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Research Article

A Study of Non-Euclidean *s*-Topology

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The present paper focuses on the characterization of compact sets of Minkowski space with a non-Euclidean s-topology which is defined in terms of Lorentz metric. As an application of this study, it is proved that the 2-dimensional Minkowski space with s-topology is not simply connected. Also, it is obtained that the n-dimensional Minkowski space with s-topology is separable, first countable, path-connected, nonregular, nonmetrizable, nonsecond countable, noncompact, and non-Lindelöf.

1. Introduction

Non-Euclidean topologies on 4-dimensional Minkowski space were first introduced by Zeeman [1] in 1967. These topologies include fine, space topology [2], time topology [3], t-topology [3], and s-topology [3]. Studying the homeomorphism group of 4-dimensional Minkowski space with fine topology, Zeeman in his paper [1] mentioned that it is Hausdorff, connected, locally connected space that is not normal, not locally compact and not first countable. His results were interesting both topologically and physically, because its homeomorphism group was the group generated by the Lorentz group, translations and dilatations which was exactly the one physicists would want it to be. Continuing the study of non-Euclidean topologies, Nanda in his papers [2, 3] mentioned that the 4-dimensional Minkowski space, with the space topology, is Hausdorff but neither normal nor locally compact nor second countable and that with each of the t-topology and s-topology is a nonnormal, noncompact Hausdorff space besides proving that the homeomorphism group of 4-dimensional Minkowski space with space, t and s-topologies, is generated by the Lorentz group, translations, and dilatations. Further, Nanda and Panda [4] introduced the notion of a non-Euclidean topology, namely, order topology, and obtained that it is a non-compact, non-Hausdorff, locally connected, connected, path connected, simply connected space. In 2007, Dossena [5] proved that the n-dimensional Minkowski space, $n > 1$, with the fine topology is separable, Hausdorff, nonnormal, nonlocally compact, non-Lindelöff and nonfirst countable. He further obtained that 2-dimensional Minkowski space with fine topology is path connected but not simply connected and characterized its compact sets. Quite recently, in 2009, Agrawal and Shrivastava [6] obtained a characterization for compact sets of Minkowski space with t-topology besides studying its topological properties. It may be noted that t-topology on 4-dimensional Minkowski space is same as that of the well-known path topology on strongly causal spacetime proposed by Hawking et al. in 1976 [7].

The present paper explores the s-topology on n-dimensional Minkowski space. Section-wise description of the work carried out in this paper is given below.

Beginning with an introduction, necessary notation and preliminaries have been provided in Section 2. In Section 3, it is proved that the s-topology on n-dimensional Minkowski space is strictly finer than the Euclidean topology by studying open sets, closed sets and subspace topologies on certain subsets of Minkowski space with s-topology. Topological properties of Minkowski space with s-topology are dealt in Sections 4, 5, and 6. In Section 7, compact subsets of Minkowski space with s-topology have been characterized. As a consequence of this study, it is proved that 2-dimensional Minkowski space with s-topology is not simply connected. Finally, Section 8 concludes the paper.

2. Notation and Preliminaries

Let Λ denote an indexing set while R, N, and K denote the set of real, natural and rational numbers, respectively. To avoid any confusion later, we mention here that the symbol Q, in this paper, denotes the indefinite characteristic quadratic form. For a subset A of a set X, $X - A$ denotes the complement of A in X. For $x, y \in R^n$, let $d_E(x, y)$ be the Euclidean distance between x and y. For $\epsilon > 0, N_\epsilon^E(x)$ denotes the ϵ-Euclidean neighborhood about x given by the set $\{y \in R^n : d_E(x, y) < \epsilon\}$. For $x, y \in R^n$, let $[x, y]$ denote the line segment joining x and y.

The *n-dimensional Minkowski Space*, denoted by M, is the n-dimensional real vector space R^n with a bilinear form $g : R^n \times R^n \rightarrow R$, satisfying the following properties:

(i) for all $x, y \in R_n$, $g(x, y) = g(y, x)$, that is, the bilinear form is symmetric

(ii) if for all $y \in R_n$, $g(x, y) = 0$, then $x = 0$, that is, the bilinear form is nondegenerate, and

(iii) there exists a basis $\{e_0, e_1, \dots, e_{n-1}\}$ for R_n with

$$g(e_i, e_j) = \begin{cases} 1 & \text{if } i = j = 0 \\ -1 & \text{if } i = j = 1, 2, \dots, n-1 \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.1}$$

The bilinear form g is called the *Lorentz inner product*.

