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Research Article

Quasifinite Representations of Classical Subalgebras of the Lie Superalgebra of Quantum Pseudodifferential Operators

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We classify the anti-involutions of the superalgebra of quantum pseudodifferential operators on the super circle $S^{1|1}$ preserving the principal gradation, producing in this way a family of Lie subalgebras minus fixed by these anti-involutions. We classify the irreducible quasifinite highest weight representations of the central extension of these Lie subalgebras.

1. Introduction

The $\mathscr{W}$-infinity algebras naturally arise in various physical theories, such as conformal field theory and the theory of quantum Hall effect (see [1, 2] and references therein). The $W_{1+\infty}$ algebra, which is the central extension of the Lie algebra $\mathscr{D}$ of differential operators on the circle, is the most fundamental among these algebras.

The difficulty in understanding the representation theory of a Lie algebra of this kind is that although $W_{1+\infty}$ admits a natural $\mathbb{Z}$-gradation, each of the graded subspaces is still infinite dimensional in contrast to the more familiar cases such as the Virasoro algebra and Kac-Moody algebras. Therefore, the study of the highest weight modules which satisfy the quasifiniteness condition, that its graded subspaces have finite dimension, becomes a nontrivial problem. The systematic study of quasifinite highest weight modules of $W_{1+\infty}$ was initiated by Kac and Radul in [2] and further studied in [1, 3–5] and many others.

By analyzing for which parabolic subalgebras of $W_{1+\infty}$ the corresponding generalized Verma modules are quasifinite, Kac and Radul [2] gave a characterization of quasifinite highest weight $W_{1+\infty}$-modules in terms of certain generating function of highest weights and these modules where constructed in terms of irreducible highest weight representations of the Lie algebra of infinite matrices.

The classification and construction of quasifinite modules for the matrix version (denoted by $W_{1+\infty}^N$), super analog, q-analog, and super q-analog of $W_{1+\infty}$, were developed in [1, 2, 4, 6], respectively.

The Lie algebra $W_{1+\infty}^N$, recently studied in [4], correspond to the central extension of the algebra of matrix differential operators on the circle. The study of the representation theory of some interesting subalgebras of $W_{1+\infty}^N$, and its q-analog and super version [1], is not complete.

Another important example is the Lie algebra W_∞ which is a particular case of a family of subalgebras $W_{\infty,p}$ of $W_{1+\infty}$, where $W_{\infty,p}$ ($p \in \mathbb{C}[x]$) is the central extension of the Lie algebra $\mathscr{D}p(t\partial_t)$ of differential operators on the circle that are a multiple of $p(t\partial_t)$.

This Lie algebra was studied by Kac and Liberati in [3]; observe that $W_\infty = W_{\infty,x}$. Following the ideas of Kac-Radul [2], in [3] they obtained the classification of the irreducible quasifinite highest weight modules over $W_{\infty,p}$. They also developed a general theory of quasifinite highest weight modules over $\mathbb{Z}$-graded Lie algebras. These general results were extended to the super version in [6].

A natural source of subalgebras comes from the subalgebras minus fixed by an anti-involution of the corresponding associative algebra, which preserve the gradation. In [7], Bloch finds an anti-involution of W_∞ and he shows a relation between the representations of the corresponding subalgebra

Table 1

	$W_{1+\infty}$	$W_{\infty,p}$	$W_{1+\infty}^N$	super $W_{1+\infty}$	q-$W_{1+\infty}$	super q-$W_{1+\infty}$
Algebra	[2]	[3, 11]	[4]	[1]	[2]	[6]
Anti-involution	[5]	[10] (particular case in [12])	Partial results in [13] and others	[8]	[9]	Present work

and certain values of the Riemann zeta function. In several and recent papers ([5, 8–10]) the authors obtained the classification of the anti-involutions of certain algebras and then they characterize the irreducible quasifinite highest weight modules of the corresponding subalgebras minus fixed by these anti-involutions, obtaining orthogonal and symplectic subalgebras of $W_{1+\infty}$, $W_{\infty,p}$, and so forth.

The main goal of this work is to present a q-analog of Cheng-Wang [8] or a super-analog of Boyallian-Liberati [9]; namely, we classify the anti-involutions of the superalgebra q-$\mathcal{SD}$ that preserve the gradation, where q-$\mathcal{SD}$ is the superalgebra of regular pseudodifferential operators on the super circle $S^{1|1}$. Then we present the classification of the irreducible quasifinite highest weight modules over the Lie subalgebras minus fixed by these anti-involutions.

Table 1 describes the map on the classification of irreducible quasifinite highest weight modules over $W_{1+\infty}$, the matrix version, super analog and q-analog, and for the subalgebras constructed from the anti-involutions, showing the place of the present results in this long-term program.

The work is organized as follows. In Section 2, we classify the anti-involutions of q-$\mathcal{SD}$ that preserve the gradation, where q-$\mathcal{SD}$ is the superalgebra of regular pseudodifferential operators on the super circle $S^{1|1}$. In Section 3, we recall the general results on quasifinite modules over a graded Lie superalgebra and we present the classification of the irreducible quasifinite highest weight modules over the Lie subalgebras minus fixed by the anti-involutions obtained in Section 2.

