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*Research Article*

# The Inviscid Limits to Piecewise Smooth Solutions for a General Parabolic System

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We study the viscous limit problem for a general system of conservation laws. We prove that if the solution of the underlying inviscid problem is piecewise smooth with finitely many noninteracting shocks satisfying the entropy condition, then there exist solutions to the corresponding viscous system which converge to the inviscid solutions away from shock discontinuities at a rate of $\varepsilon^1$ as the viscosity coefficient $\varepsilon$ vanishes.

## 1. Introduction

We consider the relation between the solutions, $u^\varepsilon$, of the system of viscous conservation laws

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon(B(u^\varepsilon)u_x^\varepsilon)_x, \quad u^\varepsilon \in R^n,\ x \in R,\ t \geq 0,\ \varepsilon > 0, \tag{1.1}$$

and the distributional solution, $u$, of the corresponding system of conservation laws without viscosity

$$u_t + f(u)_x = 0, \quad u \in R^n,\ x \in R,\ t > 0. \tag{1.2}$$

We assume that (1.2) is strictly hyperbolic, then by normalization, we have the decomposition

$$A \equiv D_u f(u) = R\Lambda L, \qquad RL = I, \tag{1.3}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_n, L = (l_1, \ldots, l_n)^t$ is a matrix whose rows are left eigenvectors of $A$, and $R = (r_1, \ldots, r_n)$ is a matrix whose columns are right eigenvectors of $A$.

For the zero dissipation limit problem, there are many significant works. When the Euler flow contains a single shock, Hoff and Liu [1] studied the isentropic case, they established the limit process from the solutions of the compressible Navier-Stokes equations to the single shock-wave solution of the corresponding compressible Euler system (so-called p-system). They show that the solutions to the isentropic Navier-Stokes equations with shock data exist and converge to the inviscid shocks as the viscosity vanishes, uniformly away from the shocks. Ignoring the initial layers, Goodman and Xin [2] gave a very detailed description of the asymptotic behavior of solutions for the general viscous systems as the viscosity tends to zero, via a method of matching asymptotics. This method can be applied to the Navier-Stokes equations (1.1), such as [3–5]. Later, Yu [6] revealed the rich structure of nonlinear wave interactions due to the presence of shocks and initial layers by a detailed pointwise analysis. As far as rarefaction wave is concerned, Xin in [7] has obtained that the solutions for the isentropic Navier-Stokes equations with weak centered rarefaction wave data exist for all time and converge to the weak centered rarefaction wave solution of the corresponding Euler system, as the viscosity tends to zero, uniformly away from the initial discontinuity. Moreover, in the case that either the initial layers are ignored or the rarefaction waves are smooth, he also obtains a rate of convergence which is valid uniformly for all time. Recently, Jiang et al. [8] improve the first part with weak centered srarefaction waves data and Zeng [9] improve the other results, respectively, in [7] to the full compressible Navier-Stokes equations, provided that the viscosity and heat-conductivity coefficients are in the same order. Furthermore, by a spectral analysis and Evans function method, Kevin Zumbrun and his collaborators have obtained many important results even for large amplitude and multidimensional case [10–14], and so forth. The case that the solutions to the Euler system containing contact discontinuity is much more subtle, there are few results in this respect[15–17].

In this paper, motivated by Goodman and Xin's work [2], we establish that the piecewise smooth solutions, $u$, of (1.2), with finitely many noninteracting shocks satisfying the entropy condition, are strong limits as $\varepsilon \to 0$ of solutions, $u^\varepsilon$, of (1.1) when the matrix $LBR$ is positive definite.

For simplicity of presentation, we only consider the case in which $u$ is a single-shock solution.

*Definition 1.1.* A function $u(x,t)$ is called a single-shock solution of (1.2) up to time $T$ if:

(i) $u(x,t)$ is a distributional solution of the hyperbolic system (1.2) in the region $R^1 \times [0,T]$;

(ii) there is a smooth curve, the shock, $x = s(t),\ 0 \le t \le T$, so that $u(x,t)$ is sufficiently smooth at any point $x \ne s(t)$;

(iii) the limits

$$\begin{aligned} \partial_x^k u(s(t)-0,t) &= \lim_{x \to s(t)^-} \partial_x^k u(x,t), \\ \partial_x^k u(s(t)+0,t) &= \lim_{x \to s(t)^+} \partial_x^k u(x,t), \end{aligned} \tag{1.4}$$

exist and are finite for $t \le T$ and $0 \le k \le 5$;

(iv) the Lax geometrical entropy condition [18] is satisfied at $x = s(t)$, that is,

$$\begin{aligned}
&\lambda_1(u(s(t)-0,t)) < \cdots < \lambda_p(u(s(t)-0,t)), \\
&\lambda_p(u(s(t)+0,t)) < \frac{d}{dt}s(t) < \lambda_p(u(s(t)-0,t)), \\
&\lambda_p(u(s(t)+0,t)) < \cdots < \lambda_n(u(s(t)+0,t)).
\end{aligned} \tag{1.5}$$

The main results of this paper are as follows.

**Theorem 1.2.** *Suppose that the system* (1.2) *is strictly hyperbolic and that the pth characteristic family is genuinely nonlinear. There exist positive constants, $\mu_0$ and $\varepsilon_0$, such that if $u(x,t)$ is a single-shock solution up to time T with*

$$\begin{aligned}
&\sum_{1\le k\le 6}\int_0^T\int \left|\partial_x^k u(x,t)\right|^2 dx\,dt < \infty, \\
&\mu \equiv \sup_{0\le t\le T}|u(s(t)+0,t) - u(s(t)-0,t)| \le \mu_0,
\end{aligned} \tag{1.6}$$

*then for each $\varepsilon \in [0,\varepsilon_0]$, there is a smooth solution, $u^\varepsilon(x,t)$, of* (1.1) *with*

$$u_x^\varepsilon \in C^1\left([0,T],H^1\right). \tag{1.7}$$

*Moreover, for any given $\eta \in (0,1)$,*

$$\sup_{0\le t\le T}\int |u^\varepsilon(x,t) - u(x,t)|^2 dx \le C_\eta \varepsilon^\eta, \tag{1.8}$$

$$\sup_{0\le t\le T,\ |x-s(t)|\ge\varepsilon^\eta} |u^\varepsilon(x,t) - u(x,t)| \le C_\eta \varepsilon, \tag{1.9}$$

*where $C_\eta$ is a positive constant depending only on $\eta$.*

*Notation.* In this paper, we use $H^l (l \ge 1)$ to denote the usual Sobolev space with the norm $\|\cdot\|_l$, and $\|\cdot\| = \|\cdot\|_0$ denotes the usual $L_2$-norm. We also use $O(1)$ to denote any positive bounded function which is independent of $\varepsilon$.

## 2. Construction of the Approximate Solution

In this section, following the method of Goodman and Xin, in [2], we construct the approximate solution $v^\varepsilon$ through different scaling and asymptotic expansions in the region near and away from the shock respectively, such that $v^\varepsilon$ approximate the piecewise smooth inviscid solution $u$ away from the shock and has a sharp change near the shock.

### *2.1. Outer and Inner Expansions and the Matching Conditions*

In the region away from the shock, $x = s(t)$, we approximate the solution of (1.1) by truncation of the formal series

$$u^{\varepsilon}(x,t) \sim u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots . \tag{2.1}$$

Substituting this into (1.1) and comparing the coefficients of powers of $\varepsilon$, we get, for $x \neq s(t)$, that

$$O(1) : u_{0t} + f(u_0)_x = 0, \tag{2.2}$$

$$O(\varepsilon) : u_{1t} + \left(f'(u_0)u_1\right)_x = (B(u_0)u_{0x})_x, \tag{2.3}$$

$$O\left(\varepsilon^2\right) : u_{2t} + \left(f'(u_0)u_2\right)_x = (B(u_0)u_{1x})_x + \left(B'(u_0)(u_1, u_{0x})\right)_x - \frac{1}{2}\left(f''(u_0)(u_1, u_1)\right)_x, \tag{2.4}$$

and so forth. The outer functions $u_0, u_1, \ldots$, are generally discontinuous at the shock, $x = s(t)$, but smooth up to the shock. The leading term, $u_0$, is the single-shock solution of (1.2) which is given in the theorem.

Near the shock, $u^{\varepsilon}$ should be represented by an inner expansion:

$$u^{\varepsilon}(x,t) \sim U_0(\xi,t) + \varepsilon U_1(\xi,t) + \varepsilon^2 U_2(\xi,t) + \cdots , \tag{2.5}$$

where

$$\xi = \frac{x - s(t)}{\varepsilon} + \delta(t,\varepsilon), \tag{2.6}$$

and $\delta(t,\varepsilon)$ is the perturbation of the shock position to be determined later.

We assume that $\delta(t,\varepsilon)$ has the form

$$\delta(t,\varepsilon) = \delta_0(t) + \varepsilon\delta_1(t) + \varepsilon^2\delta_2(t) + \cdots . \tag{2.7}$$

Substitute (2.5)–(2.7) into (1.1) to obtain

$$O\left(\frac{1}{\varepsilon}\right) : (B(U_0)U_{0\xi})_{\xi} + \dot{s}U_{0\xi} - \left(f(U_0)\right)_{\xi} = 0, \tag{2.8}$$

$$O(1) : \left\{B(U_0)U_{1\xi} + B'(U_0)\left(U_1, U_{0\xi}\right)\right\}_{\xi} + \dot{s}U_{1\xi} - \left(f'(U_0)U_1\right)_{\xi} = U_{0t} + \dot{\delta}_0 U_{0\xi}, \tag{2.9}$$

$$\begin{aligned} O(\varepsilon) : &\left\{B(U_0)U_{2\xi} + B'(U_0)\left(U_2, U_{0\xi}\right)\right\}_{\xi} + \dot{s}U_{2\xi} - \left(f'(U_0)U_2\right)_{\xi}, \\ \text{s} = {} & U_{1t} + \dot{\delta}_1 U_{0\xi} + \dot{\delta}_0 U_{1\xi} + \frac{1}{2}\left(f''(U_0)(U_1, U_1)\right)_{\xi} - \left(B'(U_0)\left(U_1, U_{1\xi}\right)\right)_{\xi}, \end{aligned} \tag{2.10}$$

and so forth, where $\dot{s} = ds/dt, \dot{\delta}_0 = d\delta_0/dt$, and so forth. The inner approximation is supposed to be valid in a small zone of size $O(\varepsilon)$ around $x = s(t)$.

In a matching zone, we expect the outer and the inner expansion agree with each other. Using the Taylor series to express the outer solutions in terms of $\xi$, we obtain the following "matching conditions" as $\xi \to \pm\infty$:

$$U_0(\xi, t) = u_0(s(t) \pm 0, t) + o(1), \tag{2.11}$$

$$U_1(\xi, t) = u_1(s(t) \pm 0, t) + (\xi - \delta_0)\partial_x u_0(s(t) \pm 0, t) + o(1), \tag{2.12}$$

$$\begin{aligned} U_2(\xi, t) = {} & u_2(s(t) \pm 0, t) + (\xi - \delta_0)\partial_x u_1(s(t) \pm 0, t) - \delta_1 \partial_x u_0(s(t) \pm 0, t) \\ & + \frac{1}{2}(\xi - \delta_0)^2 \partial_x^2 u_0(s(t) \pm 0, t) + o(1), \end{aligned} \tag{2.13}$$

and so forth.