Elements of M are referred to as *events*. If $x \equiv \sum_{i=0}^{n-1} x_i e_i$ is an event, then the coordinate x_0 is called the *time component* and the coordinates $x_1, \dots, x_{n-1}$ are called the *spatial components* of x relative to the basis $\{e_0, e_1, \dots, e_{n-1}\}$. In terms of components, the Lorentz inner product $g(x, y)$ of two events $x \equiv \sum_{i=0}^{n-1} x_i e_i$ and $y \equiv \sum_{i=0}^{n-1} y_i e_i$ is defined by $x_0 y_0 - \sum_{i=1}^{n-1} x_i y_i$. Lorentz inner product induces an indefinite characteristic quadratic form Q on M given by $Q(x) = g(x, x)$. Thus $Q(x) = x_0^2 - \sum_{i=1}^{n-1} x_i^2$. The group of all linear operators T on M which leave the quadratic form Q invariant, that is, $Q(x) = Q(T(x))$, for all $x \in M$, is called the *Lorentz group*.

A event $x \in M$ is called *spacelike*, *lightlike* (also called *null*) or timelike vector according as $Q(x)$ is negative, zero, or positive. The sets $C^S(x) = \{y \in M : y = x \text{ or } Q(y-x) < 0\}, C^L(x) = \{y \in M : Q(y-x) = 0\}$, $C^T(x) = \{y \in M : y = x \text{ or } Q(y-x) > 0\}$ are likewise, respectively called the *space cone*, *light cone* (or *null cone*), and *time cone* at x. For given $x, y \in M$, the set $\{x + t(y-x) | t \in R\}$ is called a *spacelike straight line* or *light ray* or *timelike straight line* joining x and y according as $Q(y-x)$ is negative or zero or positive. For further details, we refer to [8].

The Euclidean topology on the n-dimensional Minkowski space M is the topology generated by the basis $B = \{N_\epsilon^E(x) : \epsilon > 0, x \in M\}$. M with the Euclidean topology will be denoted by M^E.

The s-topology on the n-dimensional Minkowski space M is defined by specifying the local base of neighborhoods at each point of $x \in M$ given by the collection $\mathcal{N}(x) = \{N_\epsilon^s(x) : \epsilon > 0\}$, where $N_\epsilon^s(x) = N_\epsilon^E(x) \cap C^S(x)$. We call $N_\epsilon^s(x)$ the s-neighborhood of radius ϵ. M endowed with s-topology is denoted by M^s. For a subset A of M, A^s (A^E) denotes the subspace A of M^s (M^E).

3. Important Subsets and Subspaces of M^s

In this section, besides proving that the s-topology on M is strictly finer than the Euclidean topology on M, important subsets and subspaces of M^s, which will use in the following sections, are studied.

Lemma 3.1. *Let M be the n-dimensional Minkowski space and $x \in M$. Then $C^T(x) - \{x\}$, and $C^S(x) - \{x\}$ are open in M^E and $C^L(x)$ is closed in M^E.*

Proof. For $u \in M^E$, define $f : M^E \to R$ by $f(u) = (u_0 - x_0)^2 - \sum_{i=1}^{n-1}(u_i - x_i)^2$. Then f is continuous and $f^{-1}(0,\infty) = C^T(x) - \{x\}$, $f^{-1}(-\infty, 0) = C^S(x) - \{x\}$ and $f^{-1}\{0\} = C^L(x)$. Since $(0,\infty)$ and $(-\infty,0)$ are open and $\{0\}$ is closed in M^E, the results follow. □

In the following lemma, it is proved that the s-neighborhoods are open in M^s.

Lemma 3.2. *Let M be the n-dimensional Minkowski space and $x \in M$. Then $N_\epsilon^s(x), \epsilon > 0$ is open in M^s.*

Proof. It is sufficient to show that $N_\epsilon^s(x)$ is a neighborhood of each of its point. For this, let $y \in N_\epsilon^s(x)$ and $y \neq x$. Then $y \in C^S(x) - \{x\}$. By Lemma 3.1, $N_\epsilon^s(x) - \{x\} \equiv N_\epsilon^E(x) \cap (C^S(x) - \{x\})$, is open in M^E. Hence there exists a δ-Euclidean neighborhood $N_\delta^E(y)$ of y such that $N_\delta^E(y) \subseteq N_\epsilon^s(x) - \{x\}$. This implies that $N_\delta^s(y) \subseteq N_\delta^E(y) \subseteq N_\epsilon^s(x)$. Therefore, $N_\epsilon^s(x)$ is a neighborhood of y. Since $N_\epsilon^s(x)$ is a neighborhood of x, the result follows. □

In the following proposition a subset of M is obtained which is open in M^s but not in M^E.

