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Research Article

Numerical Solution of Singular Lane-Emden Equation

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A new approach for solving the nonlinear Lane-Emden type equations has been proposed. The method is based on Legendre wavelets approximations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique, and the results have been compared with the exact solution.

1. Introduction

The Lane-Emden type equations are nonlinear ordinary differential equations on semi-infinite domain. They are categorized as singular initial value problems. These equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytrophic theory of stars essentially follows out of thermodynamic considerations that deal with the issue of energy transport, through the transfer of material between different levels of the star. These equations are one of the basic equations in the theory of stellar structure and have been the focus of many studies. The general form of the Lane-Emden equations is the following form:

$$y''(x) + \frac{m}{x} y'(x) + f(x, y) = g(x), \quad 0 < x \le 1,\ m \ge 0, \tag{1}$$

with the following initial conditions:

$$y(0) = A, \qquad y'(0) = B, \tag{2}$$

where $f(x, y)$ is a continuous real-value function and $g(x)$ is an analytical function. Equation (1) was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents [1, 2]. The solution of the Lane-Emden equation, as well as those of a variety of nonlinear problems in quantum mechanics and astrophysics such as the scattering length calculations in the variable phase approach, is numerically challenging because of the singular point at the origin. Bender et al. [3] proposed a new perturbation technique based on an artificial parameter δ; the method is often called δ-method. El-Gebeily and O'Regan [4] used the quasilinearization approach to solve the standard Lane-Emden equation. This method approximates the solution of a nonlinear differential equation by treating the nonlinear terms as a perturbation about the linear ones, and unlike perturbation theories, it is not based on the existence of some small parameters. Approximate solutions to the above problems were presented by Shawagfeh [5] and Wazwaz [6, 7] by applying the Adomian method which provides a convergent series solution. Nouh [8] accelerated the convergence of a power series solution of the Lane-Emden equation by using an Euler-Abel transformation and Padé approximation. Mandelzweig and Tabakin [9] applied Bellman and Kalaba's quasilinearization method, and Ramos [10] used a piecewise linearization technique based on the piecewise linearization of the Lane-Emden equation. Bozkhov and Gilli Martins [11] and later Momoniat and Harley [12] applied the Lie Group method successfully to generalized Lane-Emden equations of the first kind. Exact solutions of generalized Lane-Emden solutions of the first kind are investigated by Goenner and Havas [13]. Liao [14] solved Lane-Emden type equations by applying a homotopy analysis method. He [15] obtained an approximate analytical solution of the Lane-Emden equation by applying a variational approach which uses a semi-inverse method. Ramos [16] presented a series approach to the Lane-Emden equation and gave the comparison with homotopy perturbation method. Özis and Yildirim [17, 18] gave the solutions of a class of singular second-order IVPs of Lane-Emden type by using homotopy perturbation and variational iteration method. Parand et al. [19–22] presented three numerical techniques to solve higher ordinary differential equations such as Lane-Emden. Their approach was based on the rational Chebyshev, rational Legendre Tau, and Hermite functions collocation methods. In this paper, the new approximate analytical method will be introduced for exact solution of Lane-Emden equation.

This paper is arranged as follows.

In Section 2, the properties of Legendre wavelets and the way to construct the wavelet technique for this type of equation are described. In Section 3, the proposed method is applied to some types of Lane-Emden equations, and a comparison is made with the existing analytic or exact solutions that were reported in other published works in the literature. Finally, we give a brief conclusion in the last section.

2. Legendre Wavelets Applied to Singular IVPs of Lane-Emden Type Equation

2.1. Wavelets and Legendre Wavelets. Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \qquad a, b \in \mathbb{R},\ a \neq 0. \tag{3}$$

If we restrict the parameter a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, and n, k positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(x) = |a_0|^{k/2}\psi\left(a_0^k x - nb_0\right), \tag{4}$$

where $\psi_{k,n}(x)$ forms a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(x)$ forms an orthonormal basis.

Legendre wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments, $n = 1,2,3,\ldots,2^{k-1}$: k can assume any positive integer, m is the order for Legendre polynomials, and they are defined on the interval $[0,1)$ as follows:

$$\psi_{nm}(x) = \begin{cases} \sqrt{m+\dfrac{1}{2}}\,2^{k/2}P_m\left(2^k x - 2n + 1\right), & \dfrac{n-1}{2^{k-1}} \le x < \dfrac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $m = 0,1,\ldots,M-1$, $n = 1,2,3,\ldots,2^{k-1}$. The coefficient $\sqrt{m+(1/2)}$ is for orthonormality. Here, $P_m(x)$ are the well-known Legendre polynomials of order m which are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulae:

$$\begin{gathered} P_0(x) = 1, \qquad P_1(x) = x, \\ P_{m+1}(x) = \left(\frac{2m+1}{m+1}\right)xP_m(x) - \left(\frac{m}{m+1}\right)P_{m-1}(x), \quad m = 1, 2, \ldots \end{gathered} \tag{6}$$

2.2. Function Approximation. A function $f(x)$ defined on the interval $[0,1)$ may be expanded by Legendre wavelet as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{7}$$

where

$$c_{nm} = \left(f(x), \psi_{nm}(x)\right). \tag{8}$$

In (8), $(\cdot,\cdot)$ denotes the inner product.