2. Anti-Involution of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ Preserving Its Principal Gradation

Let $q \in \mathbb{C}^{\times}$ with $|q| \neq 1$. Now, T_q denotes the following operator on $\mathbb{C}[z^{-1}, z]$:

$$T_q(f(z)) = f(qz). \tag{1}$$

We denote by $\mathfrak{S}_q^{\mathrm{as}}$ the associative algebra of all pseudodifferential operators, that is, the operators on $\mathbb{C}[z^{-1}, z]$ of the form

$$E = \sum_{k\in\mathbb{Z}} e_k(z) T_q^k, \quad \text{where } e_i(z) \in \mathbb{C}\left[z^{-1}, z\right] (\text{sum is finite}). \tag{2}$$

Any pseudodifferential operator can be written as linear combinations of elements of the form $z^n f(T_q)$, where $f \in \mathbb{C}[w^{-1}, w]$ and $n \in \mathbb{Z}$. The product in $\mathfrak{S}_q^{\mathrm{as}}$ is given by

$$z^n f(T_q) \cdot z^m g(T_q) = z^{n+m} f(q^m T_q) g(T_q). \tag{3}$$

Letting $wt z^n f(T_q) = n$, we define the *principal $\mathbb{Z}$-gradation* of $\mathfrak{S}_q^{\mathrm{as}}$.

Moreover, we denote by $M(1|1)$ the set of 2×2 supermatrices with coefficients in $\mathbb{C}$, viewed as the associative superalgebra of linear transformations of the complex $(1|1)$-dimensional superspace $\mathbb{C}^{(1|1)}$. And we denote by E_{ij} the 2×2 matrix with 1 in the ij-place and 0 everywhere else. Declaring E_{11}, E_{22} even and E_{12}, E_{21} odd elements, we endow $M(1|1)$ with a $\mathbb{Z}_2$-gradation where $|M|$ denotes the parity of the homogeneous element $M \in M(1|1)$.

We denote by $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ the associative superalgebra of 2×2 supermatrices with entries in $\mathfrak{S}_q^{\mathrm{as}}$, namely,

$$\mathcal{S}\mathfrak{S}_q^{\mathrm{as}} = \mathfrak{S}_q^{\mathrm{as}} \bigotimes M(1|1), \tag{4}$$

and the product is given by the usual matrix multiplication. Let $\mathcal{S}\mathfrak{S}_q$ denote the Lie superalgebra obtained from $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ by taking the usual bracket.

Now, we introduce the linear map $\mathrm{Str}_0 : \mathcal{S}\mathfrak{S}_q \to \mathbb{C}$ as

$$\mathrm{Str}_0\left(\sum_{ij} f_{ij}(T_q) E_{ij}\right) = \left(f_{11}(T_q)\right)_0 - \left(f_{22}(T_q)\right)_0, \tag{5}$$

where $(f(T_q))_0 = f_0$ if $f(T_q) = \sum_{k\in\mathbb{Z}} f_k T_q^k$, and we define the 2-cocycle ψ in $\mathcal{S}\mathfrak{S}_q$ by

$$\psi\left(z^n f(T_q) E_{ij}, z^m g(T_q) E_{kl}\right) = \begin{cases} -(-1)^i \sum_{r=0}^{n-1} \left(f\left(q^{-n+r} T_q\right) g\left(q^r T_q\right)\right)_0 \delta_{jk}\delta_{il}, & \text{if } n = -m > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

Then we denote by $\widehat{\mathcal{S}\mathfrak{S}_q}$ the one-dimensional central extension of $\mathcal{S}\mathfrak{S}_q$ with central charge C corresponding to the 2-cocycle ψ, namely, $\widehat{\mathcal{S}\mathfrak{S}_q} = \mathcal{S}\mathfrak{S}_q \bigoplus \mathbb{C}C$, where the bracket is given by

$$\begin{aligned} &\left[z^n f(T_q) E_{ij}, z^m g(T_q) E_{kl}\right] \\ &\quad = z^{n+m}\left(f(q^m T_q) g(T_q) \delta_{jk} E_{il} - (-1)^{|E_{ij}||E_{kl}|} \right. \\ &\qquad \left. \times f(T_q) g(q^n T_q) \delta_{li} E_{kj}\right) \\ &\qquad + \psi\left(z^n f(T_q) E_{ij}, z^m g(T_q) E_{kl}\right) C. \end{aligned} \tag{7}$$

Letting $wt(z^n f(T_q) E_{12} + z^{n+1} g(T_q) E_{21}) = n + 1/2$, $wt z^n f(T_q) E_{ii} = n$ and $wt\, C = 0$, where $i = 1, 2$, $n \in \mathbb{Z}$, defines the *principal* $(1/2)\mathbb{Z}$*-gradation* of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$, $\mathcal{S}\mathfrak{S}_q$ and $\widehat{\mathcal{S}\mathfrak{S}_q}$.

This equips $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$, $\mathcal{S}\mathfrak{S}_q$, and $\widehat{\mathcal{S}\mathfrak{S}}_q$ with $(1/2)\mathbb{Z}$-gradations compatible with their $\mathbb{Z}_2$-gradation; thus,

$$\widehat{\mathcal{S}\mathfrak{S}}_q = \left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_{\bar{0}} \bigoplus \left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_{\bar{1}},$$
$$\left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} \left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_n, \qquad \left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_{\bar{1}} = \bigoplus_{n\in\mathbb{Z}} \left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_{n+1/2}, \tag{8}$$

where

$$\begin{aligned}
&\left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_n \\
&\quad = \left\{z^n f_1\left(T_q\right) E_{11} + z^n f_2\left(T_q\right) E_{22} :\right. \\
&\qquad \left. f_i \in \mathbb{C}\left[w^{-1}, w\right], i = 1, 2\right\} + \delta_{n,0}\mathbb{C}C, \\
&\left(\widehat{\mathcal{S}\mathfrak{S}}_q\right)_{n+1/2} \\
&\quad = \left\{z^n f_1\left(T_q\right) E_{12} + z^{n+1} f_2\left(T_q\right) E_{21} : f_i \in \mathbb{C}\left[w^{-1}, w\right],\right. \\
&\qquad \left. i = 1, 2\right\}.
\end{aligned} \tag{9}$$