Lemma 3.3. *Let M be the n-dimensional Minkowski space and $x \in M$. Then*

(i) *$C^S(x)$ is not open in M^E,*

(ii) *$C^S(x)$ is open in M^s.*

Proof. (i) We assert that x is not an interior point of $C^S(x)$ in M^E. To prove the assertion, consider the Euclidean neighbourhood $N_\epsilon^E(x)$ of radius ϵ containing x. Then it is easy to see that $N_\epsilon^E(x)$ is not contained in $C^S(x)$. Since $x \in C^S(x)$, the result follows.

(ii) Let $y \in C^S(x)$. Then either $y \in C^S(x) - \{x\}$ or $y = x$. If $y \in C^S(x) - \{x\}$, then, by Lemma 3.1, there exists a δ, such that $N_\delta^E(y) \subseteq C^S(x) - \{x\}$ and hence $N_\delta^s(y) \subseteq C^S(x)$. If $y = x$, then for any ϵ, $N_\epsilon^s(x) \subseteq C^S(x)$. Hence in either case, x is an interior point of $C^S(x)$ in M^s. This proves the result. □

It is known that on 4-dimensional Minkowski space, s-topology is finer than the Euclidean topology. In the following proposition, we prove this result for the n-dimensional Minkowski space. In fact, the s-topology is shown to be strictly finer than the Euclidean topology.

Proposition 3.4. *Let M be the n-dimensional Minkowski space. Then the s-topology on M is strictly finer than the Euclidean topology on M.*

Proof. Let G be open in M^E and $x \in G$. Then there exists a Euclidean neighbourhoood $N_\epsilon^E(x)$ of x, such that $N_\epsilon^E(x) \subseteq G$. Hence $N_\epsilon^s(x) \subseteq G$. This proves that G is open in M^s. Hence the s-topology on M is finer than the Euclidean topology on M. That it is strictly finer than the Euclidean topology follows from Lemma 3.3 (i) and (ii). □

Lemma 3.5. *(i) Let σ be a spacelike straight line joining p and $x \in M$. Then $u-v$ is a spacelike vector for $u, v \in \sigma$.*

(ii) Let τ be a timelike straight line joining p and $x \in M$. Then $u - v$ is a timelike vector for $u, v \in \tau$.

(iii) Let λ be a light ray joining p and $x \in M$. Then $u - v$ is a lightlike vector for $u, v \in \lambda$.

Proof. (i) For $u, v \in \sigma$, there exist $\alpha, \beta \in R$ such that $u = p + \alpha(x - p)$ and $v = p + \beta(x - p)$. Then $Q(u - v) = (\alpha - \beta)^2 Q(x - p)$. This implies that $u - v$ is a spacelike vector, as $Q(x - p) < 0$.

(ii) Similar to that of (i).

(iii) Similar to that of (i). □

Remark 3.6. Lemma 3.5 (i), (ii) and (iii) can be reinterpreted as follows.

(i) If σ is a spacelike straight line, then for $w \in \sigma$, σ is contained in $C^S(w)$.

(ii) If τ is a timelike straight line, then for $w \in \tau$, τ is contained in $C^T(w)$.

(iii) If λ is a light ray, then for $w \in \lambda$, λ is contained in $C^L(w)$.

Proposition 3.7. *Let M be the n-dimensional Minkowski space. Then spacelike straight lines, timelike straight lines, and light rays are closed in M^s.*

Proof. It follows from Proposition 3.4 and the facts that the spacelike straight lines, timelike straight lines, and light rays are all closed in M^E. □

It is mentioned in [3] that the s-topology on the 4-dimensional Minkowski space induces Euclidean topology on every spacelike hyperplane. In the following proposition, it is proved that the s-topology on n-dimensional Minkowski space induces Euclidean topology on every spacelike straight line.

