2^{1/2}v_t(t)\right|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2 + \widehat{M}\left(\|u(t)\|^2 + \|v(t)\|^2\right)\right] \\ &+ |u_t(t)|^2 + |v_t(t)|^2 = \left(\frac{\partial K_1}{\partial t}, u_t^2(t)\right) + \left(\frac{\partial K_2}{\partial t}, v_t^2(t)\right),\end{aligned} \tag{4.2}$$

where

$$\widehat{M}(s) = \int_0^s M(\tau)d\tau. \tag{4.3}$$

Using (2.6) and considering $\delta > 0$ sufficiently small, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left[\left|K_1^{1/2}u_t(t)\right|^2 + \left|K_2^{1/2}v_t(t)\right|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2 + \widehat{M}\left(\|u(t)\|^2 + \|v(t)\|^2\right)\right] \\ &+ (1 - \delta - K_0 C(\delta))\left(|u_t(t)|^2 + |v_t(t)|^2\right) \le 0,\end{aligned} \tag{4.4}$$

where $K_0 = \max\{\overline{K_1}, \overline{K_2}\}$ with

$$\overline{K_i} = \max_{s\in[t,t+1]}\{\text{ess sup}\, K_i(x,s)\}, \quad i = 1,2. \tag{4.5}$$

Integrating (4.4) from 0to$t$, we have

$$E(t) + (1 - \delta - K_0 C(\delta))\int_0^t\left(|u_t(s)|^2 + |v_t(s)|^2\right)ds \le E(0), \tag{4.6}$$

where

$$E(t)=\frac{1}{2}\left[\left|K_1^{1/2}u_t(t)\right|^2+\left|K_2^{1/2}v_t(t)\right|^2+|\Delta v(t)|^2+|\Delta v(t)|^2+\widehat{M}\left(\|u(t)\|^2+\|v(t)\|^2\right)\right] \tag{4.7}$$

is the energy associated with the system (1.1)–(1.5). From (4.4) we conclude that

$$\frac{d}{dt}E(t)\leq 0\quad \forall t\in(0,\infty), \tag{4.8}$$

that is, $E(t)$ is bounded and increasing in $(0,\infty)$.

Integrating (4.4) from $\tau_1$toτ_2, $0<\tau_1<\tau_2<\infty$, we arrive at

$$E(\tau_2)+(1-\delta-K_0C(\delta))\int_{\tau_1}^{\tau_2}\left(|u_t(s)|^2+|v_t(s)|^2\right)ds\leq E(\tau_1). \tag{4.9}$$

Taking $\tau_1=t$ and $\tau_2=t+1$ in (4.9), we get

$$\int_t^{t+1}\left(|u_t(s)|^2+|v_t(s)|^2\right)ds\leq\frac{1}{1-\delta-K_0C(\delta)}[E(t)-E(t+1)]=F^2(t). \tag{4.10}$$

Therefore, there exist two points $t_1\in[t,t+1/4]$ and $t_2\in[t+3/4,t+1]$, such that

$$|u_t(t_i)|+|v_t(t_i)|\leq 4F(t),\quad i=1,2. \tag{4.11}$$

Making the inner product in $L^2(\Omega)$ of (1.1) and (1.2) by $u(t)$ and $v(t)$, respectively, and summing up the result we obtain

$$\begin{aligned}&\frac{d}{dt}(K_1u_t(t),u(t))+\frac{d}{dt}(K_2v_t(t),v(t))-\left|\sqrt{K_1}u_t(t)\right|^2-\left|\sqrt{K_2}v_t(t)\right|^2\\&\quad+|\Delta u(t)|^2+|\Delta v(t)|^2+M\left(\|u(t)\|^2+\|v(t)\|^2\right)\left(\|u(t)\|^2+\|v(t)\|^2\right)+(u_t(t),u(t))\\&\quad+(v_t(t),v(t))=\left(\frac{\partial K_1}{\partial t}u_t(t),u(t)\right)+\left(\frac{\partial K_2}{\partial t}v_t(t),v(t)\right).\end{aligned} \tag{4.12}$$

Integrating from t_1 to t_2 and using (2.6), and (2.7) we have

$$\begin{aligned}&\left(1-\frac{\beta}{\lambda_1}\right)\int_{t_1}^{t_2}\left(|\Delta u(s)|^2+|\Delta v(s)|^2\right)ds\\&\quad\leq(K_1u_t(t_1),u(t_1))-(K_1u_t(t_2),u(t_2))+(K_2v_t(t_1),v(t_1))-(K_2v_t(t_2),v(t_2))\\&\qquad+(1+\delta+K_0C(\delta))\int_{t_1}^{t_2}(|u_t(s)||u(s)|+|v_t(s)||v(s)|)ds\\&\qquad+K_0\int_{t_1}^{t_2}\left(|u_t(s)|^2+|v_t(s)|^2\right)ds.\end{aligned} \tag{4.13}$$

Let us consider $C > 0$ such that

$$|u(s)| \leq C|\Delta u(s)|, \qquad |v(s)| \leq C|\Delta v(s)| \tag{4.14}$$

and we take $d > 0$ sufficiently small Then we have

$$\begin{aligned}
&(1+\delta+K_0C(\delta))(|u_t(s)||u(s)|+|v_t(s)||v(s)|)\\
&\quad\leq \frac{(1+\delta+K_0C(\delta))^2}{d}\left(|u_t(s)|^2+|v_t(s)|^2\right)+d\left(|\Delta u(s)|^2+|\Delta v(s)|^2\right),\\
&|(K_1u_t(t_1),u(t_1))+(K_2v_t(t_1),v(t_1))-(K_1u_t(t_2),u(t_2))-(K_2v_t(t_2),v(t_2))|\\
&\quad\leq CK_0 \operatorname*{ess\,sup}_{s\in[t,t+1]} |\Delta u(s)|(|u_t(t_1)|+|u_t(t_2)|)+CK_0 \operatorname*{ess\,sup}_{s\in[t,t+1]} |\Delta v(s)|(|v_t(t_1)|+|v_t(t_2)|).
\end{aligned} \tag{4.15}$$