An *anti-involution* σ of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ is an involutive anti-automorphism of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$; that is, $\sigma : \mathcal{S}\mathfrak{S}_q^{\mathrm{as}} \to \mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ with $\sigma^2 = Id$, $\sigma(bA + B) = b\sigma(A) + \sigma(B)$ and $\sigma(AB) = (-1)^{|A||B|}\sigma(B)\sigma(A)$, where $A, B \in \mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$, and $b \in \mathbb{C}$.

The main result of this section is the following theorem with the classification of all anti-involutions of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ that preserve the principal $(1/2)\mathbb{Z}$-gradation.

Theorem 1. *Any anti-involution σ of $\mathcal{S}\mathfrak{S}_q^{as}$ which preserves the principal gradation is one of the following ($f \in \mathbb{C}[w^{-1}, w]$, $n \in \mathbb{Z}$):*

(a)

$$\begin{aligned}
&\sigma_{a,b,c,k}\left(z^n f\left(T_q\right) E_{11}\right) \\
&\quad = a^n q^{(n-1)nk} z^n f\left(bq^{-n}T_q^{-1}\right) T_q^{2nk} E_{11}, \\
&\sigma_{a,b,c,k}\left(z^n f\left(T_q\right) E_{22}\right) \\
&\quad = a^n q^{(n-2)nk} z^n f\left(bq^{-n+1}T_q^{-1}\right) T_q^{2nk} E_{22}, \\
&\sigma_{a,b,c,k}\left(z^n f\left(T_q\right) E_{12}\right) \\
&\quad = a^n c q^{n^2 k} z^{n+1} f\left(bq^{-n}T_q^{-1}\right) T_q^{(2n+1)k} E_{21}, \\
&\sigma_{a,b,c,k}\left(z^n f\left(T_q\right) E_{21}\right) \\
&\quad = -a^n c^{-1} q^{((n-1)^2-n)k} z^{n-1} f\left(bq^{-n+1}T_q^{-1}\right) T_q^{(2n-1)k} E_{12},
\end{aligned} \tag{10}$$

with $a, b, c \in \mathbb{C}^\times$, $k \in \mathbb{Z}$ such that $-ab^k = q^k$;

(b)

$$\begin{aligned}
&\sigma_{a,b,c,k,l}\left(z^n f\left(T_q\right) E_{11}\right) \\
&\quad = a^n q^{(n(n-1)/2)k} z^n f\left(bq^{-n}T_q^{-1}\right) T_q^{nk} E_{22}, \\
&\sigma_{a,b,c,k,l}\left(z^n f\left(T_q\right) E_{22}\right) \\
&\quad = a^n q^{(n(n-1)/2)k+nl} z^n f\left(bq^{-n}T_q^{-1}\right) T_q^{nk} E_{11}, \\
&\sigma_{a,b,c,k,l}\left(z^n f\left(T_q\right) E_{12}\right) \\
&\quad = a^n c q^{(n(n-1)/2)k+nl} z^n f\left(bq^{-n}T_q^{-1}\right) T_q^{nk+l} E_{12}, \\
&\sigma_{a,b,c,k,l}\left(z^n f\left(T_q\right) E_{21}\right) \\
&\quad = -a^n c^{-1} q^{(n(n-1)/2)k} z^n f\left(bq^{-n}T_q^{-1}\right) T_q^{nk-l} E_{21},
\end{aligned} \tag{11}$$

with $a, b, c \in \mathbb{C}^\times$, $k, l \in \mathbb{Z}$ such that $a^2 b^k = q^{k-l}$ and $b^l c^2 = 1$.

We divide the proof of Theorem 1 into several results.

Let σ be an anti-involution of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ which preserves the principal gradation; then, σ defines linear maps $\sigma_{i,j} : \mathfrak{S}_q^{\mathrm{as}} \to \mathfrak{S}_q^{\mathrm{as}}$ preserving the principal gradation

$$\sigma\left(z^n f\left(T_q\right) E_{ii}\right) = \sigma_{i,1}\left(z^n f\left(T_q\right)\right) E_{11} + \sigma_{i,2}\left(z^n f\left(T_q\right)\right) E_{22}. \tag{12}$$

Lemma 2. *Let σ be an anti-involution of $\mathcal{S}\mathfrak{S}_q^{as}$ preserving the principal gradation. Then σ satisfies one of the following conditions:*

(a)

$$\begin{aligned}
\sigma\left(z^n f\left(T_q\right) E_{11}\right) &= \sigma_{1,1}\left(z^n f\left(T_q\right)\right) E_{11}, \\
\sigma\left(z^n f\left(T_q\right) E_{22}\right) &= \sigma_{2,2}\left(z^n f\left(T_q\right)\right) E_{22}, \\
\sigma\left(E_{12}\right) &= z f_{21}\left(T_q\right) E_{21}, \\
\sigma\left(E_{21}\right) &= z^{-1} g_{12}\left(T_q\right) E_{12};
\end{aligned} \tag{13}$$