Proposition 3.8. *Let M be the n-dimensional Minkowski space. Then the subspace topology on a spacelike straight line induced from the s-topology on M is same as the subspace topology induced from the Euclidean topology.*

Proof. Let σ be the spacelike straight line joining x and y. In view of the fact that the Euclidean topology on M is coarser than s-topology, it is sufficient to show that for $\epsilon > 0$, $N_\epsilon^s(x) \cap \sigma$ is open in σ^E, for all $x \in M$. This easily follows by noting that

$$N_\epsilon^s(x) \cap \sigma = \begin{cases} N_\epsilon^E(x) \cap \sigma & \text{if } x \in \sigma \\ \left(N_\epsilon^E(x) - \{x\}\right) \cap \sigma & \text{if } x \notin \sigma. \end{cases} \tag{3.1}$$

□

It has been stated in [3] that the s-topology on the 4-dimensional Minkowski space induces discrete topology on a light ray. The following proposition generalizes this result to the n-dimensional Minkowski space.

Proposition 3.9. *Let M be the n-dimensional Minkowski space. Then the s-topology on M induces discrete topology on a light ray.*

Proof. Let λ be a light ray and $p \in \lambda$. Then from Remark 3.6 (iii), it follows that $\lambda \subseteq C^L(p)$. Hence, for $\epsilon > 0$, $N_\epsilon^s(p) \cap \lambda = \{p\}$. This proves the result. □

It has been stated in [3] that the s-topology on the 4-dimensional Minkowski space induces discrete topology on a timelike straight line. Following proposition generalizes this result to the n-dimensional Minkowski space.

Proposition 3.10. *Let M be the n-dimensional Minkowski space. Then the s-topology on M induces discrete topology on a timelike straight line.*

Proof. Similar to that of Proposition 3.9. □

4. Separability and Countability Axioms

In this section, it is proved that M^s is a separable, first countable space that is not second countable.

Proposition 4.1. *Let M be the n-dimensional Minkowski space. Then M^s is separable.*

Proof. Since K^n is countable, it remains to show that K^n is dense in M^s. Hence, it is sufficient to show that for $x \in M$ and $\epsilon > 0$, $N_\epsilon^s(x) \cap K^n \neq \phi$. If $x \in K_n$, then $N_\epsilon^s(x) \cap K^n \neq \phi$. So let $x \notin K^n$. Then $N_\epsilon^s(x) \cap K^n = (N_\epsilon^s(x) - \{x\}) \cap K^n = N_\epsilon^E(x) \cap (C^S(x) - \{x\}) \cap K^n$. From Lemma 3.1, it follows that $N_\epsilon^E(x) \cap (C^S(x) - \{x\})$ is open in M^E. Since K^n is dense in M^E, $N_\epsilon^s(x) \cap K^n \neq \phi$. This completes the proof. □

The following lemma puts an upper bound on the cardinality of the set $C(M^s, R)$.

Lemma 4.2. *Let M be the n-dimensional Minkowski space. Then the cardinality of the set $C(M^s, R)$ of all continuous real-valued functions on M^s is at most equal to $2^{\aleph_0}$.*

Proof. From Proposition 4.1, M^s is separable. Let D be a countable subset of M^s. Then $|C(D,R)|$ is at most equal to $(|R|)^{|D|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Since two continuous maps are equal if they agree on a dense subset, hence $|C(M^s,R)|$ is at most equal to $2^{\aleph_0}$. This completes the proof. □

Proposition 4.3. *Let M be the n-dimensional Minkowski space. Then M^s is first countable.*

Proof. Given $x \in M$, the collection $\eta(x) = \{N_\epsilon^s(x) : \epsilon \in K\}$ is a countable local base at x for the s-topology on M. This shows that M^s is first countable. □

Proposition 4.4. *Let M be the n-dimensional Minkowski space. Then M^s is not second countable.*

Proof. Let M^s be second countable. Then since second countability is a hereditary property, it follows that a light ray is second countable. From Proposition 3.9, the induced topology on a light ray is discrete and hence it is not second countable, a contradiction. □

5. Separation Axioms

In this section, besides studying other properties, it is proved that M^s is a nonregular space.

It is known that M^s, for $n = 4$, is T_2 and hence T_1 [3]. Indeed M^s is T_2 for all n. In the following proposition, we prove that M^s is not regular.