If the infinite series in (7) is truncated, then (7) can be written as

$$f(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^T\psi(x), \tag{9}$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \left[c_{10}, c_{11}, \ldots c_{1M-1}, c_{20}, c_{21}, \ldots c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}\right]^T, \tag{10}$$

$$\psi(x) = \left[\psi_{10}(x), \psi_{11}(x), \ldots \psi_{1M-1}(x), \psi_{20}(x), \psi_{21}(x), \ldots \psi_{2M-1}(x), \ldots, \psi_{2^{k-1}0}(x), \ldots, \psi_{2^{k-1}M-1}(x)\right]^T. \tag{11}$$

The integration of the product of two Legendre wavelets vector functions is obtained as

$$\int_0^1 \psi(x)\psi^T(x)\,dx = I, \tag{12}$$

where I is an identity matrix.

2.3. Legendre Wavelets Operational Matrix of Integration. The integration of the vector $\psi(x)$ defined in (11) can be obtained as

$$\int_0^x \psi(t)\,dt = P\psi(x), \tag{13}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of integration given by [23] as

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \ddots & \vdots \\ O & O & L & \ddots & F \\ \vdots & \ddots & \ddots & \ddots & F \\ O & \cdots & O & O & L \end{bmatrix}, \tag{14}$$

where L, F, and O are $M \times M$ matrices given by

$$L = \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & \cdots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} & \ddots & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & \cdots & \cdots & 0 & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{15}$$

The following property of the product of two Legendre wavelet vector functions will also be used:

$$\psi(t)^T \psi(t) C \approx \widehat{C}\psi(t), \tag{16}$$

where C is a vector given in (10) and $\widehat{C}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix, which is called the product operation of Legendre wavelet vector functions [23]. For $M = 3$ and $k = 1$, the matrix $\widehat{C}$ is obtained:

$$\begin{bmatrix} c_{10} & c_{11} & c_{12} \\ c_{11} & c_{10} + \frac{2c_{12}}{\sqrt{5}} & \frac{2c_{11}}{\sqrt{5}} \\ c_{12} & \frac{2c_{11}}{\sqrt{5}} & c_{10} + \frac{2\sqrt{5}c_{12}}{7} \end{bmatrix}. \tag{17}$$

2.4. Solution of Lane-Emden Equations. We multiply both sides of (1) by x,

$$xy''(x) + my'(x) + xf(x, y) = xg(x), \quad 0 < x \le 1, \ m \ge 0, \tag{18}$$

in order to use Legendre wavelets to approximate $y''(x)$ as

$$y''(x) \approx C^T \psi(x). \tag{19}$$

Integrating (19) with respect to x twice from 0 to x, we obtain

$$\begin{aligned} y'(x) &\approx C^T P\psi(x) + y'(0) \\ &= C^T P\psi(x) + U_1^T \psi(x), \end{aligned} \tag{20}$$

$$\begin{aligned} y(x) &\approx C^T P^2\psi(x) + xy'(0) + y(0) \\ &= C^T P^2\psi(x) + U_0^T \psi(x), \end{aligned} \tag{21}$$

where coefficients U_0 and U_1 are known and can be obtained from the initial conditions, C and $\psi(x)$ are defined similarly to (10) and (11), and P is $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration, defined in (14).

Also consider the following approximations:

$$\begin{aligned} xy''(x) &\approx Y_1^T \psi(x), \\ xf(x, y) &\approx x\sum_{j=0}^{n} \frac{1}{j!}\left((x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right)^j f(x_0, y_0) \\ &= Y_2^T \psi(x), \\ xg(x) &\approx x\sum_{j=0}^{n} \frac{g^{(n)}(x_0)}{n!}(x - x_0)^n = G^T \psi(x), \end{aligned} \tag{22}$$

where Y_1 and Y_2 are column vectors with the entries of the vectors C and coefficients of G known.

Substitution of approximations (20) and (22) into (18) will be resulted to

$$Y_1^T \psi(x) + m\left(C^T P + U_1^T\right)\psi(x) + Y_2^T \psi(x) = G^T \psi(x). \tag{23}$$

Simplifying $\psi(x)$ in (23), a nonlinear system in terms of C will be obtained:

$$Y_1^T + m\left(C^T P + U_1^T\right) + Y_2^T = G^T. \tag{24}$$

The element of vector functions C can be computed by solving these systems.

3. Numerical Examples

In this section, some examples of Lane-Emden equation are considered and will be solved by introduced method.