### 2.2. The Structure of Viscous Shock Profiles

Our construction of the approximate solution depends on the properties of the viscous shock profiles, which are the solutions of the ordinary differential equation

$$(B(\phi)\phi_\xi)_\xi = -\sigma\phi_\xi + f(\phi)_\xi, \tag{2.14}$$

satisfying the boundary conditions

$$\begin{aligned} \phi(\xi) &\longrightarrow u_l \quad \text{as } \xi \longrightarrow -\infty, \\ \phi(\xi) &\longrightarrow u_r \quad \text{as } \xi \longrightarrow +\infty, \end{aligned} \tag{2.15}$$

and moving with speed $\sigma$:

$$\sigma(u_l - u_r) = f(u_l) - f(u_r). \tag{2.16}$$

Integrate the differential equation to reduce that

$$B(\phi)\phi_\xi = -\sigma(\phi(\xi) - u_l) + f(\phi(\xi)) - f(u_l). \tag{2.17}$$

It is well known that for a given state $\overline{u}$ and the $p$ wave family, if $|u_l - \overline{u}| + |\sigma - \lambda_p(\overline{u})|$ is sufficiently small, then there exists a shock profile $\phi = \phi(\xi, u_l, \sigma)$, which connects $u_l$ and $u_r$ from left to right. Using the genuine nonlinearity, by similar arguments in [2], we can obtain

$$\begin{aligned} &\partial_\xi \lambda_p(\phi) < 0, \quad \forall \xi, \\ &|\partial_\xi \phi| \le c|\partial_\xi \lambda_p(\phi)| \le c|u_l - u_r|. \end{aligned} \tag{2.18}$$

And as $\xi \to -\infty$,

$$\begin{aligned}
\phi(\xi, u_l, \sigma) - u_l &= O(1)|u_l - u_r|e^{-\alpha|\xi|}, \\
\frac{\partial\phi}{\partial u_l} - I &= O(1)e^{-\alpha|\xi|}, \\
\frac{\partial\phi}{\partial\sigma} &= O(1)e^{-\alpha|\xi|}.
\end{aligned} \tag{2.19}$$

As $\xi \to +\infty$,

$$\begin{aligned}
\phi(\xi, u_l, \sigma) - u_r &= O(1)|u_l - u_r|e^{-\alpha|\xi|}, \\
\frac{\partial\phi}{\partial u_l} - \frac{\partial u_r}{\partial u_l} &= O(1)e^{-\alpha|\xi|}, \\
\frac{\partial\phi}{\partial\sigma} - \frac{\partial u_r}{\partial\sigma} &= O(1)e^{-\alpha|\xi|}.
\end{aligned} \tag{2.20}$$

### 2.3. *Solutions of the Outer and Inner Problems*

Now we construct $u_j$ and $U_j$ order by order.

The leading order outer function, $u_0$, is the single-shock solution of the theorem. For any fixed $t$, the leading order inner solution $U_0(\xi, t)$ is exactly the viscous shock profile with $u_l(t) \equiv u(s(t) - 0, t)$, $u_r(t) \equiv u(s(t) + 0, t)$, and $\sigma = \dot{s}(t)$. So

$$U_0(\xi, t) = \phi(\xi, u_l(t), \dot{s}(t)). \tag{2.21}$$

Here we take the shift to be zero since it can be absorbed into $\delta_0(t)$.

Next we determine $u_1, U_1$, and $\delta_0(t)$ together. Substituting (2.21) into (2.9) gives

$$\left\{B(\phi)U_{1\xi} + B'(\phi)(U_1, \phi_\xi)\right\}_\xi + \dot{s}U_{1\xi} - \left(f'(\phi)U_1\right)_\xi = \dot{\delta}_0(t)\phi_\xi + \frac{\partial\phi}{\partial u_l}\dot{u}_l + \frac{\partial\phi}{\partial\dot{s}}\ddot{s}. \tag{2.22}$$

By the matching condition (2.12), we expect that

$$U_1(\xi, t) = \xi \cdot \partial_x u_0(s(t) \pm 0, t) + O(1) \quad \text{as } \xi \longrightarrow \pm\infty. \tag{2.23}$$

So we set

$$U_1(\xi, t) = V_1(\xi, t) + D_1(\xi, t), \tag{2.24}$$

where $D_1(\xi, t)$ is a smooth function satisfying

$$D_1(\xi, t) = \begin{cases} \xi \cdot \partial_x u_0(s(t) - 0, t), & \xi < -1, \\ \xi \cdot \partial_x u_0(s(t) + 0, t), & \xi > 1. \end{cases} \tag{2.25}$$

Then inserting (2.24) into (2.22) and using (2.19)–(2.20) and the identity

$$\frac{d}{dt}u_0(s(t)\pm 0,t) = \left(\dot{s}I - f'(u_0(s(t)\pm 0,t))\right)u_{0x}(s(t)\pm 0,t), \tag{2.26}$$

we obtain

$$\left\{B(\phi)V_{1\xi} + B'(\phi)(V_1,\phi_\xi)\right\}_\xi + \dot{s}V_{1\xi} - \left(f'(\phi)V_1\right)_\xi = \dot{\delta}_0(t)\phi_\xi + g(\xi,t), \tag{2.27}$$

where $|g(\xi,t)| \le c\exp\{-\alpha|\xi|\}$ for large $|\xi|$. Define $G(\xi,t) = \int_0^\xi g(\eta,t)d\eta$. Then we have

$$B(\phi)V_{1\xi} + B'(\phi)(V_1,\phi_\xi) + \dot{s}V_1 - f'(\phi)V_1 = \dot{\delta}_0(t)\phi + G(\xi,t) + c(t), \tag{2.28}$$

where $c(t) \in R^n$ are integration constants to be determined later. We express $V_1$ in terms of the basis, $r_1(\phi), r_2(\phi), \dots, r_n(\phi)$, of the right eigenvectors of $f'(\phi)$. We write

$$\begin{aligned}
V_1(\xi,t) &= \sum_{j=1}^n \alpha_j(\xi,t)r_j(\phi(\xi,t)),\\
u_1(s(t)\pm 0,t) &= \sum_{j=1}^n \beta_{j\pm}(t)r_j(u_0(s(t)\pm 0,t)),\\
\partial_x u_0(s(t)\pm 0,t) &= \sum_{j=1}^n \gamma_{j\pm}(t)r_j(u_0(s(t)\pm 0,t)).
\end{aligned} \tag{2.29}$$

Here the $\beta_{j-}$ are for $u_1(s(t)-0,t)$ and the $\beta_{j+}$ are for $u_1(s(t)+0,t)$, and so forth. Then the matching conditions (2.12) are transformed into

$$\lim_{\xi\to\pm\infty}\alpha_j(\xi,t) = \beta_{j\pm}(t) - \delta_0(t)\gamma_{j\pm}(t), \quad j = 1,\dots,n. \tag{2.30}$$

Define $\sigma_j(\phi) \equiv l_j(\phi)B(\phi)r_j(\phi) > 0$. Multiplying (2.28) by $l_j(\phi)$, and using (2.29), we obtain

$$\begin{aligned}
\alpha_{j\xi} &+ \sigma_j(\phi)^{-1}\left\{l_j(\phi)B'(\phi)(r_j(\phi),\phi_\xi) + (\dot{s} - \lambda_j(\phi))\right\}\alpha_j\\
&= \sigma_j(\phi)^{-1}l_j(\phi)\left(\dot{\delta}_0(t)\phi + G(\xi,t) + c(t)\right)\\
&\quad - \sum_{i=1}^n \sigma_j(\phi)^{-1}\alpha_i(\xi,t)l_j(\phi)B(\phi)r_{i\xi}(\phi), \quad j = 1,\dots,n,
\end{aligned} \tag{2.31}$$

and then we have the following result.

**Lemma 2.1.** *There is a smooth solution, $\alpha(\xi,t)$, to* (2.31) *with the following property:*

$$\alpha_j(\xi,t)=\begin{cases}(\dot{s}-\lambda_j(u_l))^{-1}l_j(u_l)\left[\dot{\delta}_0u_l+G_-+c(t)\right]+O(1)\exp\{-\alpha|\xi|\}, & \xi\longrightarrow-\infty,\\ (\dot{s}-\lambda_j(u_r))^{-1}l_j(u_r)\left[\dot{\delta}_0u_r+G_++c(t)\right]+O(1)\exp\{-\alpha|\xi|\}, & \xi\longrightarrow+\infty,\end{cases}\tag{2.32}$$

*for $j=1,\ldots,n$, where $G_\pm=\lim_{\xi\to\pm\infty}G(\xi,t)$, and $\alpha_0$ is a positive constant.*

*Proof.* We use the standard iteration argument. Define $\alpha^0(\xi,t)\equiv0$, and

$$\begin{aligned}\alpha_j^{k+1}(\xi,t)&=\int_{-\infty}^{\xi}\exp\left\{-\int_\eta^\xi\sigma_j(\phi)^{-1}\left(l_j(\phi)B'(\phi)(r_j(\phi),\phi_\xi)+\dot{s}-\lambda_j(\phi)\right)d\zeta\right\}\\&\quad\cdot\left\{\sigma_j(\phi)^{-1}l_j(\phi)\left(\dot{\delta}_0(t)\phi+G(\xi,t)+c(t)\right)\right.\\&\quad\left.-\sum_{i=1}^n\sigma_j(\phi)^{-1}\alpha_i(\xi,t)l_j(\phi)B(\phi)r_{i\xi}(\phi)\right\}d\eta,\quad j<p,\\ \alpha_p^{k+1}(\xi,t)&=\int_0^\xi\exp\left\{-\int_\eta^\xi\sigma_p(\phi)^{-1}\left(l_p(\phi)B'(\phi)(r_p(\phi),\phi_\xi)+\dot{s}-\lambda_p(\phi)\right)d\zeta\right\}\\&\quad\cdot\left\{\sigma_p(\phi)^{-1}l_p(\phi)\left(\dot{\delta}_0(t)\phi+G(\xi,t)+c(t)\right)\right.\\&\quad\left.-\sum_{i=1}^n\sigma_p(\phi)^{-1}\alpha_i(\xi,t)l_p(\phi)B(\phi)r_{i\xi}(\phi)\right\}d\eta,\\ \alpha_j^{k+1}(\xi,t)&=-\int_\xi^{+\infty}\exp\left\{-\int_\eta^\xi\sigma_j(\phi)^{-1}\left(l_j(\phi)B'(\phi)(r_j(\phi),\phi_\xi)+\dot{s}-\lambda_j(\phi)\right)d\zeta\right\}\\&\quad\cdot\left\{\sigma_j(\phi)^{-1}l_j(\phi)\left(\dot{\delta}_0(t)\phi+G(\xi,t)+c(t)\right)\right.\\&\quad\left.-\sum_{i=1}^n\sigma_j(\phi)^{-1}\alpha_i(\xi,t)l_j(\phi)B(\phi)r_{i\xi}(\phi)\right\}d\eta,\quad j>p.\end{aligned}\tag{2.33}$$

Set

$$\begin{aligned}M_k&=\sum_{0\le i\le n}\sup_{R\times[0,T]}\left|\alpha_i^k(\xi,t)\right|,\quad k=0,1,2,\ldots,\\ D_k&=\sum_{0\le i\le n}\sup_{R\times[0,T]}\left|\alpha_i^{k+1}(\xi,t)-\alpha_i^k(\xi,t)\right|,\quad k=0,1,2,\ldots.\end{aligned}\tag{2.34}$$

Then from the lax entropy condition (1.5), we can obtain

$$D_k \le \bar{c}\mu D_{k-1}, \quad k = 1, 2, \ldots, \tag{2.35}$$

where $\bar{c}$ is independent of $\mu$. And then for suitably small $\mu$, we have

$$M_k \le \bar{c} M_1, \quad k = 0, 1, \ldots, \tag{2.36}$$

and $\alpha^k(\xi, t)$ converges uniformly to a smooth bounded function, $\alpha(\xi, t)$, which is a solution to (2.31). The asymptotic behavior of the solution, $\alpha(\xi, t)$, follows from the formulas (2.33).