(b)

$$\begin{aligned}
\sigma\left(z^n f\left(T_q\right) E_{11}\right) &= \sigma_{1,2}\left(z^n f\left(T_q\right)\right) E_{22}, \\
\sigma\left(z^n f\left(T_q\right) E_{22}\right) &= \sigma_{2,1}\left(z^n f\left(T_q\right)\right) E_{11}, \\
\sigma\left(E_{12}\right) &= f_{12}\left(T_q\right) E_{12}, \\
\sigma\left(E_{21}\right) &= g_{21}\left(T_q\right) E_{21}.
\end{aligned} \tag{14}$$

For some $f_{ij}, g_{ij} \in \mathbb{C}[w^{-1}, w]$, where $1 \leqslant i, j \leqslant 2$ and $i \neq j$.

Proof. Since σ preserves the principal gradation, we have that

$$\begin{aligned}
\sigma\left(E_{12}\right) &= f_{12}\left(T_q\right) E_{12} + z f_{21}\left(T_q\right) E_{21}, \\
\sigma\left(E_{21}\right) &= z^{-1} g_{12}\left(T_q\right) E_{12} + g_{21}\left(T_q\right) E_{21},
\end{aligned} \tag{15}$$

for some $f_{ij}, g_{ij} \in \mathbb{C}[w^{-1}, w]$, with $1 \leqslant i, j \leqslant 2$ and $i \neq j$.

Then, since $0 = \sigma(z^n f(T_q)E_{11} \cdot z^m g(T_q)E_{22}) = \sigma(z^m g(T_q) E_{22})\sigma(z^n f(T_q)E_{11})$, we have two possibilities

$$\begin{aligned}\sigma\left(z^n f\left(T_q\right)E_{11}\right) &= \sigma_{1,1}\left(z^n f\left(T_q\right)\right)E_{11},\\ \sigma\left(z^n f\left(T_q\right)E_{22}\right) &= \sigma_{2,2}\left(z^n f\left(T_q\right)\right)E_{22},\end{aligned} \tag{16}$$

or

$$\begin{aligned}\sigma\left(z^n f\left(T_q\right)E_{11}\right) &= \sigma_{1,2}\left(z^n f\left(T_q\right)\right)E_{22},\\ \sigma\left(z^n f\left(T_q\right)E_{22}\right) &= \sigma_{2,1}\left(z^n f\left(T_q\right)\right)E_{11}.\end{aligned} \tag{17}$$

Moreover, since $0 = \sigma(E_{ij}^2) = -\sigma(E_{ij})^2$, for $1 \leqslant i, j \leqslant 2$, $i \neq j$, from (15) we have that

$$\begin{aligned}&\sigma\left(E_{12}\right) = f_{12}\left(T_q\right)E_{12} \quad \text{or } \sigma\left(E_{12}\right) = z f_{21}\left(T_q\right)E_{21},\\ &\sigma\left(E_{21}\right) = g_{21}\left(T_q\right)E_{21} \quad \text{or } \sigma\left(E_{21}\right) = z^{-1} g_{12}\left(T_q\right)E_{12}.\end{aligned} \tag{18}$$

Finally, since

$$\sigma\left(E_{11}\right) = -\sigma\left(E_{21}\right)\sigma\left(E_{12}\right), \tag{19}$$

the result follows from (16), (17), (18), and (19). □

Corollary 3. *Let σ be an anti-involution of $\mathcal{S}\mathfrak{S}_q^{as}$ which preserves the principal gradation. Then one has the following.*

(a) *If σ satisfies Lemma 2(a), then $\sigma_{i,j}(1) = \delta_{i,j}$, for all $1 \leqslant i, j \leqslant 2$.*

(b) *If σ satisfies Lemma 2(b), then $\sigma_{i,j}(1) = 1 - \delta_{i,j}$, for all $1 \leqslant i, j \leqslant 2$.*

Proof. It is clear that $\sigma(Id) = Id$; then the result follows from Lemma 2. □

Proposition 4. *Let σ be an anti-involution of $\mathcal{S}\mathfrak{S}_q^{as}$ which preserves the principal gradation and let one assume that it satisfies Lemma 2(a). Then σ is one of the $\sigma_{a,b,c,k}$ from Theorem 1(a).*

In order to prove Proposition 4 we will need the following result.

Lemma 5. *Let σ be as in Proposition 4; then,*

$$\begin{aligned}\sigma\left(z^n f\left(T_q\right)E_{ii}\right) &= a_i^n q^{(n(n-1)/2)k_i} z^n f\left(b_i q^{-n} T_q^{-1}\right) T_q^{nk_i} E_{ii},\\ \sigma\left(z^n f\left(T_q\right)E_{12}\right) &\\ &= a_1^n c q^{(n(n-1)/2)k_1 + nk} z^{n+1} f\left(b_1 q^{-n} T_q^{-1}\right) T_q^{nk_1+k} E_{21},\\ \sigma\left(z^n f\left(T_q\right)E_{21}\right) &\\ &= -a_2 c^{-1} q^{(nk_2/2-k)(n-1)} z^{n-1} f\left(b_2 q^{-n} T_q^{-1}\right) T_q^{nk_2-k} E_{12},\end{aligned} \tag{20}$$

where $a_i, b_i, c \in \mathbb{C}^\times$, $k_i, k \in \mathbb{Z}$ such that $a_i^2 b_i^{k_i} = q^{k_i}$, with $i = 1, 2$.

