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Research Article

Numerical Study of the Elastic Pendulum on the Rotating Earth

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The elastic pendulum is a simple physical system represented by nonlinear differential equations. Analytical solutions for the bob trajectories on the rotating earth may be obtained in two limiting cases: for the ideally elastic pendulum with zero unstressed string length and for the Foucault pendulum with an inextensible string. The precession period of the oscillation plane, as seen by the local observer on the rotating earth, is 24 hours in the first case and has a well-known latitude dependence in the second case. In the present work, we have obtained numerical solutions of the nonlinear equations for different string elasticities in order to study the transition from one precession period to the other. It is found that the transition is abrupt and that it occurs for a quite small perturbation of the ideally elastic pendulum, that is, for the unstressed string length equal to about 10^{-4} of the equilibrium length due to the weight of the bob.

1. Introduction

The pendulum is such a fundamental physical system that all the details and various aspects of its motion are of interest. In the limit of inextensible suspension strings and small amplitudes, the behaviour of the system is described by simple, linear equations. However, a pendulum with an elastic suspension string is described by nonlinear differential equations, which couple horizontal and vertical oscillations [1]. This coupling has been mainly investigated in the vicinity of the so-called autoparametric resonance, where the string elasticity is such that the frequency of vertical, that is, spring-mode oscillations is double the frequency of horizontal, that is, pendulum-mode oscillations [1, 2].

It is perhaps surprising that no published work about the influence of earth's rotation on the behaviour of the nonlinear elastic pendulum could be found in the available literature. In the limit of an inextensible suspension string, one obtains the well-known and popular Foucault pendulum, for which the oscillation plane rotates, that is, precesses at a constant

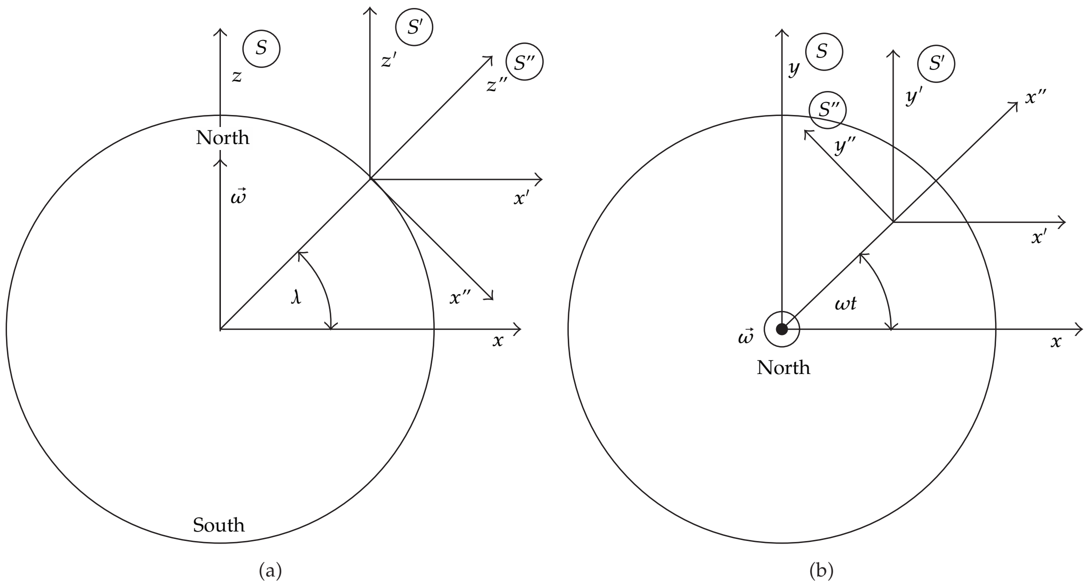

Figure 1: Views along the y, y' axes (a) and z, z' axes (b) of reference systems S and S', which have parallel axes, that are fixed relative to the distant stars. The origin of S is at earth's center, and the origins of S' and S'' are at the pendulum suspension point. The local lab system S'' has the z'' axis vertical, the x'' axis southward, and the y'' axis eastward.

rate [3]. This rate is given by $\omega \sin\lambda$, where ω is earth's angular velocity, and λ is the geographical latitude of the site. The precession period would therefore be $T = (24/\sin\lambda)$ h. In the other limit of extreme elasticity, where the force of the suspension string is proportional to the total string length (i.e., the unstressed length is equal to zero), the equation of motion becomes linear and leads to a precession period equal to earth's rotation period, that is, to 24 hours [4]. The precession period is therefore an interesting parameter of the elastic pendulum, which undergoes a transition from a latitude-independent value of 24 h in the case of an extremely soft string to the latitude-dependent value of $(24/\sin\lambda)$ h in the case of a very stiff suspension string. In an attempt to learn more about this transition, we have performed numerical studies for various elasticities of the suspension string and for different latitudes of the pendulum site. Some results and basic findings are presented below.