Example 1. Consider the following nonlinear Lane-Emden equation:

$$\begin{aligned} y''(x) + \frac{6}{x} y'(x) + 14y(x) \\ = -4y(x)\ln(y(x)), \quad 0 < x \le 1, \end{aligned} \tag{25}$$

subject to the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \tag{26}$$

which has the following analytical solution:

$$y(x) = e^{-x^2}. \tag{27}$$

We solve (25) by the method discussed in this paper with $k = 1$ and $M = 7$.

We multiply both sides of (25) by x,

$$\begin{aligned} xy''(x) + 6y'(x) + 14xy(x) \\ = -4xy(x)\ln(y(x)), \quad 0 < x \le 1. \end{aligned} \tag{28}$$

Let us consider the following approximations:

$$\begin{aligned} y''(x) &\approx C^T \psi(x) \\ y'(x) &\approx C^T P \psi(x) + U_1^T \psi(x) \\ y(x) &\approx C^T P^2 \psi(x) + U_0^T \psi(x) \\ xy''(x) &\approx Y_1^T \psi(x) \\ xy(x) &\approx Y_2^T \psi(x) \\ xy(x)\ln(y(x)) &\approx Y_3^T \psi(x). \end{aligned} \tag{29}$$

Substitution into (28) and simplifying will be resulted to:

$$Y_1^T + 6C^T P + 14Y_2^T = -4Y_3^T. \tag{30}$$

Table 1: Numerical results of Example 1.

x	Exact solution	Legendre wavelets	Absolute error
0.0	1	1.000020858	0.000020858
0.1	0.9900498337	0.9900449375	0.0000048962
0.2	0.9607894392	0.9607962885	0.0000068493
0.3	0.9139311853	0.9139303816	8.037×10^{-7}
0.4	0.8521437890	0.8521354021	0.0000083869
0.5	0.7788007831	0.7787879122	0.0000128709
0.6	0.6976763261	0.6976230812	0.0000532449
0.7	0.6126263942	0.6124194854	0.0002069088
0.8	0.5272924240	0.5266984761	0.0005939479
0.9	0.4448580662	0.4434381168	0.0014199494
1	0.3678794412	0.3648016892	0.0030777520

By solving the system (30), we have

$$\begin{aligned} c_{1,0} &= -0.75742553415, \qquad c_{1,1} = 0.8884200666, \\ c_{1,2} &= 0.02494929272, \qquad c_{1,3} = -0.09212822515, \\ c_{1,4} &= 0.002866418366, \qquad c_{1,5} = 0.003597228202, \\ c_{1,6} &= -0.00002910926654. \end{aligned} \tag{31}$$

Therefore, the approximate solution of (25) will be obtained as follows:

$$\begin{aligned} y(x) &\approx \left(C^T P^2 + U_0^T\right)\psi(x) \\ &= 0.2023447118x^6 - 0.3173001850x^5 \\ &\quad + 0.7344456412x^4 - 0.08725168885x^3 \\ &\quad - 0.984175344x^2 - 0.001172073003x \\ &\quad + 1.000020858. \end{aligned} \tag{32}$$

Table 1 shows some values of the solutions and absolute errors at some x, and plots of the exact and approximate solutions are shown in Figure 1.

Example 2. Consider the following nonlinear Lane-Emden equation:

$$y''(x) + \frac{2}{x} y'(x) + y^n(x) = 0, \qquad 0 < x \le 1, \tag{33}$$

subject to the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \tag{34}$$

where $n \ge 0$ is constant. Substituting $n = 0, 1,$ and 5 into (33) leads to the exact solution

$$\begin{aligned} y(x) &= 1 - \frac{1}{3!}x^2, \\ y(x) &= \frac{\sin(x)}{x}, \\ y(x) &= \left(1 + \frac{x^2}{3}\right)^{-1/2}, \end{aligned} \tag{35}$$

respectively.

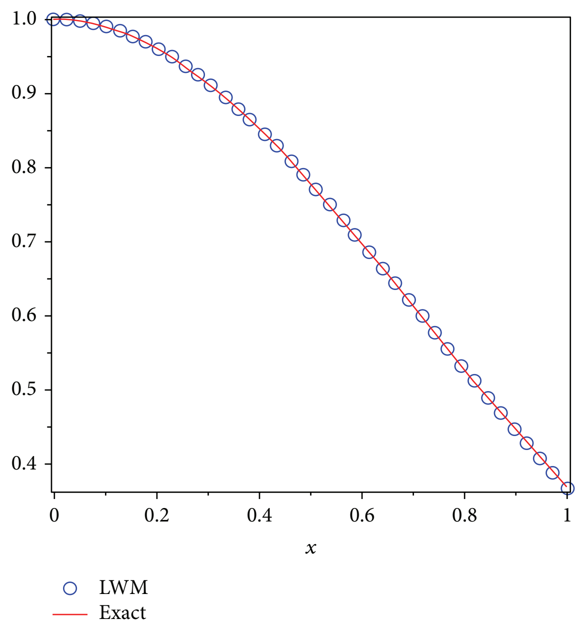


Figure 1: The exact and LWM solution of Example 1.