Proposition 5.1. *Let M be the n-dimensional Minkowski space. Then M^s is not regular.*

Proof. Let λ be a light ray passing through 0. Then by Propositions 3.4 and 3.9, λ is a closed discrete subspace of M^s. Hence $\lambda - \{0\}$ is closed in M^s. We claim that $\lambda - \{0\}$ and 0 cannot be separated by disjoint open sets. For this, let G_1 and G_2 be open sets in M^s containing 0 and $\lambda - \{0\}$, respectively. Then for some $\epsilon > 0$, $0 \in N_\epsilon^s(0) \equiv N_\epsilon^E(0) \cap C^s(0) \subseteq G_1$. Notice that $N_\epsilon^E(0) \cap \lambda \neq \{0\}$, for otherwise $\{0\}$ would be open in L^E, a contradiction. Choose $x \in N_\epsilon^E(0) \cap \lambda$, $x \neq 0$. Then $x \in \lambda - \{0\}$ and hence there exists a $\delta > 0$ such that $x \in N_\delta^s(x) \subseteq G_2$. Then it can be verified that $N_\epsilon^s(0) \cap N_\delta^s(x) \neq \phi$. Hence, $G_2 \cap G_1 \neq \phi$. This completes the proof. □

Proposition 5.2. *Let M be the n-dimensional Minkowski space. Then M^s is not normal.*

Proof. Let M^s be normal. Then since M^s is T_1, M^s is T_4. The fact that a T_4 space is regular implies that M^s is regular, a contradiction to Proposition 5.1. □

The following remark gives an alternate proof to the fact that M^s is not normal.

Remark 5.3. Let M^s be normal, λ a light ray, and $A \subseteq \lambda$. Then by Propositions 3.7 and 3.9, λ is a closed discrete subspace of M^s. Hence A and $\lambda - A$ are closed in M^s. Since M^s is normal, by Urysohn's lemma, there exists a continuous map $f : M^s \to R$ such that $f(A) = \{0\}$ and $f(\lambda - A) = \{1\}$. This implies there would be at least as many real-valued continuous functions on M^s as there are subsets of λ. Hence $|C(M^s,R)|$ would be at least $(2^{2^{\aleph_0}})$, a contradiction to Lemma 4.2.

Corollary 5.4. *Let M be the n-dimensional Minkowski space. Then M^s is not metrizable.*

Proof. Since a metrizable space is regular, the result follows from Proposition 5.1. □

6. Connectedness and Compactness

In this section, it is proved that M^s is a path-connected, noncompact, non-Lindelöf, nonlocally compact, nonparacompact, non-locally m-Euclidean space.

Proposition 6.1. *Let M be the n-dimensional Minkowski space. Then M^s is path-connected*

Proof. Let $x, y \in M$. Then either $Q(x-y) < 0$ or $Q(x-y) \geq 0$. If $Q(x-y) < 0$, define $\gamma : [0,1] \to M^s$ by $\gamma(t) = x + t(y-x)$. Then $\gamma(0) = x$ and $\gamma(1) = y$. By Proposition 3.8, $\gamma : [0,1] \to [x,y]$ is continuous. This implies that $\gamma : [0,1] \to M^s$ is continuous. Hence γ is the required path in M^s joining x and y. If $Q(x-y) \geq 0$, then choose $z \in C^s(x) \cap C^s(y)$. Define $\gamma : [0,1] \to M^s$ to be the join of $\gamma_1 : [0,1] \to M^s$ and $\gamma_2 : [0,1] \to M^s$, where

$$\begin{aligned} \gamma_1(t) &= x + t(z-x); \quad t \in [0,1] \\ \gamma_2(t) &= x + t(y-z); \quad t \in [0,1]. \end{aligned} \tag{6.1}$$

Then by Proposition 3.8, $\gamma_1 : [0,1] \to [x,z]$ and $\gamma_2 : [0,1] \to [z,y]$ are continuous. Hence $\gamma_1 : [0,1] \to M^s$ and $\gamma_2 : [0,1] \to M^s$ are continuous. Hence γ_1 and γ_2 are paths in M^s joining x, z and z, y, respectively. Since the join of two paths is a path, γ is the required path in M^s joining x and y. This completes the proof. □

Corollary 6.2. *M^s is connected.*

Proof. Since a path-connected space is connected, the result follows from Proposition 6.1. □

It has been stated in [3] that the 4-dimensional Minkowski space with s-topology is not compact. The following proposition proves this result for n-dimensional Minkowski space.