2. Method of Analysis

Instead of directly solving the equations of motion in the noninertial laboratory system by including the centrifugal and the Coriolis inertial forces, we have calculated the pendulum trajectories using a slightly different procedure. Such a procedure was followed also in deriving analytical solutions for the ideal elastic pendulum [4].

By neglecting the tidal gravitational forces of external bodies, such as the sun and moon, we have a quasi-inertial system S, with origin at the earth's centre and with the orientation of coordinate axes fixed with respect to the distant stars (Figure 1). Let the z axis of S coincide with earth's rotation axis. In the present work, the equations for the elastic pendulum are solved numerically in the noninertial reference system S', the origin of which is fixed at the point of pendulum suspension, and its coordinate axes remain parallel to the axes of the inertial system S (Figure 1). The system S' is thus circulating, but it is not rotating. The differential equations are solved in S', and the solutions are then transformed into the system of the local observer S'', where the z'' axis points vertically upward, the x'' axis points southward, and the y'' axis points eastward (Figure 1). Since the inertial forces in the circulating system S' are less complicated than those in the circulating and rotating system S'', one might expect that also the trajectories are simpler in S' than in S'' (see below). A famous historical example is given by the simple heliocentric planetary orbits as compared to their geocentric counterparts.

Neglecting dissipative forces, we have Newton's law for the motion of the pendulum bob

$$\mathbf{F}_g + \mathbf{F}_s = m\mathbf{a}, \tag{2.1}$$

where $\mathbf{F}_g$ is the force of earth's gravity, $\mathbf{F}_s$ is the force of the suspension string, m is the mass of the bob, and $\mathbf{a}$ is the bob acceleration in the inertial system S. Since the coordinate axes of systems S and S' remain parallel, the accelerations $\mathbf{a}$ and $\mathbf{a}'$, measured by observers in S and S', respectively, are simply related by $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}'$, where $\mathbf{a}_0$ is the acceleration of the origin of S' as measured by the observer in S. Therefore, $\mathbf{a}_0$ is the centripetal acceleration perpendicular to earth's rotation axis, that is, to the z axis of S.

We further assume that $\mathbf{F}_g = m\mathbf{g}_0$ is of constant magnitude and has a fixed direction in the meridional plane containing the origin of S'. As both $\mathbf{g}_0$ and $\mathbf{a}_0$ lie in the meridional plane, by appropriately choosing $t = 0$, one obtains the effective gravity in S' as

$$\mathbf{g}_{ef} = \mathbf{g}_0 - \mathbf{a}_0 = -g_{ef}(\cos\lambda\cos(\omega t), \cos\lambda\sin(\omega t), \sin\lambda). \tag{2.2}$$

Here, $g_{ef} = |\mathbf{g}_0 - \mathbf{a}_0|$, and, as already mentioned, ω is the angular velocity of the earth, and λ is the angle of $\mathbf{g}_{ef}$ with respect to the equatorial plane.

The force of the string is proportional to its dilatation and points towards the suspension point, that is, towards the origin of S',

$$\mathbf{F}_s = -k\left(r' - l_0\right)\frac{\mathbf{r}'}{r'}, \tag{2.3}$$

where $\mathbf{r}'(t)$ is the vector of the instantaneous bob position in S', and l_0 is the unstressed length of the string. The elastic constant is then $k = mg_{ef}/(l_e - l_0)$, where l_e is the equilibrium length of the string, subject to the weight mg_{ef} of the bob. The string force may therefore be written as

$$\mathbf{F}_s = -mg_{ef}\frac{r' - l_0}{l_e - l_0}\frac{\mathbf{r}'}{r'}. \tag{2.4}$$

By inserting all this into Newtons law (2.1) and rearranging the terms, we obtain a nonlinear differential equation for the position vector $\mathbf{r}'(t)$ of the bob in S',

$$\frac{d^2\mathbf{r}'}{dt^2} + \frac{g_{ef}}{l_e - l_0}\frac{r' - l_0}{r'}\mathbf{r}' = \mathbf{g}_0 - \mathbf{a}_0, \tag{2.5}$$

where $\mathbf{g}_0 - \mathbf{a}_0$ is given by (2.2) above. The solution of the differential equation in S', obtained for a given set of parameters, is then transformed into the system S'' of the local observer. The transformation consists of a rotation around the z' axis for ωt followed by a rotation around the y'' axis for $\pi/2 - \lambda$ [4]. One has to transform also the initial conditions, because they are usually given in S'', but the equation is solved in S'.