With Lemma 2.1 and the matching condition (2.30) at hand, we can determine, completely the same as in [2], $\beta_\pm(t), \delta_0(t)$, and $c(t)$, which guarantees the existence of $U_1(\xi, t)$ and $u_1(x, t)$. We give the sketch of this process. First, we use Lemma 2.1 and (2.30) for incoming indices to get a system of $(n+1)$ equations for $n+1$ unknowns, that is,

$$l_j(u_r)\left[\dot{\delta}_0 u_r + c(t)\right] = \left(\beta_{j+}(t) - \delta_0 \gamma_{j+}(t)\right)\left(\dot{s} - \lambda_j(u_r)\right) - l_j(u_r) G_+, \quad \text{for } 1 \le j \le p, \tag{2.37}$$

$$\begin{aligned} l_j(u_l)\left[\dot{s} u_r + c(t)\right] = {} & \left(\beta_{j-}(t) - \delta_0 \gamma_{j-}(t)\right)\left(\dot{s} - \lambda_j(u_l)\right) + l_j(u_l)(u_r - u_l)\dot{\delta}_0 \\ & - l_j(u_l) G_-, \quad \text{for } (p+1) \le j \le n, \end{aligned} \tag{2.38}$$

$$\begin{aligned} \left(l_j(u_l)(u_l - u_r)\right)\dot{\delta}_0 = {} & \left(l_p(u_r) - l_p(u_l)\right)\left(\dot{\delta}_0 u_r + c(t)\right) + \left(\dot{s} - \lambda_p(u_l)\right)\beta_{p-}(t) \\ & - \left(\dot{s} - \lambda_p(u_r)\right)\beta_{p+}(t) + \delta_0\left[\gamma_{p+}\left(\dot{s} - \lambda_p(u_r)\right) - \gamma_{p-}\left(\dot{s} - \lambda_p(u_l)\right)\right] \\ & + l_p(u_r) G_+ - l_l(u_l) G_-. \end{aligned} \tag{2.39}$$

Then we can solve for $\dot{\delta}_0 u_r + c(t)$ from (2.37)-(2.38). Substituting the resulting expression into (2.39), by writing $\beta_{\text{in}} = (\beta_{p_-}, \beta_{(p+1)_-}, \ldots, \beta_{n_-}, \beta_{1_+}, \ldots, \beta_{p_+})$, we arrive at an ordinary differential equation for $\delta_0$:

$$\dot{\delta}_0 + E_1(t)\delta_0 = E_2(t) \cdot \beta_{\text{in}} + G_1(t), \tag{2.40}$$

provided that $\mu$ is suitably small. Here $E_1(t), E_2(t)$, and $G_1(t)$ are smooth known functions, and $E_1(t)$ and $E_2(t)$ remain bounded even as $\mu \to 0^+$. Solving for $\delta_0$ from (2.37) up to a constant, we obtain $c(t)$ uniquely in terms of $\beta_{\text{in}}$. Then substitute the expression of $\delta_0$ and $c(t)$ into the equation of the matching condition for outgoing indices to yield the linear relations

$$\beta_{\text{out}} = E(t) \cdot \beta_{\text{in}} + F(t), \tag{2.41}$$

where $\beta_{\text{out}} = (\beta_{1_-}, \beta_{2_-}, \ldots, \beta_{(p-1)_-}, \beta_{(p+1)_+}, \ldots, \beta_{n_+})$, and $F(t) \in R^{n-1}$ is a smooth known function, $E(t)$ is a smooth $(n-1) \times (n+1)$ matrix and remains bounded even as $\mu \to 0^+$. Then the theory of linear hyperbolic equations [19, 20] shows that the problem (2.3), (2.41) has a solution smooth up to the shock provided that the initial value, $u_1(x, 0)$, is chosen to satisfy the appropriate compatibility conditions at $x = s(0)$. Thus $u_1(x, t)$ is completely determined, which in turn gives $\delta_0$ and $c(t)$ by (2.37)-(2.38), and therefore $U_1(\xi, t)$. □

Now we summarize the above discussion to achieve the following.

**Proposition 2.2.** *If $\mu$ is suitably small, then $u_1(x,t)$, $U_1(\xi,t)$ and $\delta_0$ can be established such that*

(i) *$u_1(x,t)$ and its derivatives are uniformly continuous up to $x = s(t)$, and*

$$\sum_{0\le k\le 6}\int_0^T\int\left|\partial_x^k u_1(x,t)\right|^2 dx\,dt < \infty. \tag{2.42}$$

(ii) *$U_1(\xi,t)$ and $\delta_0$ are smooth functions, and there are an $\alpha > 0$, such that*

$$U_1(\xi,t) = u_1(s(t)\pm 0,t) + (\xi-\delta_0)\partial_x u_0(s(t)\pm 0,t) + O(1)\exp\{-\alpha|\xi|\}, \quad \text{as } \xi \longrightarrow \pm\infty. \tag{2.43}$$

The above constructions can be carried out to any order. In particular, we can determine $u_i, U_i$ and $\delta_{i-1}$ ($i = 2,3$) simultaneously and similar results as in Proposition 2.2 hold for them.

### 2.4. Approximate Solutions

Now we can construct an approximate solution to (1.1) by patching the truncated outer and inner solutions in the previous discussion as in [2]. Define

$$\begin{gathered} I(x,t) = \phi\left(\frac{x-s(t)}{\varepsilon}+\delta_0+\varepsilon\delta_1+\varepsilon\delta^2,t\right)+\sum_{i=1}^{3}\varepsilon^i U_i\left(\frac{x-s(t)}{\varepsilon}+\delta_0+\varepsilon\delta_1+\varepsilon\delta^2,t\right), \\ O(x,t) = u_0(x,t)+\varepsilon u_1(x,t)+\varepsilon^2 u_2(x,t)+\varepsilon^3 u_3(x,t). \end{gathered} \tag{2.44}$$

Let $m \in C_0^\infty(R)$ satisfy $0 \le m(y) \le 1$, and

$$m(y) = \begin{cases} 1, & |y| \le 1, \\ 0, & |y| \ge 2. \end{cases} \tag{2.45}$$

Let $\gamma \in (5/7, 1)$ be a constant. Then we define the approximate solution to (1.1) as

$$v^\varepsilon(x,t) = m\left(\frac{x-s(t)}{\varepsilon^\gamma}\right)I(x,t) + \left(1-m\left(\frac{x-s(t)}{\varepsilon^\gamma}\right)\right)O(x,t) + d(x,t), \tag{2.46}$$

where $d(x,t)$ is a higher-order correction term to be determined.

Using the structures of the various orders of inner and outer solutions, we compute that

$$\begin{aligned} v_t^\varepsilon + f(v^\varepsilon)_x - \varepsilon(B(v^\varepsilon)v_x^\varepsilon)_x &= \sum_{i=3}^{n} q_i(x,t) + d_t - (B(mI+(1-m)O)d_x)_x \\ &\quad -\varepsilon q_{4x}(x,t) + \left(f(v^\varepsilon)-f(v^\varepsilon-d)\right)_x, \end{aligned} \tag{2.47}$$

where

$$
\begin{aligned}
q_1(x,t) &= (1-m)\left\{\left[f(O)-\Gamma\left(f(O)\right)\right]_x - \varepsilon\left[B(O)O_x - \Gamma(B(O)O_x)\right]_x\right\},\\
q_2(x,t) &= m\Big\{\left[f(I)-\Gamma\left(f(I)\right)\right]_x - \varepsilon\left[B(I)I_x - \Gamma(B(I)I_x)\right]_x + \varepsilon^3 U_{3t}\\
&\qquad + \varepsilon^4\left(\dot{\delta}_2 U_1 + \dot{\delta}_1 U_2 + \dot{\delta}_0 U_3 + \varepsilon\dot{\delta}_2 U_2 + \varepsilon\dot{\delta}_1 U_3 + \varepsilon^2\dot{\delta}_2 U_3\right)_x\Big\},\\
q_3(x,t) &= m_t(I-O) + m_x\left(f(I)-f(O)\right)\\
&\quad + \left\{f(mI+(1-m)O) - \left(mf(I)+(1-m)f(O)\right)\right\}_x\\
&\quad - \varepsilon m_x(B(I)I_x - B(O)O_x),\\
q_4(x,t) &= B(v^\varepsilon)(v^\varepsilon - d)_x - mB(I)I_x - (1-m)B(O)O_x + (B(v^\varepsilon) - B(v^\varepsilon - d))d_x,
\end{aligned}
\tag{2.48}
$$

where $\Gamma(f(O)), \Gamma(B(O)O_x)$ denote the truncated Taylor's expansion of $f(O), B(O)O_x$, respectively, at $u_0$, including all the terms of $O(1), O(1)\varepsilon, O(1)\varepsilon^2, O(1)\varepsilon^3$, and $\Gamma(f(I)), \Gamma(B(I)I_x)$ denote the truncated Taylor's expansion of $f(I), B(I)I_x$, respectively, at $\phi$, including all the terms of $O(1), O(1)\varepsilon, O(1)\varepsilon^2, O(1)\varepsilon^3$.

In view of our construction, we have

(i) supp $q_1(x,t) \subseteq \{(x,t) : |x - s(t)| \geq \varepsilon^\gamma, 0 \leq t \leq T\}$, and

$$
\begin{gathered}
\partial_x^l q_1(x,t) = O(1)\varepsilon^{4-l\gamma}, \qquad \left(\int_0^T \left\|q_1(\cdot,t)\right\|^2 dt\right)^{1/2} \leq O(1)\varepsilon^4,\\
\left(\int_0^T \left\|\partial_x^l q_1(\cdot,t)\right\|^2 dt\right)^{1/2} \leq O(1)\varepsilon^{4-(l-(1/2))\gamma}, \quad l = 1,2,3,
\end{gathered}
\tag{2.49}
$$

(ii) supp $q_2(x,t) \subseteq \{(x,t) : |x - s(t)| \leq 2\varepsilon^\gamma, 0 \leq t \leq T\}$, and

$$
\partial_x^l q_2(x,t) = O(1)\varepsilon^{(3-l)\gamma}, \quad l = 0,1,2,3,
\tag{2.50}
$$

(iii) supp $q_3(x,t) \subseteq \{(x,t) : \varepsilon^\gamma \leq |x - s(t)| \leq 2\varepsilon^\gamma, 0 \leq t \leq T\}$, and

$$
\partial_x^l q_3(x,t) = O(1)\varepsilon^{(3-l)\gamma},\ l = 0,1,2,3,
\tag{2.51}
$$

where we have used the estimates $\partial_x^l(I-O) = O(1)\varepsilon^{(4-l)\gamma}$ on $\{(x,t) : \varepsilon^\gamma \leq |x-s(t)| \leq 2\varepsilon^\gamma\},\ l = 0,1,2,3$.