Thus, substituting (4.15) into (4.13), we arrive at

$$\begin{aligned}
&\left(1-\frac{\beta}{\lambda_1}\right)\int_{t_1}^{t_2}\left(|\Delta u(s)|^2+|\Delta v(s)|^2\right)ds\\
&\quad\leq K_0\int_{t_1}^{t_2}\left(|u_t(s)|^2+|v_t(s)|^2\right)ds+d\int_{t_1}^{t_2}\left(|\Delta u(s)|^2+|\Delta v(s)|^2\right)ds\\
&\qquad+CK_0 \operatorname*{ess\,sup}_{s\in[t,t+1]} |\Delta u(s)|(|u_t(t_1)|+|u_t(t_2)|)+CK_0 \operatorname*{ess\,sup}_{s\in[t,t+1]} |\Delta v(s)|(|v_t(t_1)|+|v_t(t_2)|).
\end{aligned} \tag{4.16}$$

Applying (4.10) and (4.11) in (4.16), we have

$$\int_{t_1}^{t_2}\left(|\Delta u(s)|^2+|\Delta v(s)|^2\right)ds \leq C_1\left[F^2(t)+\operatorname*{ess\,sup}_{s\in[t,t+1]}(|\Delta u(s)|+|\Delta v(s)|)F(t)\right]=G^2(t), \tag{4.17}$$

where C_1 is a positive constant independent of t. Therefore, from (4.10) and (4.17) we obtain

$$\int_{t_1}^{t_2}\left(|u_t(s)|^2+|v_t(s)|^2+|\Delta u(s)|^2+|\Delta v(s)|^2\right)ds \leq F^2(t)+G^2(t). \tag{4.18}$$

Hence, there exists $t^* \in [t_1, t_2]$ such that

$$|u_t(t^*)|^2+|v_t(t^*)|^2+|\Delta u(t^*)|^2+|\Delta v(t^*)|^2 \leq 2\left[F^2(t)+G^2(t)\right]. \tag{4.19}$$

Consequently,

$$\widehat{M}\left(\|u(t)\|^2+\|v(t)\|^2\right) \leq C_2\left[F^2(t)+G^2(t)\right], \tag{4.20}$$

where

$$C_2 = 2m_0\widetilde{C}, \quad m_0 = \max_{0\leq s\leq \left(\|u(t^*)\|^2+\|v(t^*)\|^2\right)<\infty} M(s) \tag{4.21}$$

and $\widetilde{C}$ is a positive constant such that $\|u(t^*)\|^2 \leq \widetilde{C}|\Delta u(t^*)|^2$.

From (4.19) and (4.20), we have

$$E(t^*) \leq C_3\left[F^2(t) + G^2(t)\right]. \tag{4.22}$$

Since $E(t)$ is increasing, we have

$$\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s) \leq E(t^*) + (1+\delta+K_0C(\delta))\int_{t_1}^{t_2}\left(|u_t(s)|^2 + |v_t(s)|^2\right)ds. \tag{4.23}$$

Now, by (4.10), (4.22), and (4.23) we get

$$E(t) \leq C_4[E(t) - E(t+1)], \tag{4.24}$$

where C_4 is a positive constant. Then, by the Nakao lemma (see [12]) we conclude that

$$E(t) \leq b_1 e^{-\alpha_2 t}, \quad \forall t \geq 1, \tag{4.25}$$

where b_1 and α_2 are positive constants, that is,

$$\left|\sqrt{K_1}u_t(t)\right|^2 + \left|\sqrt{K_2}v_t(t)\right|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2\widehat{M}\left(\|u(t)\|^2 + \|v(t)\|^2\right) \leq 2b_1e^{-\alpha_2 t}. \tag{4.26}$$

Using (2.7) we obtain

$$\left|\sqrt{K_1}u_t(t)\right|^2 + \left|\sqrt{K_2}v_t(t)\right|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2 \leq \frac{2b_1}{m_1}e^{-\alpha_2 t}, \tag{4.27}$$

where

$$m_1 = 1 - \frac{\beta}{\lambda_1} > 0. \tag{4.28}$$

From (4.10) we have

$$\int_t^{t+1}\left(|u_t(s)|^2 + |v_t(s)|^2\right)ds \leq \frac{1}{1-\delta-K_0C(\delta)}[E(t) - E(t+1)] \leq E(t) \leq b_1e^{-\alpha_2 t}. \tag{4.29}$$

Therefore, from (4.27) and (4.29) we conclude that

$$\begin{aligned}&\left|\sqrt{K_1}u_t(t)\right|^2+\left|\sqrt{K_2}v_t(t)\right|^2+|\Delta u(t)|^2+|\Delta v(t)|^2\\&\quad+\int_t^{t+1}\left(|u_t(s)|^2+|v_t(s)|^2\right)ds\le\alpha_1e^{-\alpha_2 t},\quad\forall t\ge 1,\end{aligned}\tag{4.30}$$

where α_1 and α_2 are positive constants. Now, the proof is complete. □

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