Let us observe that the above nonlinear differential equation (2.5) reduces to a linear differential equation for the case of an ideal elastic suspension string, that is, for $l_0 = 0$. The analytical solutions for such a case have been derived in [4].

From the differential equation (2.5), we see that the suspension string is determined by two parameters: the unstressed string length l_0 and the equilibrium length l_e. The present calculations have been performed with a fixed equilibrium length $l_e = 10\,\mathrm{m}$, but with the unstressed length l_0 varying between $l_0 = 0$ (ideally elastic string) and $l_0 = l_e = 10\,\mathrm{m}$ (inextensible string). For a given mass of the bob, this corresponds to variation of the suspension string elastic constant. The two parameters, l_0 and l_e, also determine the angular frequencies ω_p and ω_s of pendulum-mode and of spring-mode oscillations, respectively. For a fixed-string length l_e, and for small amplitudes, we have the pendulum-mode angular frequency

$$\omega_p = \sqrt{\frac{g_{ef}}{l_e}}, \tag{2.6}$$

while the spring-mode angular frequency is given by

$$\omega_s = \sqrt{\frac{k}{m}} = \sqrt{\frac{g_{ef}}{l_e - l_0}}. \tag{2.7}$$

It follows that the ratio ω_s/ω_p of frequencies is determined by the ratio l_0/l_e,

$$\frac{\omega_s}{\omega_p} = \sqrt{\frac{l_e}{l_e - l_0}}\ . \tag{2.8}$$

The spring-mode frequency is thus always greater than the pendulum mode frequency and therefore determines the computation time. On the PC with an Intel Pentium, dual core, 3 GHz, and 64-bit processor, the Mathematica programme package requires between a few minutes and a few hours to calculate the bob trajectory during the first 48 hours of oscillation.

The motion of the bob has been studied for the usual initial conditions with which a Foucault pendulum is started in the system S'' of the local observer

$$\begin{gathered} \mathbf{r}''(0) = l_e\left(\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, -\cos\theta_0\right), \\ \mathbf{v}''(0) = (0, 0, 0). \end{gathered} \tag{2.9}$$

Here, θ_0 is the initial angle between the pendulum string and the negative z'' axis, and ϕ_0 is the initial angle between the oscillation plane and the $y'' = 0$ plane. All the calculations have been performed with $\theta_0 = 5°$ and with $\phi_0 = 0°$ (displacement towards the south).

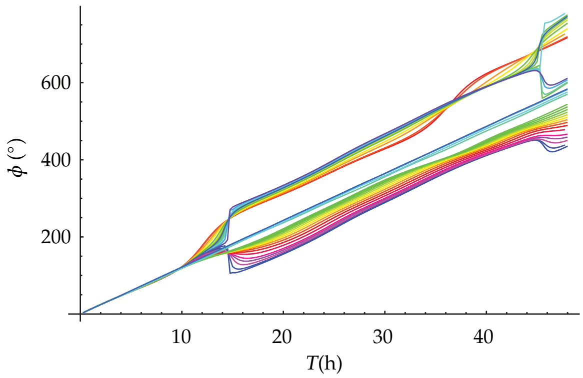

Figure 2: The dependence of the azimuth angle of the oscillation plane on time for latitude 55°. Different curves represent different string elasticities (see text).

The oscillation plane is defined with the z'' axis and with the vector of bob position $\mathbf{r}''$ at moments of maximum displacement. This amplitude vector is obtained with a special fitting procedure. The current maximum displacement vector therefore defines the oscillation plane, which is time dependent, that is, it precesses.

3. Results

For various values of the string elasticity, we show in Figure 2 the azimuth angle of the oscillation plane (defined in the previous section) as a function of time at 55° geographical latitude. The string elasticity varied by varying the unstressed string length (l_0), while keeping the equilibrium length (l_e), due to the bob weight, at a constant value of 10 meters.

In the limit of a very stiff string, that is, for $l_0 = l_e$, we obtain the Foucault pendulum with a constant precession rate, which is represented by the straight line in Figure 2. For $\lambda = 55°$, the corresponding precession velocity is 12.3° per hour, so the precession period is 29.3 hours. For softer strings ($l_0 < l_e$), the oscillation plane initially keeps up with the Foucault pendulum, but after about 15 hours, it falls behind, resulting in a longer precession time. At string elasticity corresponding to about $l_0 \approx 10^{-4} l_e$ (see below), a discontinuity occurs. Instead of slowing down after 15 hours, the precession speeds up, completing a full circle in less time. As the string elasticity parameter l_0 is reduced towards zero, the precession time tends towards 24 hours.