We now choose $d(x,t)$ to satisfy

$$
\begin{gathered}
d_t - (B(mI + (1-m)O)d_x)_x = -\sum_{i=1}^{3} q_i(x,t),\\
d(x,0) = 0,
\end{gathered}
\tag{2.52}
$$

so that $v^\varepsilon$ satisfies

$$v_t^\varepsilon + f(v^\varepsilon)_x = \varepsilon(B(v^\varepsilon)v_x^\varepsilon)_x - \varepsilon q_{4x} + \left(f(v^\varepsilon) - f(v^\varepsilon - d)\right). \tag{2.53}$$

Since $B$ is smooth and positive definite, by the standard energy estimates for the linear parabolic system and Sobolev's inequalities, we have the following results.

**Lemma 2.3.** *Let $d(x,t)$ be the solution of* (2.52). *Then the following estimates hold for all $t \in [0,T]$:*

$$\begin{aligned} \left\|\partial_x^l d(\cdot,t)\right\|_{L^\infty} &\le O(1)\varepsilon^{(7/2)\gamma-(l+(1/2))}, \quad \textit{for } l = 0,1,2,3, \\ \left\|\partial_x^l d(\cdot,t)\right\| &\le O(1)\varepsilon^{(7/2)\gamma-l}, \quad l = 0,1,2,3,4. \end{aligned} \tag{2.54}$$

*Then for $q_4$, we have*

$$\left\|\partial_x^l q_4\right\| \le O(1)\varepsilon^{(7/2)\gamma-(l+1)}, \quad l = 0,1,2. \tag{2.55}$$

And by our construction, we obtain the following.

**Lemma 2.4.** *One has*

$$\begin{aligned} v^\varepsilon(x,t) &= \begin{cases} u_0(x,t) + O(1)\varepsilon, & |x - s(t)| \ge \varepsilon^\gamma, \\ \phi(\xi,t) + O(1)\varepsilon^\gamma, & |x - s(t)| \le 2\varepsilon^\gamma, \end{cases} \\ \frac{\partial v^\varepsilon}{\partial x} &= \frac{1}{\varepsilon} m\partial_\xi\phi + O(1), \qquad \frac{\partial v^\varepsilon}{\partial t} = O(1). \end{aligned} \tag{2.56}$$

## 3. Stability Analysis

We now show that there exists an exact solution to (1.1) in a neighborhood of the approximate solution $v^\varepsilon(x,t)$, and that the asymptotic behavior of the viscous solution is given by $v^\varepsilon$ for small viscosity $\varepsilon$.

Suppose that $u^\varepsilon(x,t)$ is the exact solution to (1.1) with the initial data $u^\varepsilon(x,0) = v^\varepsilon(x,0)$. We decompose the solution as

$$u^\varepsilon(x,t) = v^\varepsilon(x,t) + \overline{w}(x,t), \quad (x,t) \in R \times [0,T]. \tag{3.1}$$

Then using the relation (2.53) for $v^\varepsilon$, we compute that

$$\begin{aligned} \overline{w}_t + \left(f'(v^\varepsilon)\overline{w}\right)_x + Q(v^\varepsilon,\overline{w})_x &= \varepsilon(B(u^\varepsilon)u_x^\varepsilon - B(v^\varepsilon)v_x^\varepsilon)_x + \varepsilon q_{4x} + \left(f(v^\varepsilon - d) - f(v^\varepsilon)\right)_x, \\ \overline{w}(x,0) &= 0, \end{aligned} \tag{3.2}$$

where $Q(v^\varepsilon,\overline{w}) = f(u^\varepsilon) - f(v^\varepsilon) - f'(v^\varepsilon)\overline{w}$ satisfies $|Q| \le O(1)\overline{w}$ for small $\overline{w}$.

Set $\overline{w}(x,t) = \widetilde{w}_x(x,t)$ in (3.2) and integrate the resulting equation with respect to $x$ to give

$$\begin{aligned} \widetilde{w}_t + \left(f'(v^\varepsilon)\widetilde{w}\right)_x + Q(v^\varepsilon, \widetilde{w}) &= \varepsilon(B(u^\varepsilon)u_x^\varepsilon - B(v^\varepsilon)v_x^\varepsilon) + \varepsilon q_4 + f(v^\varepsilon - d) - f(v^\varepsilon), \\ \widetilde{w}(x,0) &= 0, \end{aligned} \tag{3.3}$$

by making the following scalings,

$$\widetilde{w}(x,t) = \varepsilon w(y,\tau), \quad y = \frac{x - s(t)}{\varepsilon}, \ \tau = \frac{t}{\varepsilon}, \tag{3.4}$$

we transform (3.3) into

$$\begin{aligned} w_\tau - \dot{s}w_y + f'(v^\varepsilon)w_y + Q(v^\varepsilon, w_y) &= B(v^\varepsilon)w_{yy} + (B(u^\varepsilon) - B(v^\varepsilon))u_y^\varepsilon \\ &\quad + \varepsilon q_4 + f(v^\varepsilon - d) - f(v^\varepsilon), \\ w(y,0) &= 0. \end{aligned} \tag{3.5}$$

Then we only need to show that for suitably small $\varepsilon$, (3.5) has a unique "small" smooth solution up to $\tau = T/\varepsilon$. By the standard existence and uniqueness theory, and the continuous induction argument for parabolic equations [21], this will follow from the following a priori estimate.

**Proposition 3.1.** *Suppose that the Cauchy problem* (3.5) *has a solution $w \in C^1([0,\tau_0] : H^3(R^1))$ for some $\tau_0 \in (0, T/\varepsilon]$. Then there exist positive constants $\mu_1, \varepsilon_1$ and $C$, which are independent of $\varepsilon$ and $\tau_0$, such that if*

$$0 < \varepsilon < \varepsilon_1, \qquad \sup_{0\le\tau\le\tau_0} \|w(\cdot,\tau)\|_3 + \mu \le \mu_1, \tag{3.6}$$

*then*

$$\sup_{0\le\tau\le\tau_0} \|w(\cdot,\tau)\|_3^2 + \int_0^{\tau_0} \|w_y(\cdot,\tau)\|_3^2 d\tau \le C\varepsilon^{7\gamma-3}, \tag{3.7}$$

*where $\gamma$ is defined in Section 2.4.*

The proof of the proposition occupies the rest of this section. We separate it into several parts. First we diagonalize the system (3.5). Define

$$\begin{aligned} \theta(y,\tau) &= L(v^\varepsilon)w(y,\tau), \\ M(y,\tau) &= (\partial_y L(v^\varepsilon)) \cdot R(v^\varepsilon), \\ N(y,\tau) &= (\partial_\tau L(v^\varepsilon)) \cdot R(v^\varepsilon). \end{aligned} \tag{3.8}$$

Then we have

$$\theta_y = M\theta + Lw_y,$$
$$\theta_\tau = N\theta + Lw_\tau, \tag{3.9}$$
$$\theta_{yy} = (M\theta)_y + M\theta_y - M^2\theta + Lw_{yy}.$$

Using the identity (3.9), we can rewrite (3.5) as

$$\begin{aligned}
&\theta_\tau + (\Lambda - \dot{s})\theta_y + (\dot{s} - \Lambda)M\theta - N\theta + LQ(v^\varepsilon, R\theta_y + R_y\theta) \\
&\quad = L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\Big(\theta_{yy} - (M\theta)_y - M\theta_y + M^2\theta\Big) \\
&\qquad + \ L(v^\varepsilon)(B(u^\varepsilon) - B(v^\varepsilon))u_y^\varepsilon + \varepsilon L(v^\varepsilon)q_4 - \Lambda L(v^\varepsilon)d + L(v^\varepsilon)Q_1(v^\varepsilon, d).
\end{aligned} \tag{3.10}$$

In what follows, we use $c$ to denote any positive constant which is independent of $\varepsilon, y$, and $\tau$;$\bar{c}$ to denote any positive constant which is independent of $\varepsilon$ and $\mu$. And we set $\varepsilon \leq 1$.

Now we do the following estimates on transversal waves.

**Lemma 3.2.** *There exist suitably small positive constants $\mu_2$, $\varepsilon_2$, independent of $\varepsilon$ and $\tau_0$, such that*

$$\begin{aligned}
\sum_{k\neq p}\int \left|\partial_y\lambda_p(\phi)\right|\theta_k^2 dy \leq{}& \frac{d}{d\tau}\left[-\sum_{k\neq p}\int\left(\frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}}\right)\theta_k^2 dy\right] \\
&+ \left(\bar{c}\mu + c\varepsilon^\gamma\right)\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2 dy \\
&+ \left(\bar{c}\mu + c\varepsilon\right)\left\|\theta_y(\cdot,\tau)\right\|^2 + \bar{c}\mu\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 \\
&+ c\varepsilon\|\theta(\cdot,\tau)\|^2 + c\varepsilon^{7\gamma-2},
\end{aligned} \tag{3.11}$$

*provided that $\|\theta(\cdot,\tau)\|_{L^\infty}$ is bounded.*

*Proof.* Using (3.10), we compute that for $k \neq p$,

$$\begin{aligned}
&\int\left|\partial_y\lambda_p(\phi)\right|\theta_k^2 dy \\
&\quad = -\int \partial_y\lambda_p(\phi)\theta_k^2 dy = 2\int \lambda_p(\phi)\theta_k\partial_y\theta_k dy \\
&\quad = 2\int\left(\frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}}\right)\theta_k
\end{aligned}$$

$$
\begin{aligned}
&\times \Big[ -(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\theta_{ky} - \theta_{k\tau} - \{(\dot{s}-\Lambda)M\theta - N\theta + \Lambda L d + L(Q-Q_1)\}_k \\
&\quad + \Big\{ L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\Big(\theta_{yy} - (M\theta)_y - M\theta_y + M^2\theta\Big) \\
&\quad\quad + L(v^\varepsilon)(B(u^\varepsilon)-B(v^\varepsilon))u_y^\varepsilon + \varepsilon L(v^\varepsilon)q_4 \Big\}_k \Big] dy \\
&\equiv \sum_{i=1}^{12} J_i.
\end{aligned}
\tag{3.12}
$$

We now estimate $J_i$ $(1 \le j \le 12)$ separately as follows:

$$
\begin{aligned}
J_1 &= -2\int \left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\theta_k\theta_{ky}\,dy \\
&= \int \frac{\partial}{\partial y}\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\theta_k^2\,dy \\
&\quad + \int \left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)(\nabla\lambda_k(v^\varepsilon)\partial_y v^\varepsilon - \nabla\lambda_k(\phi)\partial_y\phi)\theta_k^2\,dy \\
&\equiv I_1 + I_2.
\end{aligned}
\tag{3.13}
$$

By Lemma 2.4,

$$
\begin{aligned}
I_1 &\le \int \left|\frac{\partial}{\partial y}\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\right| \left|(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\right|\theta_k^2\,dy \le c\int |\partial_y\phi|\left|(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\right|\theta_k^2\,dy \\
&\le c\int_{|y|\le\varepsilon^{\gamma-1}} |\partial_y\lambda_p(\phi)|\left|(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\right|\theta_k^2\,dy + c\int_{|y|\ge\varepsilon^{\gamma-1}} |\partial_y\phi|\left|(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\right|\theta_k^2\,dy \\
&\le c\varepsilon^\gamma \int |\partial_y\lambda_p(\phi)|\theta_k^2\,dy + c\varepsilon\|\theta_k(\cdot,\tau)\|^2.
\end{aligned}
\tag{3.14}
$$