For $n = 0$, we solve (33) by the Legendre wavelet method with $k = 1$ and $M = 3$. For this equation, we find

$$c_{10} = -\frac{1}{6}, \qquad c_{11} = 0, \qquad c_{13} = 0. \tag{36}$$

Therefore, we have

$$y(x) \approx \left(C^T P^2 + U_0^T\right)\psi(x) = -\frac{x^2}{6} + 1, \tag{37}$$

which is the exact solution.

For $n = 1$, we solve (33) by the method discussed in this paper with $k = 1$ and $M = 10$. We have

$$\begin{aligned} c_{10} &= -0.3011686789, \qquad c_{11} = 0.02752116388, \\ c_{12} &= 0.006713250014, \qquad c_{13} = -0.0002154247382, \\ c_{14} &= -0.00002541095207, \qquad c_{15} = 5.308742329 \times 10^{-7}, \\ c_{16} &= 4.155816200 \times 10^{-5}, \qquad c_{17} = -8.1255913 \times 10^{-10}, \\ c_{18} &= 9.51835270 \times 10^{-11}, \\ c_{19} &= -8.086072988 \times 10^{-11}. \end{aligned} \tag{38}$$

Table 2: Numerical results of Example 2 for $n = 1$.

x	Exact solution	Legendre wavelets	Absolute error
0.0	1	1	0
0.1	0.9983341665	0.9983341665	0
0.2	0.9933466540	0.9933466540	0
0.3	0.9850673556	0.9850673556	0
0.4	0.9735458558	0.9735458558	0
0.5	0.9588510772	0.9588510772	0
0.6	0.9410707892	0.9410707890	2×10^{-10}
0.7	0.0920310982	0.0920310989	2×10^{-10}
0.8	0.8966951136	0.8966951137	1×10^{-10}
0.9	0.8703632328	0.8703632329	1×10^{-10}
1	0.8414709848	0.8414709848	0

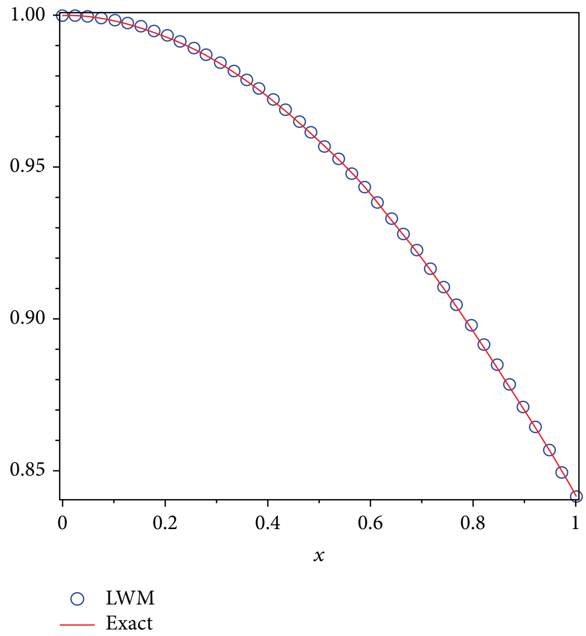
Therefore, the following solution will result:

$$\begin{aligned} y(x) &\approx \left(C^T P^2 + U_0^T\right)\psi(x) \\ &= -1.367445662 \times 10^{-7} x^9 + 0.000003078850787 x^8 \\ &\quad - 4.245445355 \times 10^{-7} x^7 \\ &\quad - 0.0001980757263 x^6 - 1.650008216 \times 10^{-7} x^5 \\ &\quad + 0.008333382135 x^4 \\ &\quad - 8.134931025 \times 10^{-9} x^3 - 0.1666666660 x^2 \\ &\quad + 4.268843531 \times 10^{-11} x + 1. \end{aligned} \tag{39}$$

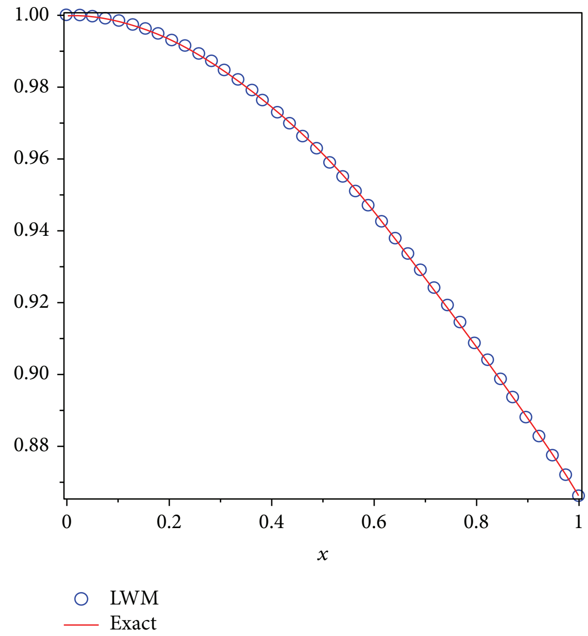
Table 2 shows that the Legendre wavelet solution is very near to the exact solution. Figure 2(a) shows that Legendre wavelet solution coincides with the exact solution.