In the limit of an ideally elastic string ($l_0 = 0$), the equation of motion of the pendulum bob (2.5) becomes linear and may be solved analytically. As has been shown in a previous publication [4], the solution, as seen by an observer in S', is quite simple. The trajectory is an ellipse with fixed orientation relative to the distant stars, but with the center of the ellipse moving on a circle. However, the local observer S'', with whom we are mainly concerned, will see a more complicated bob trajectory, which is obtained by a time-dependent transformation of the ellipse into the local system S'' (with z'' axis vertical, y'' axis eastward, and x'' axis southward). Consider, for example, an elliptical orbit, initially in the horizontal plane at latitude 45°. For the local observer in S'', the trajectory would be such that after 12 hours the ellipse will appear to be in a vertical plane. Alternatively, the relatively complicated bob trajectory as seen by the local observer S'' may be attributed to the inertial forces (centrifugal and Coriolis) present in the noninertial system S''. However, it seems easier to understand

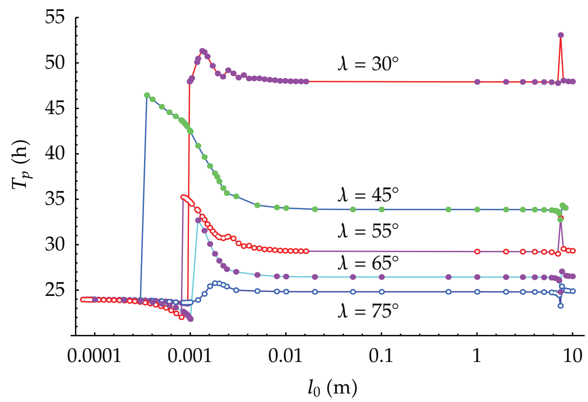

Figure 3: The dependence of pendulum precession time on the suspension string elasticity, that is, on the length l_0 of the unstressed string at fixed equilibrium length $l_e = 10$ m. The curves connect points corresponding to a given geographical latitude.

the 24-hour period by considering the periodic transformation of the elliptical orbit from the system S' to the system S''. The addition of a small perturbation to the ideally elastic pendulum, in the form of a small nonzero unstressed string length l_0, might be expected to contribute some modifications to the trajectory. Such trajectories therefore represent the faster branch in Figure 2.

In order to learn more about the transition from the 24-hour precession time to the precession time of the Foucault pendulum, we plot in Figure 3 the time required for the oscillation plane to return to its initial position and precession velocity as a function of string elasticity at different geographical latitudes. In Figure 2, for example, one obtains these precession times at the interception of the curves with a horizontal line going through 360°. (We note that for latitudes below 45° the oscillation plane does not necessarily complete a full rotation, but may oscillate between positive and negative azimuth angles). We see in Figure 3 that the transition between the two limiting values of precession time is abrupt and occurs at surprisingly low values of the elasticity parameter $l_0 \approx 10^{-4} l_e$. The exact value of the elasticity at which the transition occurs seems to depend on latitude (Figure 3). This, as well as a dependence of the transition on initial conditions and on earth's angular velocity, remains to be the subject of further studies.

Finally, we should point out that in addition to the discontinuity in precession time at $l_0/l_e \approx 10^{-4}$, our calculations show a structure at the elasticity given by $l_0/l_e = 3/4$ (see Figure 3). This structure corresponds to the well-known parametric resonance for which $\omega_s/\omega_p = 2$ (2.8).

4. Conclusions

We have performed calculations of the precession of the elastic pendulum and have found a most interesting behaviour of the precession time as a function of suspension string elasticity. In the limit of an ideally elastic string with zero unstressed string length ($l_0 = 0$), the precession is periodic with a period of 24 hours. Introduction of a small perturbation ($l_0 = 10^{-4} l_e$) causes a discontinuous transition from the 24-hour precession time to the latitude-dependent precession time of the Foucault pendulum ($T = (24/\sin\lambda)$ h). In the broad range of l_0 between $10^{-4} l_e$ and l_e, the precession time mainly has the value as for the inextensible Foucault pendulum. An exception is the structure observed at $l_0/l_e = 3/4$, which is due to the parametric resonance ($\omega_s/\omega_p = 2$). We may therefore conclude that the agreement of our numerical results with limiting cases where analytical solutions exist, as well as with the well-known parametric resonance, allows for confidence in the correctness of the procedure and the results.

Surely, the reader is wondering why should there be a discontinuity of the pendulum precession time and why should it appear at that particular value of string elasticity. The authors do not have answers to these questions, but they do believe that answers will be found by further investigation of the long-term behaviour of the elastic pendulum. Maybe, this simple system can teach us something about more complex systems.

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