For the second term $I_2$, since

$$
\begin{aligned}
\left|\nabla\lambda_k(v^\varepsilon)\partial_y v^\varepsilon - \nabla\lambda_k(\phi)\partial_y\phi\right| &= \left|\nabla\lambda_k(v^\varepsilon)m\partial_y\phi - \nabla\lambda_k(\phi)\partial_y\phi + O(1)\varepsilon\right| \\
&\le \left|\nabla\lambda_k(\phi)(m-1)\partial_y\phi\right| \\
&\quad + \left|(\nabla\lambda_k(v^\varepsilon)-\nabla\lambda_k(\phi))m\partial_y\phi\right| + O(1)\varepsilon \\
&\le O(1)\varepsilon + O(1)\varepsilon^\gamma m\left|\partial_y\lambda_p(\phi)\right|,
\end{aligned}
\tag{3.15}
$$

$$
\partial_y v^\varepsilon(y,\tau) = m\Big(\varepsilon^{1-\gamma}y\Big)\partial_y\phi + O(1)\varepsilon,
\tag{3.16}
$$

which follows from Lemma 2.4. Then we have

$$I_2 \le \int \left| \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right| \left| \nabla \lambda_k(v^\varepsilon)\partial_y v^\varepsilon - \nabla \lambda_k(\phi)\partial_y \phi \right| \theta_k^2 dy$$
$$\le c\mu\varepsilon^\gamma \int \left| \partial_y \lambda_p(\phi) \right| \theta_k^2 dy + c\mu\varepsilon \| \theta_k(\cdot,\tau) \|. \tag{3.17}$$

Consequently, we obtain

$$J_1 = I_1 + I_2 \le c\varepsilon^\gamma \int \left| \partial_y \lambda_p(\phi) \right| \theta_k^2 dy + c\varepsilon \| \theta_k(\cdot,\tau) \|^2. \tag{3.18}$$

Using the estimate $(d/d\tau)((\lambda_p(\phi) - \dot{s})/\lambda_k(\phi) - \dot{s}) = O(1)\varepsilon$, we find

$$J_2 = -2 \int \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k \theta_{k\tau} dy$$
$$= -\frac{d}{d\tau} \int \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k^2 dy + \int \frac{d}{d\tau} \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k^2 dy \tag{3.19}$$
$$\le -\frac{d}{d\tau} \int \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k^2 dy + c\varepsilon \| \theta_k(\cdot,\tau) \|^2.$$

Notice that the facts

$$M = m\left(\varepsilon^{1-\gamma} y\right)(\nabla L(v^\varepsilon)\partial_y \phi) R(v^\varepsilon) + O(1)\ \varepsilon$$
$$= m\left(\varepsilon^{1-\gamma} y\right)(\nabla L(\phi)\partial_y \phi) R(\phi) + O(1)\varepsilon^\gamma m\left(\varepsilon^{1-\gamma} y\right)\partial_y \phi + O(1)\varepsilon, \tag{3.20}$$

$$\left| \dot{s} - \lambda_p(\phi) \right| \le \bar{c}\mu, \tag{3.21}$$

we arrive at

$$J_3 = -2 \int \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k [(\dot{s} - \Lambda(v^\varepsilon)) M\theta]_k dy$$
$$\le (\bar{c}\mu + c\varepsilon^\gamma) \int m \left| \partial_y \lambda_p(\phi) \right| |\theta|^2 dy + c\varepsilon \| \theta(\cdot,\tau) \|^2, \tag{3.22}$$
$$J_4 = 2 \int \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k [N\theta]_k dy \le c\mu\varepsilon \| \theta(\cdot,\tau) \|^2,$$

where we have used the estimate $|N(y,\tau)| \le O(1)\varepsilon$. From Lemma 2.3,

$$\begin{aligned} J_5 &= -2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k(\Lambda L d)_k dy \\ &\le \mu\varepsilon\|\theta_k(\cdot,\tau)\|^2 + c\varepsilon^{-1}\int d^2 dy \\ &\le \mu\varepsilon\|\theta_k(\cdot,\tau)\|^2 + c\varepsilon^{7\gamma-2}. \end{aligned} \tag{3.23}$$

Using Lemma 2.4 again, we have

$$|L(v^\varepsilon)(Q-Q_1)| \le (\bar{c}+c\varepsilon^\gamma)\left(|d|^2 + \left|R(v^\varepsilon)\theta_y + R_y(v^\varepsilon)\theta\right|^2\right). \tag{3.24}$$

Then it follows from Lemma 2.3 and (3.16) that

$$\begin{aligned} J_6 &= -2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k[L(Q-Q_1)]_k dy \\ &\le \mu(\bar{c}+c\varepsilon^\gamma)\left\{\int|\theta_k||d|^2dy + (\bar{c}+c\varepsilon^\gamma)\int|\theta_k|\left|\theta_y\right|^2dy + \int|\theta_k|\left|R_y(v^\varepsilon)\theta\right|^2dy\right\} \\ &\le \mu(\bar{c}+c\varepsilon^\gamma)\left\{\frac{1}{2}\varepsilon\|\theta_k(\cdot,\tau)\|^2 + \frac{1}{2}\varepsilon^{-1}\int|d|^4dy + (\bar{c}+c\varepsilon^\gamma)\|\theta_k\|_{L^\infty}\left\|\theta_y(\cdot,\tau)\right\|^2 \right.\\ &\qquad\left. +(\bar{c}\mu+c\varepsilon^\gamma)\|\theta_k\|_{L^\infty}\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2dy + c\varepsilon^2\|\theta_k\|_{L^\infty}\|\theta(\cdot,\tau)\|^2\right\} \\ &\le c\mu\varepsilon^{14\gamma-3} + \mu\left(\bar{c}+c\varepsilon^{2\gamma}\right)\|\theta_k(\cdot,\tau)\|_{L^\infty}\left\|\theta_y(\cdot,\tau)\right\|^2 \\ &\quad + c\mu\varepsilon(1+\varepsilon\|\theta_k(\cdot,\tau)\|_{L^\infty})\|\theta(\cdot,\tau)\|^2 \\ &\quad + \mu(\bar{c}\mu+c\varepsilon^\gamma)\|\theta_k(\cdot,\tau)\|_{L^\infty}\int m\left|\partial_y\lambda_p(\phi)\right|^2|\theta|^2dy. \end{aligned} \tag{3.25}$$

In view of (3.16)–(3.21),

$$\begin{aligned} J_7 &= 2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k\left[L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\theta_{yy}\right]_k dy \\ &= -2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_{ky}\left[L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\theta_y\right]_k dy \\ &\quad -2\int\theta_k\left[\left(\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\right)_y\theta_y\right]_k dy \\ &\le \mu(\bar{c}+c\varepsilon^\gamma)\left\|\theta_y(\cdot,\tau)\right\|^2 + \bar{c}\int\left(\left|\partial_y\phi\right| + \left|\partial_y v^\varepsilon\right|\right)|\theta_k|\left|\theta_y\right|dy \end{aligned}$$

$$
\begin{aligned}
&\le (\overline{c}\mu_0 + c\varepsilon^\gamma)\left\|\theta_y(\cdot,\tau)\right\|^2 + \overline{c}\int\left|\partial_y\lambda_p(\phi)\right||\theta_k|\left|\theta_y\right|dy + c\varepsilon\int|\theta_k|\left|\theta_y\right|dy \\
&\le \frac{1}{2}\int\left|\partial_y\lambda_p(\phi)\right||\theta_k|^2dy + (\overline{c}\mu + c\varepsilon)\left\|\theta_y(\cdot,\tau)\right\|^2 + c\varepsilon\|\theta_k(\cdot,\tau)\|^2, \\
J_8 &= -2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k\left[L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)(M\theta)_y\right]_k dy \\
&= 2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)_y\theta_k[L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)M\theta]_k dy \\
&\quad + 2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_{ky}[L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)M\theta]_k dy \\
&\le (\overline{c} + c\varepsilon^\gamma)\int\left|\partial_y\phi\right||\theta_k||M\theta|dy + \mu\left\|\theta_y(\cdot,\tau)\right\|^2 + \mu(\overline{c} + c\varepsilon^\gamma)\int|M\theta|^2dy \\
&\le \mu\left\|\theta_y(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\int\left|\partial_y\lambda_p(\phi)\right||\theta_k|^2dy + c\varepsilon\|\theta(\cdot,\tau)\|^2.
\end{aligned}
\tag{3.26}
$$

Same bounds hold for $J_9$ and $J_{10}$.

Applying Cauchy inequality and (3.21), $J_{11}$ can be estimated as

$$
\begin{aligned}
J_{11} &= 2\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k\left[L(v^\varepsilon)(B(u^\varepsilon)-B(v^\varepsilon))u_y^\varepsilon\right]_k dy \\
&\le \overline{c}\mu\int\left|\theta_k[L(v^\varepsilon)(B(u^\varepsilon)-B(v^\varepsilon))w_{yy}]_k\right|dy \\
&\quad + \overline{c}\mu\int\left|\theta_k\left[L(v^\varepsilon)(B(u^\varepsilon)-B(v^\varepsilon))v_y^\varepsilon\right]_k\right|dy \\
&\equiv K_1 + K_2.
\end{aligned}
\tag{3.27}
$$

Lemma 2.4 yields,

$$
\begin{aligned}
K_1 &\le \mu(\overline{c} + c\varepsilon^\gamma)\int|\theta_k|\left|w_y\right|\left|\partial_y^2 w\right|dy \\
&\le (\overline{c}\mu + c\varepsilon^\gamma)\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\|\theta_k\|_{L^\infty}^2\int\left|w_y\right|^2dy \\
&\le (\overline{c}\mu + c\varepsilon^\gamma)\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\|\theta_k\|_{L^\infty}^2\left(\left\|\theta_y(\cdot,\tau)\right\|^2 + \int|M\theta|^2dy\right) \\
&\le (\overline{c}\mu + c\varepsilon^\gamma)\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\|\theta_k\|_{L^\infty}^2 \\
&\quad \times\left\{\left\|\theta_y(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2dy + c\varepsilon\|\theta(\cdot,\tau)\|^2\right\},
\end{aligned}
\tag{3.28}
$$

where we used the fact $w_y = R\theta_y - RM\theta$. Similarly,

$$\begin{aligned}
K_2 &\leq \mu(\bar{c} + c\varepsilon^\gamma) \int |\theta_k| |w_y| |\partial_y v^\varepsilon| dy \\
&\leq (\bar{c}\mu + c\varepsilon^\gamma) \int |\theta_k| |\theta_y| |\partial_y v^\varepsilon| dy + (\bar{c}\mu + c\varepsilon^\gamma) \int |M| |\theta|^2 |\partial_y v^\varepsilon| dy \\
&\leq (\bar{c}\mu + c\varepsilon^\gamma) \|\theta_y(\cdot,\tau)\|^2 + (\bar{c}\mu + c\varepsilon^\gamma) \int |\theta|^2 |\partial_y v^\varepsilon|^2 dy \\
&\quad + (\bar{c}\mu + c\varepsilon^\gamma) \int |M| |\theta|^2 |\partial_y v^\varepsilon| dy \\
&\leq (\bar{c}\mu + c\varepsilon^\gamma) \|\theta_y(\cdot,\tau)\|^2 + (\bar{c}\mu + c\varepsilon^\gamma) \int m |\partial_y \lambda_p(\phi)| |\theta|^2 dy + c\varepsilon \|\theta(\cdot,\tau)\|^2.
\end{aligned} \tag{3.29}$$