Proof. It is easy to check that $\sigma_{i,i}$ is a anti-involution of $\mathfrak{S}_q^{as}$ with $i = 1, 2$; then, from Section 3 in [9] and hypothesis, we obtain that

$$\sigma\left(z^n f\left(T_q\right)E_{ii}\right) = a_i^n q^{(n(n-1)/2)k_i} z^n f\left(b_i q^{-n} T_q^{-1}\right) T_q^{nk_i} E_{ii}, \tag{21}$$

with $k_i \in \mathbb{Z}$, $a_i, b_i \in \mathbb{C}^\times$ such that $a_i^2 b_i^{k_i} = q^{k_i}$, $i = 1, 2$. Besides, from Corollary 3(a) and hypothesis, we have that

$$E_{11} = -\sigma\left(E_{21}\right)\sigma\left(E_{12}\right) = -g_{12}\left(qT_q\right) f_{21}\left(T_q\right)E_{11}. \tag{22}$$

Therefore $f_{21}(T_q) = cT_q^k$ and $g_{12}(T_q) = -c^{-1} q^k T_q^{-k}$ with $c \in \mathbb{C}^\times$, $k \in \mathbb{Z}$; then,

$$\begin{aligned}\sigma\left(E_{12}\right) &= cz T_q^k E_{21},\\ \sigma\left(E_{21}\right) &= -c^{-1} q^k z^{-1} T_q^{-k} E_{12}.\end{aligned} \tag{23}$$

Moreover,

$$\sigma\left(z^n f\left(T_q\right)E_{ij}\right) = \sigma\left(E_{ij}\right)\sigma\left(z^n f\left(T_q\right)E_{ii}\right), \tag{24}$$

with $i, j = 1, 2$, $i \neq j$. The proof follows from (21) and by replacing (21) and (23) in (24). □

Proof of Proposition 4. From Lemma 5, we have that

$$\begin{aligned}E_{12} &= \sigma^2\left(E_{12}\right) = c\sigma\left(E_{21}\right)\sigma\left(zT_q^k E_{22}\right)\\ &= -a_2 b_2^k q^{-k} T_q^{k_2-2k} E_{12},\end{aligned} \tag{25}$$

$$\begin{aligned}E_{12} &= \sigma^2\left(E_{12}\right) = c\sigma\left(zT_q^k E_{11}\right)\sigma\left(E_{21}\right)\\ &= -a_1 b_1^k q^{k-k_1} T_q^{k_1-2k} E_{12}.\end{aligned} \tag{26}$$

Then from (25)-(26), we obtain that

$$k_1 = k_2 = 2k, \tag{27}$$

and also from (26)-(27) we obtain

$$-a_1 b_1^k = q^k. \tag{28}$$

Then, from (27) and again using Lemma 5, we have that

$$\sigma\left(z^n T_q^l E_{12}\right) = a_1^n b_1^l c q^{(nk-l)n} z^{n+1} T_q^{(2n+1)k-l} E_{21}, \tag{29}$$

$$\begin{aligned}\sigma\left(z^n T_q^l E_{12}\right) &= \sigma\left(z^n T_q^l E_{22}\right)\sigma\left(E_{12}\right)\\ &= a_2^n b_2^l c q^{(nk-l)(n+1)} z^{n+1} T_q^{(2n+1)k-l} E_{21},\end{aligned} \tag{30}$$

for all $n, l \in \mathbb{Z}$. Comparing (29) with (30), we obtain that

$$a_1^n b_1^l = a_2^n b_2^l q^{nk-l}, \quad \forall n, l \in \mathbb{Z}; \tag{31}$$

in particular,

$$\begin{aligned}&\text{if } n = 0, \quad l = 1, \quad b_2 = b_1 q,\\ &\text{if } n = 1, \quad l = 0, \quad a_2 = a_1 q^{-k}.\end{aligned} \tag{32}$$

From (28) and replacing (27) and (32) in (20), we obtain that $\sigma = \sigma_{a_1,b_1,c,k}$, finishing the proof. □

Proposition 6. *Let σ be an anti-involution of $\mathcal{S}\mathfrak{S}_q^{as}$ which preserves the principal graduation, and let one assume that it satisfies Lemma 2(b). Then σ is one of the $\sigma_{a,b,c,k,l}$ from Theorem 1(b).*

In order to prove Proposition 6 we will need the following result.

Lemma 7. *Let σ be as in Proposition 6, then*

$$\begin{aligned}
\sigma\left(z^n f\left(T_q\right) E_{11}\right) &= a_1^n q^{(n(n-1)/2)k_1} z^n f\left(b_1 q^{nm_1} T_q^{m_1}\right) T_q^{nk_1} E_{22},\\
\sigma\left(z^n f\left(T_q\right) E_{22}\right) &= a_2^n q^{(n(n-1)/2)k_2} z^n f\left(b_2 q^{nm_2} T_q^{m_2}\right) T_q^{nk_2} E_{11},\\
\sigma\left(z^n f\left(T_q\right) E_{12}\right) &= a_1^n c q^{(n(n-1)/2)k_1+nl} z^n f\left(b_1 q^{nm_1} T_q^{m_1}\right) T_q^{nk_1+l} E_{12},\\
\sigma\left(z^n f\left(T_q\right) E_{21}\right) &= -a_2 c^{-1} q^{(n(n-1)/2)k_2-nl} z^n f\left(b_2 q^{nm_2} T_q^{m_2}\right) T_q^{nk_2-l} E_{21},
\end{aligned} \tag{33}$$

where $a_i, b_i, c \in \mathbb{C}^\times$, $k_i, m_i, l \in \mathbb{Z}$, and $i = 1, 2$.