For solving (33) by Legendre wavelets with $k = 1$, $M = 12$, and $n = 5$, we find

$$\begin{aligned} c_{10} &= -0.2165063510, \qquad c_{11} = 0.08910161508, \\ c_{12} &= 0.009961765824, \qquad c_{13} = -0.005784759597, \\ c_{14} &= 0.0001927423729, \qquad c_{15} = 0.0001784630080, \\ c_{16} &= -0.00002233585057, \\ c_{17} &= -0.000003109896876, \\ c_{18} &= 8.759973237 \times 10^{-7}, \qquad c_{19} = 5.92941957 \times 10^{-9}, \\ c_{110} &= -2.264665721 \times 10^{-8}, \\ c_{111} &= 1.961578560 \times 10^{-9}. \end{aligned} \tag{40}$$



(a)



(b)

FIGURE 2: (a) The exact and LWM solution of Example 2 for $n = 1$. (b) The exact and LWM solution of Example 2 for $n = 5$.

The approximate solution of $y(x)$ is as follows:

$$\begin{aligned} y(x) &\approx \left(C^T P^2 + U_0^T\right)\psi(x) \\ &= 0.000007988889774x^{11} + 0.0004944909178x^{10} \\ &\quad - 0.003163180797x^9 + 0.006776529202x^8 \\ &\quad - 0.002268670185x^7 \\ &\quad - 0.01058534314x^6 - 0.00028270820x^5 \\ &\quad + 0.04171831604x^4 \\ &\quad - 0.000005679982758x^3 - 0.1666663304x^2 \\ &\quad - 8.584262110 \times 10^{-9}x + 1. \end{aligned} \tag{41}$$

Table 3 shows that the Legendre wavelet solution is very near to the exact solution. Figure 2(b) shows that Legendre wavelet solution coincides with the exact solution.

TABLE 3: Numerical results of Example 2 for $n = 5$.

x	Exact solution	Legendre wavelets	Absolute error
0.0	1	1	0
0.1	0.9983374884	0.9983374884	0
0.2	0.9933992679	0.9933992677	2×10^{-10}
0.3	0.9853292781	0.9853292781	0
0.4	0.9743547036	0.9743547036	0
0.5	0.9607689228	0.9607689228	0
0.6	0.9449111826	0.9449111825	1×10^{-10}
0.7	0.9271455411	0.9271455408	3×10^{-10}
0.8	0.9078412992	0.9078412989	3×10^{-10}
0.9	0.8873565093	0.8873565094	1×10^{-10}
1	0.8660254038	0.8660254038	0

Example 3. Consider the following nonlinear Lane-Emden equation:

$$y''(x) + \frac{2}{x}y'(x) - 2\left(2x^2 + 3\right)y = 0, \quad 0 < x \le 1, \tag{42}$$

subject to the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \tag{43}$$

which has the following analytical solution:

$$y(x) = e^{x^2}. \tag{44}$$

Solving (42) by Legendre wavelets method with $k = 1$ and $M = 12$, we have

$$\begin{aligned} &c_{10} = 5.436563650, && c_{11} = 3.464101614, \\ &c_{12} = 1.517605100, && c_{13} = 0.4100561906, \\ &c_{14} = 0.1035241990, && c_{15} = 0.02034845798, \\ &c_{16} = 0.003837769593, && c_{17} = 0.0006174959039, \\ &c_{18} = 0.0000958708425, && c_{19} = 0.00001333846701, \\ &c_{1,10} = 0.000001773800247, \\ &c_{1,11} = 2.395987744 \times 10^{-7}. \end{aligned} \tag{45}$$

Table 4: Numerical results of Example 3.

x	Exact solution	Legendre wavelets	Absolute error
0.0	1	0.9999999958	4.2×10^{-9}
0.1	1.010050167	1.010050166	1×10^{-9}
0.2	1.040810774	1.040810774	0
0.3	1.094174284	1.094174282	2×10^{-9}
0.4	1.173510871	1.173510871	0
0.5	1.284025417	1.284025415	2×10^{-9}
0.6	1.433329415	1.433329414	1×10^{-9}
0.7	1.632316220	1.632316219	1×10^{-9}
0.8	1.896480879	1.896480878	1×10^{-9}
0.9	2.247907987	2.247907986	1×10^{-9}
1	2.718281828	2.718281824	4×10^{-9}