Proposition 6.3. *Let M be the n-dimensional Minkowski space. Then M^s is not compact.*

Proof. It follows from Proposition 3.4 and the fact that M^E is not compact. □

Proposition 6.4. *Let M be the n-dimensional Minkowski space. Then M^s is not Lindelöf.*

Proof. Let M^s be Lindelöf and λ a light ray. Then by Proposition 3.9, λ is a discrete subspace of M^s and hence it is not Lindelöf. The fact that Lindelöfness is closed hereditary, together with Proposition 3.7, implies that λ is Lindelöf, a contradiction. □

Proposition 6.5. *Let M be the n-dimensional Minkowski space. Then M^s is not paracompact.*

Proof. Since a paracompact Hausdorff space is normal [9], hence M^s is not paracompact from Proposition 5.2. □

Proposition 6.6. *Let M be the n-dimensional Minkowski space. Then M^s is not locally compact.*

Proof. Since a Hausdorff locally compact space is regular [9], the result follows from Proposition 5.1. □

Proposition 6.7. *Let M be the n-dimensional Minkowski space. Then M^s is not locally m-Euclidean.*

Proof. It follows from Proposition 6.6 and the fact that a locally m-Euclidean space is locally compact [9]. □

7. Compact Sets and Simple Connectedness

The concept of Zeno sequences was originally defined by Zeeman [1] for 4-dimensional Minkowski space with fine topology. In this section, we develop the notion of Zeno sequence in n-dimensional Minkowski Space with s-topology to characterize the compact subsets of M^s. As a consequence of this study the two dimensional Minkowski space with s-topology is proved to be not simply connected. The study of Zeno sequences is also used to obtain a sufficient condition for continuity of maps from a topological space into M^s.

Definition 7.1. Let $z \in M$ and let $(z_n)_{n\in N}$ be a sequence of distinct terms in M such that $z_n \neq z$, for every $n \in N$. Then $(z_n)_{n\in N}$ is called a Zeno sequence in M^s converging to $z \in M$, if $(z_n)_{n\in N}$ converges to z in M^E but not in M^s. The image of a Zeno sequence $(z_n)_{n\in N}$ will mean the set $Z = \{z_n | n \in N\}$. The completed image of a Zeno sequence $(z_n)_{n\in N}$ will mean the set $Z \cup \{z\}$.

Example 7.2. Let $z \in M$. Consider the collection $\{\lambda_n : \lambda_n$ is a light ray passing through $z, n \in N\}$. For $n \in N$, choose $z_n \in \lambda_n$ such that $0 < d_E(z_n, z) < 1/n$ and $z_n \neq z_i$, for $i = 1, 2, \ldots, n-1, n > 1$. Then $(z_n)_{n\in N}$ converges to z in M^E but not in M^s, since any s-neighborhood about z contains no z_n. Hence $(z_n)_{n\in N}$ is a Zeno sequence in M^s.

Example 7.3. Let $z \in M$. Consider the collection $\{\tau_n : \tau_n$ is a timelike straight line passing through $z, n \in N\}$. For $n \in N$, choose $z_n \in \tau_n$ such that $0 < d_E(z_n, z) < 1/n$ and $z_n \neq z_i$, for $i = 1, 2, \ldots, n-1, n > 1$. Then $(z_n)_{n\in N}$ converges to z in M^E but not in M^s, since any s-neighborhood about z contains no z_n. Hence $(z_n)_{n\in N}$ is a Zeno sequence in M^s.

Example 7.4. Let $z \in M$. Consider the collection $\{\sigma_n : \sigma_n$ is a spacelike straight line passing through $z, n \in N\}$. For $n \in N$, choose $z_n \in \sigma_n$ such that $0 < d_E(z_n, z) < 1/n$ and $z_n \neq z_i$, for $i = 1, 2, \ldots, n-1, n > 1$. Then $(z_n)_{n\in N}$ converges to z in M^E and in M^s. Hence $(z_n)_{n\in N}$ is a not a Zeno sequence in M^s.