Proof. From hypothesis and Corollary 3(b), we have that

$$\begin{aligned}
E_{11} &= \sigma\left(T_q E_{22} T_q^{-1} E_{22}\right) = \sigma_{2,1}\left(T_q^{-1}\right) \sigma_{2,1}\left(T_q\right) E_{11},\\
E_{22} &= \sigma\left(T_q E_{11} T_q^{-1} E_{11}\right) = \sigma_{1,2}\left(T_q^{-1}\right) \sigma_{1,2}\left(T_q\right) E_{22},\\
E_{11} &= \sigma\left(z E_{22} z^{-1} E_{22}\right) = \sigma_{2,1}\left(z^{-1}\right) \sigma_{2,1}(z) E_{11},\\
E_{22} &= \sigma\left(z E_{11} z^{-1} E_{11}\right) = \sigma_{1,2}\left(z^{-1}\right) \sigma_{1,2}(z) E_{22},
\end{aligned} \tag{34}$$

$$E_{11} = -\sigma\left(E_{12}\right) \sigma\left(E_{21}\right) = -f_{12}\left(T_q\right) g_{21}\left(T_q\right) E_{11}. \tag{35}$$

Then, using (34), we have that

$$\begin{aligned}
\sigma_{i,j}\left(T_q^{\pm 1}\right) &= b_i^{\pm 1} T_q^{\pm m_i},\\
\sigma_{i,j}(z) &= a_i z T_q^{k_i},\\
\sigma_{i,j}\left(z^{-1}\right) &= a_i^{-1} q^{k_i} z^{-1} T_q^{-k_i},
\end{aligned} \tag{36}$$

and by induction we obtain that

$$\begin{aligned}
\sigma\left(T_q^m E_{ii}\right) &= b_i^m T_q^{m_i m} E_{jj}, \quad \forall m \in \mathbb{Z},\\
\sigma\left(z^n E_{ii}\right) &= a_i^n q^{(n(n-1)/2)k_i} z^n T_q^{nk_i} E_{jj}, \quad \forall n \in \mathbb{Z},
\end{aligned} \tag{37}$$

where $a_i, b_i \in \mathbb{C}^\times$, $k_i, m_i \in \mathbb{Z}$, $i, j = 1, 2$, $i \neq j$. Then, from (37) we have that

$$\begin{aligned}
\sigma\left(z^n f\left(T_q\right) E_{ii}\right) &= \sigma\left(f\left(T_q\right) E_{ii}\right) \sigma\left(z^n E_{ii}\right)\\
&= a_i^n q^{(n(n-1)/2)k_i} z^n f\left(b_i q^{nm_i} T_q^{m_i}\right) T_q^{nk_i} E_{jj},
\end{aligned} \tag{38}$$

with $i, j = 1, 2$, $i \neq j$.

On the other hand, from (35) we have that $f_{12}(T_q) = cT_q^l$ and $g_{21}(T_q) = -c^{-1}T_q^{-l}$ with $c \in \mathbb{C}^\times$ and $l \in \mathbb{Z}$; therefore,

$$\sigma\left(E_{12}\right) = cT_q^l E_{12}, \qquad \sigma\left(E_{21}\right) = -c^{-1} T_q^{-l} E_{21}. \tag{39}$$

Moreover,

$$\sigma\left(z^n f\left(T_q\right) E_{ij}\right) = \sigma\left(E_{ij}\right) \sigma\left(z^n f\left(T_q\right) E_{ii}\right), \tag{40}$$

with $i, j = 1, 2$, $i \neq j$. The proof follows from (38) and by replacing (38) and (39) in (40). □

Proof of Proposition 6. By Lemma 7, we have that

$$\begin{aligned}
T_q E_{11} &= \sigma^2\left(T_q E_{11}\right) = b_1 b_2^{m_1} T_q^{m_1 m_2} E_{11},\\
z E_{ii} &= \sigma^2\left(z E_{ii}\right) = a_i a_j b_j^{k_i} q^{m_j k_i} z T_q^{m_j k_i + k_j} E_{ii},\\
E_{12} &= \sigma^2\left(E_{12}\right) = b_1^l c^2 E_{12},
\end{aligned} \tag{41}$$

$$\sigma\left(z E_{12}\right) = a_1 c q^l z T_q^{k_1+l} E_{12}, \tag{42}$$

$$\sigma\left(z E_{12}\right) = \sigma\left(z E_{22}\right) \sigma\left(E_{12}\right) = a_2 c z T_q^{k_1+l} E_{12}. \tag{43}$$

Then, using (41), we have that

$$m_1 = m_2 = 1 \quad \text{or } m_1 = m_2 = -1, \tag{44}$$

$$b_1 = b_2^{-m_1}, \tag{45}$$

$$a_i a_j b_j^{k_i} q^{m_j k_i} = 1, \tag{46}$$

$$m_j k_i = -k_j, \tag{47}$$

$$b_1^l c^2 = 1, \tag{48}$$

where $i, j = 1, 2$, $i \neq j$. Suppose that $m_1 = m_2 = 1$; then, from (45) and (47) we have that $b_1 = b_2^{-1}$, $k_1 = -k_2$, and by replacing them in (46), we obtain that $q^{2k_1} = 1$, but since q is not a root of unity, we necessarily have that $k_1 = k_2 = 0$. Then, using Lemma 7 we get

$$\sigma\left(z T_q E_{11}\right) = a_1 b_1 q z T_q E_{22}; \tag{49}$$

moreover,

$$\sigma\left(z T_q E_{11}\right) = q^{-1} \sigma\left(z E_{11}\right) \sigma\left(T_q E_{11}\right) = a_1 b_1 q^{-1} z T_q E_{22}. \tag{50}$$