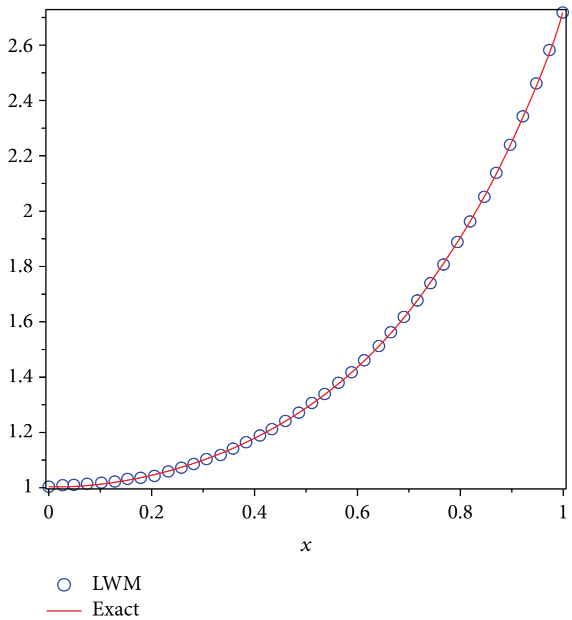


Figure 3: The exact and LWM solution of Example 3.

Therefore, the solution of the Lane-Emden equation will be obtained as follows:

$$\begin{aligned} y(x) &\approx \left(C^T P^2 + U_0^T\right)\psi(x) \\ &= 0.02527877277x^{11} - 0.08422571788x^{10} \\ &\quad + 0.1670609573x^9 - 0.1386060456x^8 \\ &\quad + 0.1250499043x^7 + 0.109479685x^6 \\ &\quad + 0.017184049x^5 + 0.4967026721x^4 \\ &\quad + 0.000380562x^3 + 0.9999763578x^2 \\ &\quad + 6.313645775 \times 10^{-7}x + 0.9999999958. \end{aligned} \tag{46}$$

Table 4 shows that the Legendre wavelet solution is very near to the exact solution. Figure 3 shows that Legendre wavelet solution coincides with the exact solution.

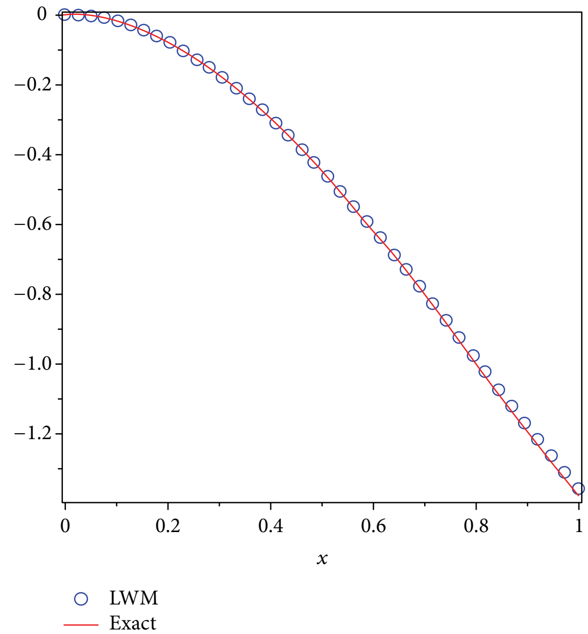


Figure 4: The exact and LWM solution of Example 4.

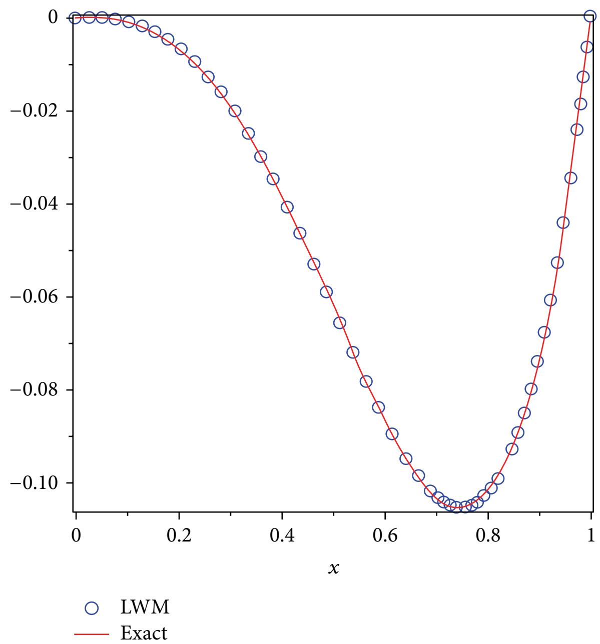


Figure 5: The exact and LWM solution of Example 5.