Thus, combining the above two inequality together, we obtain

$$\begin{aligned}
J_{11} &= K_1 + K_2 \\
&\leq (\bar{c}\mu + c\varepsilon^\gamma) \left\| \partial_y^2 w(\cdot,\tau) \right\|^2 + \left(1 + \|\theta_k\|_{L^\infty}^2\right) \\
&\quad \times \left\{ (\bar{c}\mu + c\varepsilon^\gamma) \|\theta_y(\cdot,\tau)\|^2 + (\bar{c}\mu + c\varepsilon^\gamma) \int m |\partial_y \lambda_p(\phi)| |\theta|^2 dy + c\varepsilon \|\theta(\cdot,\tau)\|^2 \right\}.
\end{aligned} \tag{3.30}$$

Finally,

$$\begin{aligned}
J_{12} &= 2\varepsilon \int \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k \left[ L(v^\varepsilon) q_4 \right]_k dy \\
&\leq \varepsilon \|\theta_k(\cdot,\tau)\|^2 + c\mu^2 \varepsilon \int |q_4|^2 dy \\
&\leq \varepsilon \|\theta_k(\cdot,\tau)\|^2 + c\varepsilon^{7\gamma-2}.
\end{aligned} \tag{3.31}$$

Summing all the inequalities for $k \neq p$, we arrive at

$$\begin{aligned}
\sum_{k \neq p} \int |\partial_y \lambda_p(\phi)| \theta_k^2 dy &\leq \frac{d}{d\tau} \left[ -\sum_{k \neq p} \int \left( \frac{\lambda_p(\phi) - \dot{s}}{\lambda_k(\phi) - \dot{s}} \right) \theta_k^2 dy \right] \\
&\quad + (\bar{c}\mu + c\varepsilon^\gamma) \int m |\partial_y \lambda_p(\phi)| \theta_p^2 dy \\
&\quad + (\bar{c}\mu + c\varepsilon^\gamma) \|\theta_y(\cdot,\tau)\|^2 + (\bar{c}\mu + c\varepsilon^\gamma) \left\| \partial_y^2 w(\cdot,\tau) \right\|^2 \\
&\quad + c\varepsilon \|\theta(\cdot,\tau)\|^2 + c\varepsilon^{7\gamma-2},
\end{aligned} \tag{3.32}$$

provided that $\|\theta(\cdot,\tau)\|_{L^\infty}$ is bounded.

We complete the proof of Lemma 3.2. □

**Lemma 3.3.** *Suppose that the conditions in Proposition 3.1 be satisfied. Then*

$$\|w(\cdot,\tau)\|_1^2 + \int_0^\tau \|w_y(\cdot,\tau)\|_1^2 d\tau + \int_0^\tau \int m\left|\partial_y \lambda_p(\phi)\right| |w|^2 dy\, d\tau \le c\varepsilon^{7\gamma-3}, \tag{3.33}$$

*for all $\tau \in [0,\tau_0]$, where $c$ is independent of $\tau_0$ and $\varepsilon$.*

*Proof.* Multiplying (3.10) on the left by $\theta^t$ and integrating over $R^1$, we obtain after integration by parts that

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\|\theta(\cdot,\tau)\|^2 = {}& \int \theta^t L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\theta_{yy}dy - \int \theta^t(\Lambda - \dot{s})\theta_y dy \\
& - \int \theta^t(\dot{s} - \Lambda)M\theta\, dy - \int \theta^t N\theta dy - \int \theta^t LQ(v^\varepsilon, R\theta_y + R_y\theta)dy \\
& + \int \theta^t LQ_1(v^\varepsilon, d)dy - \int \theta^t \Lambda L d\, dy \\
& + \int \theta^t L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\Big(-(M\theta)_y - M\theta_y + M^2\theta\Big)dy \\
& + \int \theta^t L(v^\varepsilon)(B(u^\varepsilon) - B(v^\varepsilon))u_y^\varepsilon dy + \varepsilon \int \theta^t L(v^\varepsilon)q_4 dy.
\end{aligned} \tag{3.34}$$

Next we estimate each term on the right hand side above. First, it follows from (3.16) that

$$\begin{aligned}
&\int \theta^t L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\theta_{yy}dy \\
&\quad = -\int \theta_y^t L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\theta_y dy - \int \theta^t (L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon))_y \theta_y dy \\
&\quad \le (-C_0 + c\varepsilon^\gamma)\left\|\theta_y(\cdot,\tau)\right\|^2 + (\overline{c} + c\varepsilon^\gamma)\int \left|\partial_y v^\varepsilon\right||\theta|\left|\theta_y\right|dy \\
&\quad \le (-C_0 + c\varepsilon^\gamma)\left\|\theta_y(\cdot,\tau)\right\|^2 + \frac{C_0}{10}\left\|\theta_y(\cdot,\tau)\right\|^2 + (\overline{c} + c\varepsilon^\gamma)\int \left|\partial_y v^\varepsilon\right|^2 |\theta|^2 dy \\
&\quad \le (-C_0 + c\varepsilon^\gamma)\left\|\theta_y(\cdot,\tau)\right\|^2 + \frac{C_0}{10}\left\|\theta_y(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\int m\left|\partial_y \lambda_p(\phi)\right| |\theta|^2 dy \\
&\qquad + c\varepsilon\|\theta(\cdot,\tau)\|^2.
\end{aligned} \tag{3.35}$$

Here $C_0 > 0$ is the minimum of the eigenvalues, valued at $u_0$ and $\phi$, of $(1/2)(LBR + (LBR)^t)$ and $(1/2)(B + B^t)$. And

$$\begin{aligned}
-\int \theta^t(\Lambda - \dot{s})\theta_y dy &= \frac{1}{2}\int \sum_{i=1}^{n} \partial_y \lambda_i(v^\varepsilon)\theta_i^2 dy \\
&= \frac{1}{2}\int m\left(\sum_{i=1}^{n} \partial_y \lambda_i(\phi)\theta_i^2\right) dy \\
&\quad + \frac{1}{2}\int m\left(\sum_{i=1}^{n} (\nabla\lambda_i(v^\varepsilon) - \nabla\lambda_i(\phi))\partial_y\phi\theta_i^2\right) dy + \varepsilon\int O(1)\theta^2 dy \\
&\le -\frac{1}{2}\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2 dy + c\varepsilon^\gamma \int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2 dy \\
&\quad + (\bar{c} + c\varepsilon^\gamma)\sum_{k\neq p}\int m\left|\partial_y\lambda_p(\phi)\right|\theta_k^2 dy + c\varepsilon\|\theta(\cdot,\tau)\|^2.
\end{aligned} \tag{3.36}$$

By virtue of (3.16)–(3.21), one finds

$$\begin{aligned}
-\int \theta^t(\dot{s} - \Lambda(v^\varepsilon))M\theta dy &= \int \theta^t(\Lambda(\phi) - \dot{s})M\theta dy + \int \theta^t(\Lambda(v^\varepsilon) - \Lambda(\phi))M\theta dy \\
&\le \left(\frac{1}{8} + \bar{c}\mu + c\varepsilon^\gamma\right)\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2 dy \\
&\quad + (\bar{c} + c\varepsilon^\gamma)\sum_{k\neq p}\int m\left|\partial_y\lambda_p(\phi)\right|\theta_k^2 dy + c\varepsilon\|\theta(\cdot,\tau)\|^2.
\end{aligned} \tag{3.37}$$

Now the remaining terms on the right hand side of (3.34) can be estimated in a similar way to that in the proof of Lemma 3.2. We list them below:

$$\begin{aligned}
&\int \theta^t N\theta\, dy \le c\varepsilon\|\theta(\cdot,\tau)\|^2, \\
&-\int \theta^t L(Q - Q_1) dy \le c\varepsilon^{14\gamma-3} + (\bar{c} + c\varepsilon^\gamma)\|\theta\|_{L^\infty}\left\|\theta_y(\cdot,\tau)\right\|^2 + c\varepsilon(1 + \varepsilon\|\theta\|_{L^\infty})\|\theta(\cdot,\tau)\|^2 \\
&\quad + (\bar{c}\mu + c\varepsilon^\gamma)\|\theta\|_{L^\infty}\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2 dy.
\end{aligned} \tag{3.38}$$

Lemma 2.3 leads to

$$-\int \theta^t\Lambda L d\, dy \le \varepsilon\|\theta(\cdot,\tau)\|^2 + c\varepsilon^{-1}\int d^2 dy \le \varepsilon\|\theta(\cdot,\tau)\|^2 + c\varepsilon^{7\gamma-2}. \tag{3.39}$$

Continuing, we compute that

$$
\begin{aligned}
&-\int \theta^t L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)(M\theta)_y dy,\\
&\quad = \int \theta_y^t L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)M\theta\, dy + \int \theta^t (L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon))_y M\theta\, dy\\
&\quad \le \frac{C_0}{10}\left\|\theta_y(\cdot,\tau)\right\|^2 + (\overline{c}+c\varepsilon^\gamma)\int |M\theta|^2 dy + (\overline{c}\mu_0 + c\varepsilon^\gamma)\int |M||\theta|^2 dy\\
&\quad \le \frac{C_0}{10}\left\|\theta_y(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2 dy + c\varepsilon\|\theta(\cdot,\tau)\|^2.
\end{aligned}
\tag{3.40}
$$

Same bounds hold for $\int \theta^t L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)(-M\theta_y + M^2\theta)dy$. As before, using Lemma 2.4 and (3.16)-(3.20), we obtain

$$
\begin{aligned}
&\int \theta^t L(v^\varepsilon)(B(u^\varepsilon)-B(v^\varepsilon))u_y^\varepsilon dy\\
&\quad = \int \theta^t L(v^\varepsilon)(B(u^\varepsilon)-B(v^\varepsilon))w_{yy} dy + \int \theta^t L(v^\varepsilon)(B(u^\varepsilon)-B(v^\varepsilon))v_y^\varepsilon dy\\
&\quad \le \left(\frac{C_0}{10} + (\overline{c}+c\varepsilon^\gamma)\|\theta\|_{L^\infty}\right)\left\|\theta_y(\cdot,\tau)\right\|^2 + \|\theta\|_{L^\infty}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2\\
&\qquad + (\overline{c}\mu + c\varepsilon^\gamma)\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2 dy + c\varepsilon\|\theta(\cdot,\tau)\|^2.
\end{aligned}
\tag{3.41}
$$

In view of (2.55),

$$
\varepsilon\int \theta^t L(v^\varepsilon)q_4 dy \le \varepsilon\|\theta(\cdot,\tau)\|^2 + c\varepsilon\int \left|q_4\right|^2 dy \le \varepsilon\|\theta(\cdot,\tau)\|^2 + c\varepsilon^{7\gamma-2}.
\tag{3.42}
$$

Collecting all the estimates previously we have achieved, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|\theta(\cdot,\tau)\|^2 + \frac{3}{5}C_0\left\|\theta_y(\cdot,\tau)\right\|^2 + \frac{3}{8}\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2 dy\\
&\quad \le (\overline{c}+c\varepsilon^\gamma)\|\theta\|_{L^\infty}\left\|\theta_y(\cdot,\tau)\right\|^2 + \|\theta\|_{L^\infty}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2\\
&\qquad + (\overline{c}\mu + c\varepsilon^\gamma)\sum_{k\neq p}\int m\left|\partial_y\lambda_p(\phi)\right|\theta_k^2 dy + c\varepsilon\|\theta(\cdot,\tau)\|^2 + c\varepsilon^{7\gamma-2},
\end{aligned}
\tag{3.43}
$$

provided that $\|\theta(\cdot,\tau)\|_{L^\infty}$ is bounded.