Comparing (49) with (50), we obtain that $q^2 = 1$ which is a contradiction. Therefore, by (44) and the aforementioned result, we get that

$$m_1 = m_2 = -1. \tag{51}$$

Then replacing (51) in (45) and (47) we obtain that

$$b_1 = b_2, \qquad k_1 = k_2. \tag{52}$$

On the other hand, comparing (42) with (43), we have that

$$a_2 = a_1 q^l, \tag{53}$$

and by replacing (51), (52), and (53) in (46), we obtain that

$$a_1^2 b_1^{k_1} = q^{k_1 - l}. \tag{54}$$

Using (48) and (54) and by replacing (51), (52), and (53) in (33), we obtain that $\sigma = \sigma_{a_1,b_1,c,k_1,l}$, finishing the proof. □

Proof of Theorem 1. It is straightforward to check that the two cases are anti-involutions. Reciprocally, from Lemma 2 and Propositions 4 and 6, it is clear that any anti-involution of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ which preserves the principal gradation satisfies (a) or (b), finishing the proof. □

Now, given an anti-involution σ of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$, one can check that the set of points minus σ-fixed is a subalgebra of $\mathcal{S}\mathfrak{S}_q$. Moreover, if σ preserves the principal $(1/2)\mathbb{Z}$-graduation, this subalgebra inherits the $(1/2)\mathbb{Z}$-graduation. These subalgebras are described in the last part of this section.

We define the following automorphisms of $\mathcal{S}\mathfrak{S}_q^{\mathrm{as}}$ by

$$\begin{gathered}\Theta_s(M) = \left(z^{-s} Id\right) \cdot M \cdot \left(z^s Id\right), \quad \forall M \in \mathcal{S}\mathfrak{S}_q^{\mathrm{as}},\\ \Phi_s\big|_{(\mathcal{S}\mathfrak{S}_q^{\mathrm{as}})_{\bar 0}} = id,\\ \Phi_s\left(z^n f\left(T_q\right) E_{12} + z^{n+1} g\left(T_q\right) E_{21}\right)\\ = s z^n f\left(T_q\right) E_{12} + s^{-1} z^{n+1} g\left(T_q\right) E_{21},\end{gathered} \tag{55}$$

with $s \in \mathbb{C}^\times$. On the other hand, given $n, m \in \mathbb{Z}$, we denote

$$\begin{aligned}&\mathbb{C}\left[w^{-1}, w\right]^{(n,m)}\\ &:= \left\{f \in \mathbb{C}\left[w^{-1}, w\right] : f(w) = -(-1)^n f\left(w^{-1}\right) w^m\right\}.\end{aligned} \tag{56}$$

Remark 8. We see that $f(w) \in \mathbb{C}[w^{-1}, w]^{(n,m)}$ with $f(w) = \sum f_j w^j$ if and only if $f_j = -(-1)^n f_{m-j}$, for all $j \in \mathbb{Z}$.

The Case $\sigma_{a,b,c,k}$. Let $a, b, c \in \mathbb{C}^\times$, $k \in \mathbb{Z}$, be such that $-ab^k = q^k$. We denote by $\mathcal{S}\mathfrak{S}_{q,k}^{a,b,c}$ the Lie subalgebra of $\mathcal{S}\mathfrak{S}_q$ consisting of $-\sigma_{a,b,c,k}$-fixed points; then, it inherits a $(1/2)\mathbb{Z}$-gradation from $\mathcal{S}\mathfrak{S}_q$; therefore, $\mathcal{S}\mathfrak{S}_{q,k}^{a,b,c} = \bigoplus_{j \in (1/2)\mathbb{Z}} (\mathcal{S}\mathfrak{S}_{q,k}^{a,b,c})_j$, where

$$\left(\mathcal{S}\mathfrak{S}_{q,k}^{a,b,c}\right)_j = \left\{M \in \left(\mathcal{S}\mathfrak{S}_q\right)_j : \sigma_{a,b,c,k}(M) = -M\right\}. \tag{57}$$

Moreover, we denote $\mathcal{S}\mathfrak{S}_{q,k} := \mathcal{S}\mathfrak{S}_{q,k}^{-q^k,1,1}$ and $\sigma_{q,k} := \sigma_{-q^k,1,1,k}$.

The following lemma gives a description of $\mathcal{S}\mathfrak{S}_{q,m}^{a,b,c}$.