Proposition 7.5. *Let $(z_n)_{n\in N}$ be a Zeno sequence in M^s converging to z. Then $(z_n)_{n\in N}$ admits a subsequence whose image is closed in M^s but not in M^E.*

Proof. Since $(z_n)_{n\in N}$ does not converge to z in M^s, there exists an open set U in M^s containing z that leaves outside infinitely many terms of the sequence $(z_n)_{n\in N}$. Let $(z_{n_k})_{k\in N}$ be the subsequence of $(z_n)_{n\in N}$ formed by these infinitely many terms and let A be its image. Clearly, $(z_{n_k})_{k\in N}$ converges to z in M^E. Since $z \notin A$, A is not closed in M^E. To see that A is closed in M^s, notice first that any point of M other than z is not a limit point of A in M^E and hence in M^s. Further, since $U \cap A$ is empty, z is not a limit point of A in M^s. Thus A has no limit point in M^s. This completes the proof. □

In the following proposition, it is proved that a compact subset of M^s cannot contain a Zeno sequence.

Proposition 7.6. *Let C be a subset of M and C^s be a compact. Then C does not contain image of a Zeno sequence.*

Proof. To the contrary, let $(z_n)_{n\in N}$ be a Zeno sequence converging to z. Then from Proposition 7.5, $(z_n)_{n\in N}$ admits a subsequence whose image, say A, is closed in M^s but not in M^E. Then A is compact in M^s. This implies that A is compact M^E and hence closed in M^E, a contradiction to Proposition 7.5. □

Lemma 7.7. *Let C be a subset of M, such that C does not contain the completed image of any Zeno sequence. Then for $p \in C$ and every open set G_p^s in M^s containing p, there exists an open set G_p^E containing p of M^E such that $C \cap G_p^E \subseteq C \cap G_p^s$.*

Proof. Suppose for some $p \in C$ and an open set G_p^s in M^s containing p, there is no open set G_p^E in M^E such that $C\cap G_p^E \subset C\cap G_p^s$. For each $n \in N$, choose $x_n \in C\cap N_{1/n}^E(p)$ such that $x_n \notin C\cap G_p^s$ and $x_n \neq x_i$, for $i = 1, 2, \ldots, n-1, n > 1$. Then $(x_n)_{n\in N}$ is a Zeno sequence in M^s converging to p, which is a contradiction since completed image of $(x_n)_{n\in N}$ is contained in C. □

The following proposition determines a class of subsets C of M for which $C^s = C^E$.

Proposition 7.8. *Let C be a subset of M, such that C does not contain completed image of any Zeno sequence. Then $C^s = C^E$.*

Proof. From Lemma 7.7, it follows that the subspace Euclidean topology on C is finer than the subspace s-topology on C. Proposition 3.4 now completes the proof. □

The following proposition characterizes the compact subset of M^s.

Proposition 7.9. *Let C be subset of M such that C does not contain the completed image of any Zeno sequence. Then C^E is compact if and only if C^s is compact.*

Proof. It follows from Proposition 7.8. □

The following proposition characterizes the continuous maps from a topological space into M^s.

Proposition 7.10. *Let X be a topological space and $f : X \to M^E$ a map such that $f(X)$ does not contain completed image of any Zeno sequence. Then $f : X \to M^E$ is continuous iff $f : X \to M^s$ is continuous.*

Proof. Let $f : X \to M^s$ be continuous. Then by Proposition 3.4, $f : X \to M^E$ is continuous. Conversely, let $f : X \to M^E$ is continuous. Then $f : X \to f(X)^E$ is continuous and hence by Proposition 7.8 $f : X \to f(X)^s$ is continuous. This proves that $f : X \to M^s$ is continuous. □

Lemma 7.11. *Let G be an open set in M^E and $z \in G$. Then there exists a Zeno sequence in M^s converging to z with its terms in G.*

Proof. Clearly $N_\epsilon^E(z) \subseteq G$, for some $\epsilon > 0$. For $n \in N$, choose $z_n \in N_\epsilon^E(z) \cap C^L(z)$ such that $d_E(z_n, z) < \epsilon/n$ and $z_n \neq z_i$, for $i = 1, 2, \ldots, n-1, n > 1$. Then $(z_n)_{n\in N}$ converges to z in M^E but not in M^s. This proves that $(z_n)_{n\in N}$ is a Zeno sequence, as required. □

Proposition 7.12. *Let M be the 2-dimensional Minkowski space. Then M^s is not simply connected.*