Insert (3.11) into (3.43) to arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|\theta(\cdot,\tau)\|^2+\frac{3}{5}C_0\left\|\theta_y(\cdot,\tau)\right\|^2+\frac{3}{8}\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2dy \\
&\quad\le(\bar{c}+c\varepsilon^{\gamma})\|\theta\|_{L^\infty}\left\|\theta_y(\cdot,\tau)\right\|^2+\|\theta\|_{L^\infty}\left\|\partial_y^2w(\cdot,\tau)\right\|^2 \\
&\qquad+\left(\bar{c}\mu+c\varepsilon^{\gamma}\right)\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2dy \\
&\qquad+(\bar{c}+c\varepsilon^{\gamma})\left\{\frac{d}{d\tau}\left[-\sum_{k\neq p}\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k^2dy\right]\right. \\
&\qquad\qquad+\left(\bar{c}\mu+c\varepsilon^{\gamma}\right)\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2dy \\
&\qquad\qquad+\left(\bar{c}\mu+c\varepsilon\right)\left\|\theta_y(\cdot,\tau)\right\|^2+\bar{c}\mu\left\|\partial_y^2w(\cdot,\tau)\right\|^2 \\
&\qquad\qquad\left.+c\varepsilon\|\theta(\cdot,\tau)\|^2+c\varepsilon^{7\gamma-2}\right\} \\
&\qquad+c\varepsilon\|\theta(\cdot,\tau)\|^2+c\varepsilon^{7\gamma-2},
\end{aligned}
\tag{3.44}
$$

which gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|\theta(\cdot,\tau)\|^2+\frac{3}{5}C_0\left\|\theta_y(\cdot,\tau)\right\|^2+\frac{3}{8}\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2dy \\
&\quad\le(\bar{c}+c\varepsilon^{\gamma})\left(\|\theta\|_{L^\infty}+\bar{c}\mu+c\varepsilon^{\gamma}\right)\left\|\theta_y(\cdot,\tau)\right\|^2 \\
&\qquad+\left(\|\theta\|_{L^\infty}+\bar{c}\mu+c\varepsilon^{\gamma}\right)\left\|\partial_y^2w(\cdot,\tau)\right\|^2 \\
&\qquad+(\bar{c}+c\varepsilon^{\gamma})\left\{\frac{d}{d\tau}\left[-\sum_{k\neq p}\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k^2dy\right]\right. \\
&\qquad\qquad\left.+\left(\bar{c}\mu+c\varepsilon^{\gamma}\right)\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2dy+c\varepsilon\|\theta(\cdot,\tau)\|^2+c\varepsilon^{7\gamma-2}\right\}.
\end{aligned}
\tag{3.45}
$$

Differentiating (3.5) with respect to $y$, multiplying the resulting equation by $\partial_y w^t$ on the left and integrating over $R^1$, we obtain after integration by parts that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\left\|w_y(\cdot,\tau)\right\|^2=&-\int\partial_y^2w^tB(v^{\varepsilon})\partial_y^2w\,dy \\
&+\int\partial_y^2w^t\left(f'(v^{\varepsilon})w_y+Q(v^{\varepsilon},w_y)-\left(f(v^{\varepsilon}-d)-f(v^{\varepsilon})\right)-\varepsilon q_4\right)dy
\end{aligned}
$$

$$
\begin{aligned}
&-\int \partial_y^2 w^t (B(u^\varepsilon) - B(v^\varepsilon)) \left(\partial_y^2 w + v_y^\varepsilon\right) dy \\
&\leq -C_0 \left\|\partial_y^2 w(\cdot,\tau)\right\|^2 \\
&\quad + \frac{C_0}{4}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + (\overline{c} + c\varepsilon^\gamma)\int \left(\left|w_y\right|^2 + \left|w_y\right|^4 + |d|^2 + \varepsilon^2 \left|q_4\right|^2\right) dy \\
&\quad + \overline{c}\int \left|(B(u^\varepsilon) - B(v^\varepsilon))\partial_y^2 w\right|^2 dy + \overline{c}\int \left|(B(u^\varepsilon) - B(v^\varepsilon)) v_y^\varepsilon\right|^2 dy.
\end{aligned}
\tag{3.46}
$$

Using Sobolev's inequality, we have

$$
\int \left|w_y\right|^4 dy \leq \left\|w_y\right\|_{L^\infty}^2 \left\|w_y\right\|^2 \leq \overline{c}\left\|w_y\right\|^3 \left\|\partial_y^2 w\right\| \leq \frac{C_0}{8}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + \overline{c}\left\|w_y(\cdot,\tau)\right\|^6, \tag{3.47}
$$

and so

$$
\begin{aligned}
\int \left(\left|w_y\right|^2 + \left|w_y\right|^4\right) dy &\leq \frac{C_0}{8}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + \overline{c}\left(1 + \left\|w_y(\cdot,\tau)\right\|^4\right)\left\|w_y(\cdot,\tau)\right\|^2 \\
&\leq \frac{C_0}{8}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + \overline{c}\left(1 + \mu_1^4\right)\left\|w_y(\cdot,\tau)\right\|^2.
\end{aligned}
\tag{3.48}
$$

Lemma 2.3 gives

$$
\int \left(|d|^2 + \varepsilon^2 \left|q_4\right|^2\right) dy \leq c\varepsilon^{7\gamma - 1}. \tag{3.49}
$$

The last term can be estimated as

$$
\begin{aligned}
&\overline{c}\int \left|(B(u^\varepsilon) - B(v^\varepsilon))\partial_y^2 w\right|^2 dy + \overline{c}\int \left|(B(u^\varepsilon) - B(v^\varepsilon)) v_y^\varepsilon\right|^2 dy \\
&\quad \leq c\int \left|w_y\right|^2 \left|\partial_y^2 w\right|^2 dy + (\overline{c} + c\varepsilon^\gamma)\int \left|w_y\right|^2 \left|\partial_y v^\varepsilon\right|^2 dy \\
&\quad \leq c\|w(\cdot,\tau)\|_2^2 \left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\left\|w_y(\cdot,\tau)\right\|^2 \\
&\quad \leq c\|w(\cdot,\tau)\|_2^2 \left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + (\overline{c}\mu + c\varepsilon^\gamma)\left\|\theta_y(\cdot,\tau)\right\|^2 \\
&\qquad + \left(\overline{c}\mu^2 + c\varepsilon^\gamma\right)\int m\left|\partial_y \lambda_p(\phi)\right| |\theta|^2 dy + c\varepsilon\|\theta(\cdot,\tau)\|^2.
\end{aligned}
\tag{3.50}
$$

By taking $\|w\|_2$ to be sufficiently small, we get

$$\frac{1}{2}\frac{d}{d\tau}\|w_y(\cdot,\tau)\|^2+\frac{C_0}{2}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 \le \overline{c}\left(1+\mu_1^4\right)\|\theta_y(\cdot,\tau)\|^2+(\overline{c}\mu+c\varepsilon^\gamma)\|\theta_y(\cdot,\tau)\|^2$$
$$+(\overline{c}\mu(1+\mu)+c\varepsilon^\gamma)\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2dy+c\varepsilon\|\theta(\cdot,\tau)\|^2. \tag{3.51}$$

Denote the constant $\overline{c}(1+\mu_1^4)$ in this inequality by $\overline{c_1}$. Insert (3.11) into (3.51) to give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|w_y(\cdot,\tau)\|^2+\frac{C_0}{2}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2-\overline{c_1}\|\theta_y(\cdot,\tau)\|^2\\
&\quad\le\left(\overline{c}\mu\left(1+\mu^2\right)+c\varepsilon^\gamma\right)\|\theta_y(\cdot,\tau)\|^2+\left(\overline{c}\mu^2\left(1+\mu^2\right)+c\varepsilon^\gamma\right)\left\|\partial_y^2 w(\cdot,\tau)\right\|^2\\
&\qquad+(\overline{c}\mu(1+\mu)+c\varepsilon^\gamma)\frac{d}{d\tau}\left[-\sum_{k\neq p}\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k^2dy\right]\\
&\qquad+\left(\overline{c}\mu\left(1+\mu^2\right)+c\varepsilon^\gamma\right)\int m\left|\partial_y\lambda_p(\phi)\right|\theta_p^2dy+c\varepsilon\|\theta(\cdot,\tau)\|^2+c\varepsilon^{7\gamma-2}.
\end{aligned}\tag{3.52}$$

Multiplying suitably small constants to (3.11) and (3.52), respectively, then adding the resulting inequalities to (3.45) and taking $\|\theta\|_{L^\infty}$, $\mu$ and $\varepsilon$ sufficiently small, we can obtain the following inequality:

$$\begin{aligned}
&\frac{d}{d\tau}\left(\|\theta(\cdot,\tau)\|^2+\|w_y(\cdot,\tau)\|^2\right)+\|\theta_y(\cdot,\tau)\|^2+\left\|\partial_y^2 w(\cdot,\tau)\right\|^2+\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2dy\\
&\quad\le\left(\overline{c}\left(1+\mu^2\right)+c\varepsilon^\gamma\right)\frac{d}{d\tau}\left[-\sum_{k\neq p}\int\left(\frac{\lambda_p(\phi)-\dot{s}}{\lambda_k(\phi)-\dot{s}}\right)\theta_k^2dy\right]+c\varepsilon\|\theta(\cdot,\tau)\|^2+c\varepsilon^{7\gamma-2}.
\end{aligned}\tag{3.53}$$

Integrating the above inequality to give

$$\begin{aligned}
&\|\theta(\cdot,\tau)\|^2+\|w_y(\cdot,\tau)\|^2+\int_0^\tau\left(\|\theta_y(\cdot,\tau)\|^2+\left\|\partial_y^2 w(\cdot,\tau)\right\|^2\right)d\tau+\int_0^\tau\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2dy\,d\tau\\
&\quad\le\left(\overline{c}\left(1+\mu^2\right)+c\varepsilon^\gamma\right)\mu\sum_{k\neq p}\|\theta_k(\cdot,\tau)\|^2+c\varepsilon\int_0^\tau\|\theta(\cdot,\tau)\|^2+c\tau\varepsilon^{7\gamma-2}.
\end{aligned}\tag{3.54}$$

Take $\mu$ suitably small to yield

$$\begin{aligned}
&\|\theta(\cdot,\tau)\|^2+\|w_y(\cdot,\tau)\|^2+\int_0^\tau\left(\|\theta_y(\cdot,\tau)\|^2+\left\|\partial_y^2 w(\cdot,\tau)\right\|^2\right)d\tau+\int_0^\tau\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2dy\,d\tau\\
&\quad\le c\varepsilon\int_0^\tau\|\theta(\cdot,\tau)\|^2+c\tau\varepsilon^{7\gamma-2}.
\end{aligned}\tag{3.55}$$