Example 4. Consider the following nonlinear Lane-Emden equation:

$$y''(x) + \frac{2}{x}y'(x) + 4\left(2e^y + e^{y/2}\right) = 0, \quad 0 < x \leq 1, \tag{47}$$

subject to the initial conditions

$$y(0) = 0, \qquad y'(0) = 0, \tag{48}$$

which has the following analytical solution:

$$y(x) = -2\ln\left(1+x^2\right). \tag{49}$$

We solve (47) by the method discussed in this paper with $k = 1$ and $M = 6$. We multiply both sides of (47) by x,

$$xy''(x) + 2y'(x) + 4x\left(2e^{y} + e^{y/2}\right) = 0, \qquad 0 < x \le 1. \tag{50}$$

Let us consider the following approximations:

$$\begin{aligned} y''(x) &\approx C^T\psi(x) \\ y'(x) &\approx C^T P\psi(x) + U_1^T\psi(x) \\ y(x) &\approx C^T P^2\psi(x) + U_0^T\psi(x) \\ xy''(x) &\approx Y_1^T\psi(x) \\ x\left(2e^{y} + e^{y/2}\right) &\approx Y_2^T\psi(x). \end{aligned} \tag{51}$$

Substitution into (50) and simplifying will be resulted to

$$Y_1^T + 2C^T P + 4Y_2^T = 0. \tag{52}$$

By solving the system (52), we have

$$\begin{aligned} &c_{10} = -1.789983487, \qquad c_{11} = 1.607415602, \\ &c_{12} = 0.1463990297, \qquad c_{13} = -0.04963748601, \\ &c_{14} = 0.04188454297, \qquad c_{15} = -0.007221576268. \end{aligned} \tag{53}$$

The approximate solution of $y(x)$ is as follows:

$$\begin{aligned} y(x) = C^T P^2\psi(x) &= -0.1160867335x^5 \\ &+ 0.3967806389x^4 + 0.4849181405x^3 \\ &- 2.139040298x^2 + 0.01441559362x \\ &- 0.000\,3173071437. \end{aligned} \tag{54}$$

Table 5 shows some values of the solutions and absolute errors at some x's, and plots of the exact and approximate solutions are shown in Figure 4.

Example 5. Consider the following nonlinear Lane-Emden equation:

$$\begin{aligned} &y''(x) + \frac{8}{x}y'(x) + xy(x) \\ &\quad = x^5 - x^4 + 44x^2 - 30x, \quad 0 < x \le 1, \end{aligned} \tag{55}$$

subject to the initial conditions

$$y(0) = 0, \qquad y'(0) = 0, \tag{56}$$

which has the following analytical solution:

$$y(x) = x^4 - x^3. \tag{57}$$

Table 5: Numerical results of Example 4.

x	Exact solution	Legendre wavelets	Absolute error
0.0	0	−0.0003173071437	0.0003173071437
0.1	−0.01990066171	−0.01974271542	0.00015794629
0.2	−0.07844142630	−0.07851875395	0.00007732765
0.3	−0.1723553925	−0.1724816336	0.0001262411
0.4	−0.2968400102	−0.2967939002	0.0000461100
0.5	−0.4462871026	−0.4460837377	0.0002033649
0.6	−0.6149693994	−0.6145842735	0.0003851259
0.7	−0.7975522400	−0.7962728813	0.0012793587
0.8	−0.9893924836	−0.9850104864	0.0043819972
0.9	−1.186653691	−1.174680868	0.011972823
1	−1.386294361	−1.359329965	0.026964396

Table 6: Numerical results of Example 5.

x	Exact solution	Legendre wavelets	Absolute error
0.0	0	$5.809942853 \times 10^{-12}$	5.8×10^{-12}
0.1	−0.0009	−0.0009000000166	1.6×10^{-11}
0.2	−0.0064	−0.006400000015	1.5×10^{-11}
0.3	−0.0189	−0.01890000000	0
0.4	−0.0384	−0.03839999999	1×10^{-11}
0.5	−0.0625	−0.06250000004	4×10^{-11}
0.6	−0.0864	−0.08640000001	1×10^{-11}
0.7	−0.1029	−0.1029000001	1×10^{-10}
0.8	−0.1024	−0.1024000002	2×10^{-10}
0.9	−0.0729	−0.07290000033	3.3×10^{-10}
1	0	$-3.888737334 \times 10^{-10}$	3.9×10^{-10}

We solve (55) by the method discussed in this paper with $k = 1$ and $M = 6$. This implies that

$$\begin{aligned} &c_{10} = 0.9999999996, \qquad c_{11} = 1.732050807, \\ &c_{12} = 0.8944271910, \qquad c_{13} = 7.079278793 \times 10^{-10}, \\ &c_{14} = -7.669124566 \times 10^{-10}, \\ &c_{15} = 9.994056888 \times 10^{-10}. \end{aligned} \tag{58}$$

The approximate solution of $y(x)$ is as follows:

$$\begin{aligned} y(x) \approx C^T P^2\psi(x) &= -2.36323956 \times 10^{-10}x^5 \\ &+ 1.000000001x^4 - 1.000000003x^3 \\ &+ 2 \times 10^{-9}x^2 - 3.946836763 \times 10^{-10}x \\ &+ 5.809942853 \times 10^{-12}. \end{aligned} \tag{59}$$

Table 6 shows that the Legendre wavelet solution is very near to the exact solution. Figure 5 shows that Legendre wavelet solution coincides with the exact solution.