Proof. Since M^s is path connected, it is sufficient to prove that the fundamental group of M^s at some fixed base point is nontrivial. For this, fix the base point at $(0,0)$ denoted by O. Choose distinct ordered pairs of spacelike vectors (u^i, v^i) for $i = 1,2$ such that $u^i - v^i$ is a spacelike vector. For $i = 1,2$, let $\gamma_i : [0,1] \to M^s$ be defined by

$$\gamma_i^s(t) = \begin{cases} O + 3tu^i; & s \in \left[0, \dfrac{1}{3}\right] \\ u^i + (3t-1)\left(v^i - u^i\right); & s \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right] \\ v^i + (3t-2)\left(O - v^i\right); & s \in \left[\dfrac{1}{3}, 1\right] \end{cases} \tag{7.1}$$

Then in view of Proposition 3.8 and the fact that the join of paths is a path, it follows that $\gamma_i : [0,1] \to M^s$ is a path, for $i = 1,2$. Since $\gamma_i(0) = \gamma_i(1)$, for $i = 1,2$, hence γ_i's are loops based at O. We claim that γ_1 is not path homotopic to γ_2. Suppose, on the contrary, that they are path-homotopic. Let $H : [0,1] \times [0,1] \to M^s$ be a path homotopy between γ_1 and γ_2 and T_1, T_2 be the compact triangles in M^E with boundaries $\gamma_1([0,1])$ and $\gamma_2([0,1])$, respectively. Then since $(u^1, v^1) \neq (u^2, v^2)$ at least one of $\text{int}(T_1) - T_2$ or $\text{int}(T_2) - T_1$ is nonempty, where $\text{int}(A)$ denotes the interior of set A in M^E. Let $\text{int}(T_1) - T_2 \neq \phi$. If $p \in \text{int}(T_1) - T_2$, then $p \in H([0,1] \times [0,1])$, for otherwise H would be a path homotopy between T_1 and T_2 in the punctured plane $M^E - \{p\}$ which is not possible as T_1 winds around p while T_2 does not. Hence $\text{int}(T_1) - T_2 \subseteq H([0,1] \times [0,1])$. Since $\text{int}(T_1) - T_2$ is open in M^E, from Lemma 7.11 $\text{int}(T_1) - T_2$ contains image of a Zeno sequence in M^s converging to p. This is a contradiction to Proposition 7.6, since $H([0,1] \times [0,1])$ is compact in M^s. This completes the proof. □

8. Conclusion

The present paper is focused on a detailed topological study of the physically relevant s-topology on 4-dimensional Minkowski space. Often the mathematical structure of a physical theory, especially the topology on the underlying space, is never completely determined by the physics of the processes it seeks to describe. This nonuniqueness of the topology on underlying space motivates to identify and study those topologies that are significant from the perspective of the physical theory. One of the most important physical theories is the Einstein's special theory of relativity, formulated on 4-dimensional Minkowski space, the underlying space for s-topology.

References

[1] E. C. Zeeman, "The topology of Minkowski space," *Topology*, vol. 6, no. 2, pp. 161–170, 1967.
[2] S. Nanda, "Topology for Minkowski space," *Journal of Mathematical Physics*, vol. 12, no. 3, pp. 394–401, 1971.
[3] S. Nanda, "Weaker versions of Zeeman's conjectures on topologies for Minkowski space," *Journal of Mathematical Physics*, vol. 13, no. 1, pp. 12–15, 1972.
[4] S. Nanda and H. K. Panda, "Minkowski space with order topology is simply connected," *International Journal of Theoretical Physics*, vol. 12, no. 6, pp. 393–399, 1975.
[5] G. Dossena, "Some results on the Zeeman topology," *Journal of Mathematical Physics*, vol. 48, no. 11, Article ID 113507, p. 13, 2007.

[6] G. Agrawal and S. Shrivastava, "t-topology on the n-dimensional Minkowski space," *Journal of Mathematical Physics*, vol. 50, no. 5, Article ID 053515, 6 pages, 2009.
[7] S. W. Hawking, A. R. King, and P. J. McCarthy, "A new topology for curved space-time which incorporates the causal, differential, and conformal structures," *Journal of Mathematical Physics*, vol. 17, no. 2, pp. 174–181, 1976.
[8] G. L. Naber, *The Geometry of Minkowski Spacetime*, vol. 92 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1992, An introduction to the mathematics of the special theory of relativity.
[9] J. R. Munkres, *Topology—A First Course*, Pearson Education Inc., Singapore, 2000.