It follows from the Gronwall's inequality that

$$\begin{aligned}
&\|\theta(\cdot,\tau)\|^2+\|w_y(\cdot,\tau)\|^2+\int_0^\tau\left(\|\theta_y(\cdot,\tau)\|^2+\left\|\partial_y^2 w(\cdot,\tau)\right\|^2\right)d\tau\\
&\quad+\int_0^\tau\int m\left|\partial_y\lambda_p(\phi)\right||\theta|^2dy d\tau\le c\varepsilon^{7\gamma-3}.
\end{aligned}\tag{3.56}$$

In particular,

$$\varepsilon\int_0^\tau\|\theta(\cdot,\tau)\|^2d\tau\le c\varepsilon^{7\gamma-3}.\tag{3.57}$$

By the fact $w=R\theta$ and $w_y=R\theta_y-RM\theta$, we obtain

$$\|w(\cdot,\tau)\|_1^2+\int_0^\tau\|w_y(\cdot,\tau)\|_1^2d\tau+\int_0^\tau\int m\left|\partial_y\lambda_p(\phi)\right||w|^2dy\,d\tau\le c\varepsilon^{7\gamma-3}.\tag{3.58}$$

□

**Lemma 3.4.** *Let the conditions in Proposition 3.1 be satisfied. Then*

$$\left\|\partial_y^2 w(\cdot,\tau)\right\|_1^2+\int_0^\tau\left\|\partial_y^3 w(\cdot,\tau)\right\|_1^2d\tau\le c\varepsilon^{7\gamma-3},\quad \textit{for } \tau\in[0,\tau_0],\tag{3.59}$$

*with the constant $c$ independent of $\tau_0$ and $\varepsilon$.*

*Proof.* Applying $\partial_y^l$ to (3.5), multiplying on the left of the resulting equation by $\partial_y^l w^t$, and integrating over $R^1$, we compute that

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\left\|\partial_y^l w(\cdot,\tau)\right\|^2&=-\int\partial_y^{l+1}w^t\partial_y^{l-1}\left(B(v^\varepsilon)\partial_y^2 w\right)dy\\
&\quad+\int\partial_y^{l+1}w^t\partial_y^{l-1}\{f'(v^\varepsilon)w_y+Q(v^\varepsilon,w_y)\\
&\qquad-\left(f(v^\varepsilon-d)-f(v^\varepsilon)\right)-\varepsilon q_4\}dy
\end{aligned}$$

$$
\begin{aligned}
&-\int \partial_y^{l+1} w^t \partial_y^{l-1}\left\{(B(u^\varepsilon)-B(v^\varepsilon))\partial_y^2 w\right\}dy \\
&-\int \partial_y^{l+1} w^t \partial_y^{l-1}\left\{(B(u^\varepsilon)-B(v^\varepsilon))v_y^\varepsilon\right\}dy.
\end{aligned}
\tag{3.60}
$$

In the case $l = 1$, this gives

$$
\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 &= -\int \partial_y^3 w^t \left(B(v^\varepsilon)\partial_y^2 w\right)_y dy \\
&\quad + \int \partial_y^3 w^t \Big\{ \left(f'(v^\varepsilon) w_y\right)_y + Q\left(v^\varepsilon, w_y\right)_y \\
&\qquad\qquad -\left(f(v^\varepsilon - d) - f(v^\varepsilon)\right)_y - \varepsilon q_{4y} \Big\} dy \\
&\quad - \int \partial_y^3 w^t \left\{(B(u^\varepsilon)-B(v^\varepsilon))\partial_y^2 w\right\}_y dy \\
&\quad - \int \partial_y^3 w^t \left\{(B(u^\varepsilon)-B(v^\varepsilon))v_y^\varepsilon\right\}_y dy \\
&\equiv \sum_{i=1}^4 R_i.
\end{aligned}
\tag{3.61}
$$

Using Cauchy inequality, we obtain

$$
\begin{aligned}
R_1 &= -\int \partial_y^3 w^t \left(B(v^\varepsilon)\partial_y^2 w\right)_y dy \\
&= -\int \partial_y^3 w^t B(v^\varepsilon)\partial_y^3 w \, dy - \int \partial_y^3 w^t (B(v^\varepsilon))_y \partial_y^2 w \, dy \\
&\le -C_0\left\|\partial_y^3 w(\cdot,\tau)\right\|^2 + \frac{C_0}{8}\left\|\partial_y^3 w(\cdot,\tau)\right\|^2 + c\left\|\partial_y^2 w(\cdot,\tau)\right\|^2.
\end{aligned}
\tag{3.62}
$$

Lemma 2.3 and Sobolev's inequality give

$$
\begin{aligned}
R_2 &\le \frac{C_0}{8}\left\|\partial_y^3 w(\cdot,\tau)\right\|^2 \\
&\quad + c\int\left(\left|w_y\right|^2 + \left|\partial_y^2 w\right|^2 + \left|w_y\right|^4 + \left|w_y\right|^2\left|\partial_y^2 w\right|^2 + |d|^2 + \left|d_y\right|^2 + \varepsilon^2\left|q_{4y}\right|^2\right)dy \\
&\le \frac{C_0}{8}\left\|\partial_y^3 w(\cdot,\tau)\right\|^2 + c\left(1 + \left\|w_y\right\|_{L^\infty}^2\right)\left(\left\|w_y\right\|^2 + \left\|\partial_y^2 w\right\|^2\right) + c\varepsilon^{7\gamma-1} \\
&\le \frac{C_0}{8}\left\|\partial_y^3 w(\cdot,\tau)\right\|^2 + c\left(1 + \|w\|_2^2\right)\left(\left\|w_y\right\|^2 + \left\|\partial_y^2 w\right\|^2\right) + c\varepsilon^{7\gamma-1} \\
&\le \frac{C_0}{8}\left\|\partial_y^3 w(\cdot,\tau)\right\|^2 + c\left(\left\|w_y\right\|^2 + \left\|\partial_y^2 w\right\|^2\right) + c\varepsilon^{7\gamma-1},
\end{aligned}
\tag{3.63}
$$

provided that $\|w\|_2$ is bounded. For the last two terms, we have

$$\begin{aligned}
R_3 &= -\int \partial_y^3 w^t (B(u^\varepsilon) - B(v^\varepsilon))\partial_y^3 w dy - \int \partial_y^3 w^t (B(u^\varepsilon) - B(v^\varepsilon))_y \partial_y^2 w dy \\
&\le c\int |w_y| \left|\partial_y^3 w\right|^2 dy + c\int \left|\partial_y^3 w\right| \left(|w_y| + \left|\partial_y^2 w\right|\right) \left|\partial_y^2 w\right| dy \\
&\le c\|w_y\|_{L^\infty} \left\|\partial_y^2 w(\cdot,\tau)\right\|_1^2 + c\left\|\partial_y^2 w\right\|_{L^\infty} \left\|\partial_y^2 w(\cdot,\tau)\right\| \left\|\partial_y^3 w(\cdot,\tau)\right\| \\
&\le c\|w\|_2 \left\|\partial_y^2 w(\cdot,\tau)\right\|_1^2 + \left\|\partial_y^2 w(\cdot,\tau)\right\|^{3/2} \left\|\partial_y^3 w(\cdot,\tau)\right\|^{3/2} \\
&\le c\|w\|_2 \left\|\partial_y^2 w(\cdot,\tau)\right\|_1^2 + \left\|\partial_y^2 w(\cdot,\tau)\right\| \left(\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + \left\|\partial_y^3 w(\cdot,\tau)\right\|^2\right) \\
&\le c\|w\|_2 \left\|\partial_y^3 w(\cdot,\tau)\right\|^2, \\
R_4 &\le \frac{C_0}{8} \left\|\partial_y^3 w(\cdot,\tau)\right\|^2 + c\|w_y(\cdot,\tau)\|_1^2.
\end{aligned} \tag{3.64}$$

Choose $\|w\|_2$ sufficiently small to yield

$$\frac{d}{d\tau}\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + \left\|\partial_y^3 w(\cdot,\tau)\right\|^2 \le c\left\|w_y(\cdot,\tau)\right\|_1^2 + c\varepsilon^{7\gamma-1}. \tag{3.65}$$

Integrate the above inequality, and use Lemma 3.3 to give

$$\left\|\partial_y^2 w(\cdot,\tau)\right\|^2 + \int_0^\tau \left\|\partial_y^3 w(\cdot,\tau)\right\|^2 \le c\varepsilon^{7\gamma-3}. \tag{3.66}$$

Similarly, for the case $l = 2$, we can obtain

$$\left\|\partial_y^3 w(\cdot,\tau)\right\|^2 + \int_0^\tau \left\|\partial_y^4 w(\cdot,\tau)\right\|^2 \le c\varepsilon^{7\gamma-3}. \tag{3.67}$$

This finishes the proof of Lemma 3.4. □

Combining the results of Lemmas 3.3-3.4, we complete the proof of Proposition 3.1.

## 4. Proof of Theorem 1.2

Using Proposition 3.1 and the standard continuous induction argument, we conclude the following.

**Proposition 4.1.** *There exist positive constants* $\eta_0$, $\mu_0$, *and* $C$, *which are independent of* $\varepsilon$ *such that if* $0 < \varepsilon < \varepsilon_0$, *and* $0 \le \mu \le \mu_0$, *then the Cauchy problem* (3.5) *has a unique solution* $w \in C^1([0, T/\varepsilon] : H^3(R^1))$. *Furthermore, the following inequality holds*

$$\sup_{0\le t\le T/\varepsilon} \|w(\cdot,t)\|_3^2 + \int_0^{T/\varepsilon} \left\|w_y(\cdot,t)\right\|_3^2 dt \le C\varepsilon^{7\gamma-3}. \tag{4.1}$$

Now we choose $\gamma \in (\eta, 1) \cap ((5/7), 1)$. Then due to (4.1), we have

$$\begin{aligned}\sup_{0\le t\le T} \|(u^\varepsilon - v^\varepsilon)(\cdot,t)\|^2 &= \sup_{0\le t\le T} \|\widetilde{w}_x(\cdot,t)\|^2 \\ &= \varepsilon \sup_{0\le \tau\le T/\varepsilon} \left\|w_y(\cdot,\tau)\right\|^2 \le C\varepsilon^{7\gamma-2} \le C\varepsilon^3.\end{aligned} \tag{4.2}$$

On the other hand, it follows from Lemma 2.4 that

$$\sup_{0\le t\le T} \|(v^\varepsilon - u_0)(\cdot,t)\|^2 \le C\varepsilon^\gamma \le C\varepsilon^\eta. \tag{4.3}$$

Consequently,

$$\sup_{0\le t\le T} \|(u^\varepsilon - u_0)(\cdot,t)\|^2 \le \sup_{0\le t\le T} \|(u^\varepsilon - v^\varepsilon)(\cdot,t)\|^2 + \sup_{0\le t\le T} \|(v^\varepsilon - u_0)(\cdot,t)\|^2 \le c\varepsilon^\eta, \tag{4.4}$$

which gives (1.8). Finally,

$$\|(u^\varepsilon - v^\varepsilon)(\cdot,t)\|_{L^\infty} = \left\|w_y(\cdot,t)\right\|_{L^\infty} \le c\left\|w_y(\cdot,t)\right\|^{1/2}\left\|\partial_y^2 w(\cdot,t)\right\|^{1/2} \le c\varepsilon^{(7\gamma-3)/2} \le c\varepsilon. \tag{4.5}$$

This yields (1.9) by using Lemma 2.4 again. We complete the proof of Theorem 1.2.

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