4. Conclusion

In this research, we have presented the Legendre wavelet method for solving nonlinear singular Lane-Emden equation. The Legendre wavelets operational matrix of integration is used to solve Lane-Emden equation. The present method reduces Lane-Emden equation into a set of algebraic equations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique, and the results have been compared with the exact solution.

References

[1] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Dover Publications, New York, NY, USA, 1967.

[2] O. U. Richardson, *The Emission of Electricity From Hot Bodies*, Longmans Green and Company, London, UK, 1921.

[3] C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., "A new perturbative approach to nonlinear problems," *Journal of Mathematical Physics*, vol. 30, no. 7, pp. 1447–1455, 1989.

[4] M. El-Gebeily and D. O'Regan, "A quasilinearization method for a class of second order singular nonlinear differential equations with nonlinear boundary conditions," *Nonlinear Analysis: Real World Applications*, vol. 8, pp. 174–186, 2007.

[5] N. T. Shawagfeh, "Non-perturbative approximate solution for Lane-Emden equation," *Journal of Mathematical Physics*, vol. 34, no. 9, pp. 4364–4369, 1993.

[6] A.-M. Wazwaz, "A new algorithm for solving differential equations of Lane-Emden type," *Applied Mathematics and Computation*, vol. 118, no. 2-3, pp. 287–310, 2001.

[7] A.-M. Wazwaz, "A new method for solving singular initial value problems in the second-order ordinary differential equations," *Applied Mathematics and Computation*, vol. 128, no. 1, pp. 45–57, 2002.

[8] M. I. Nouh, "Accelerated power series solution of polytropic and isothermal gas spheres," *New Astronomy*, vol. 9, no. 6, pp. 467–473, 2004.

[9] V. B. Mandelzweig and F. Tabakin, "Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs," *Computer Physics Communications*, vol. 141, no. 2, pp. 268–281, 2001.

[10] J. I. Ramos, "Linearization methods in classical and quantum mechanics," *Computer Physics Communications*, vol. 153, no. 2, pp. 199–208, 2003.

[11] Y. Bozhkov and A. C. Gilli Martins, "Lie point symmetries and exact solutions of quasilinear differential equations with critical exponents," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 57, no. 5-6, pp. 773–793, 2004.

[12] E. Momoniat and C. Harley, "Approximate implicit solution of a Lane-Emden equation," *New Astronomy*, vol. 11, pp. 520–526, 2006.

[13] H. Goenner and P. Havas, "Exact solutions of the generalized Lane-Emden equation," *Journal of Mathematical Physics*, vol. 41, no. 10, pp. 7029–7042, 2000.

[14] S. Liao, "A new analytic algorithm of Lane-Emden type equations," *Applied Mathematics and Computation*, vol. 142, no. 1, pp. 1–16, 2003.

[15] J.-H. He, "Variational approach to the Lane-Emden equation," *Applied Mathematics and Computation*, vol. 143, no. 2-3, pp. 539–541, 2003.

[16] J. I. Ramos, "Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method," *Chaos, Solitons and Fractals*, vol. 38, no. 2, pp. 400–408, 2008.

[17] T. Özis and A. Yildirim, "Solutions of singular IVP's of Lane-Emden type by homotopy pertutbation method," *Physics Letters A*, vol. 369, pp. 70–76, 2007.

[18] T. Özis and A. Yildirim, "Solutions of singular IVPs of Lane-Emden type by the variational iteration method," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 6, pp. 2480–2484, 2009.

[19] K. Parand and M. Razzaghi, "Rational Chebyshev tau method for solving higher-order ordinary differential equations," *International Journal of Computer Mathematics*, vol. 81, no. 1, pp. 73–80, 2004.

[20] K. Parand and M. Razzaghi, "Rational Legendre approximation for solving some physical problems on semi-infinite intervals," *Physica Scripta*, vol. 69, pp. 353–357, 2004.

[21] K. Parand, M. Shahini, and M. Dehghan, "Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type," *Journal of Computational Physics*, vol. 228, no. 23, pp. 8830–8840, 2009.

[22] K. Parand, M. Dehghan, A. R. Rezaei, and S. M. Ghaderi, "An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method," *Computer Physics Communications*, vol. 181, no. 6, pp. 1096–1108, 2010.

[23] M. Razzaghi and S. Yousefi, "The Legendre wavelets operational matrix of integration," *International Journal of Systems Science*, vol. 32, no. 4, pp. 495–502, 2001.