Lemma 9. *Let a, b, c, k be as aforementioned; then, $\mathcal{S}\mathfrak{S}_{q,k}^{a,b,c} \simeq \mathcal{S}\mathfrak{S}_{q,k}$ and*

$$\begin{aligned}&\left(\mathcal{S}\mathfrak{S}_{q,k}\right)_n\\ &\quad = \left\{z^n f\left(q^{n/2} T_q\right) E_{11} + z^n g\left(q^{(n-1)/2} T_q\right) E_{22} : \right.\\ &\qquad \left. f, g \in \mathbb{C}\left[w^{-1}, w\right]^{(n,2nk)}\right\},\\ &\left(\mathcal{S}\mathfrak{S}_{q,k}\right)_{n+1/2}\\ &\quad = \left\{z^n f\left(T_q\right) E_{12} - (-1)^n q^{(n+1)nk} z^{n+1} f\left(q^{-n} T_q^{-1}\right) T_q^{(2n+1)k} E_{21} : \right.\\ &\qquad \left. f \in \mathbb{C}\left[w^{-1}, w\right]\right\},\end{aligned} \tag{58}$$

for all $n \in \mathbb{Z}$.

Proof. First, it is easy to check that

$$\begin{gathered}\Theta_{-s} \sigma_{a,b,c,k} \Theta_s = \sigma_{aq^{-2ks}, bq^{2s}, cq^{-ks}, k},\\ \Phi_{-t} \sigma_{a,b,c,k} \Phi_t = \sigma_{a,b,ct^2,k}.\end{gathered} \tag{59}$$

Then if we take $s, t \in \mathbb{C}^\times$ such that $b^{-1} = q^{2s}$ and $t^2 = c^{-1} q^{ks}$, from (59) and the relation between a, b, c, k, we obtain that $\mathcal{S}\mathfrak{S}_{q,k}^{a,b,c} \simeq \mathcal{S}\mathfrak{S}_{q,k}$.

On the other hand, by Theorem 1(a) and linearity of $\sigma_{q,k}$ we obtain that (for $n \in \mathbb{Z}$) $z^n f(T_q) E_{11} + z^n g(T_q) E_{22} \in (\mathcal{S}\mathfrak{S}_{q,k})_n$ if and only if $z^n f(T_q) E_{11}, z^n g(T_q) E_{22} \in (\mathcal{S}\mathfrak{S}_{q,k})_n$. Now, let $z^n h(T_q) E_{11} \in (\mathcal{S}\mathfrak{S}_{q,k})_n$ be with $h(w) = \sum h_j w^j$; then,

$$\begin{aligned}-z^n h\left(T_q\right) E_{11} &= \sigma_{q,k}\left(z^n h\left(T_q\right) E_{11}\right)\\ &= (-1)^n q^{n^2 k} z^n h\left(q^{-n} T_q^{-1}\right) T_q^{2nk} E_{11}\end{aligned} \tag{60}$$

if and only if $q^{-(n/2)j} h_j = -(-1)^n q^{(-n/2)(2nk-j)} h_{2nk-j}$, for all $j \in \mathbb{Z}$, which is equivalent to $f(w) = h(q^{-n/2} w) \in \mathbb{C}[w^{-1}, w]^{(n,2nk)}$ (see Remark 8). Therefore,

$$\begin{gathered}z^n h\left(T_q\right) E_{11} = z^n f\left(q^{n/2} T_q\right) E_{11},\\ \text{with } f(w) \in \mathbb{C}\left[w^{-1}, w\right]^{(n,2nk)}.\end{gathered} \tag{61}$$

Similarly, we prove that $z^n h(T_q) E_{22} \in (\mathcal{S}\mathfrak{S}_{q,k})_n$ if and only if

$$\begin{gathered}z^n h\left(T_q\right) E_{22} = z^n g\left(q^{(n-1)/2} T_q\right) E_{22},\\ \text{where } g(w) \in \mathbb{C}\left[w^{-1}, w\right]^{(n,2nk)}.\end{gathered} \tag{62}$$

Now, we suppose that $z^n f(T_q)E_{12} + z^{n+1}g(T_q)E_{21} \in (\mathcal{S}\mathfrak{S}_{q,k})_{n+1/2}$; then,

$$\begin{aligned} &- z^n f\left(T_q\right) E_{12} - z^{n+1} g\left(T_q\right) E_{21} \\ &\quad = \sigma_{q,k}\left(z^n f\left(T_q\right) E_{12} + z^{n+1} g\left(T_q\right) E_{21}\right) \\ &\quad = (-1)^n q^{n^2 k}\left(z^n g\left(q^{-n}T_q^{-1}\right) T_q^{(2n+1)k} E_{12}\right. \\ &\qquad\qquad \left. + q^{nk} z^{n+1} f\left(q^{-n}T_q^{-1}\right) T_q^{(2n+1)k} E_{21}\right), \end{aligned} \tag{63}$$

if and only if $g(T_q) = -(-1)^n q^{(n+1)nk} f(q^{-n}T_q^{-1})T_q^{(2n+1)k}$, finishing the proof. □

The Case $\sigma_{a,b,c,k,l}$. Let $a,b,c \in \mathbb{C}^\times$, $k,l \in \mathbb{Z}$, be such that $a^2 b^k = q^{k-l}$ and $b^l c^2 = 1$. We denote by $\mathcal{S}\mathfrak{S}^{a,b,c}_{q,k,l}$ the Lie subalgebra of $\mathcal{S}\mathfrak{S}_q$ consisting of $-\sigma_{a,b,c,k,l}$-fixed; then, it inherits a $(1/2)\mathbb{Z}$-gradation from $\mathcal{S}\mathfrak{S}_q$; thus $\mathcal{S}\mathfrak{S}^{a,b,c}_{q,k,l} = \bigoplus_{j\in(1/2)\mathbb{Z}}(\mathcal{S}\mathfrak{S}^{a,b,c}_{q,k,l})_j$, where

$$\left(\mathcal{S}\mathfrak{S}^{a,b,c}_{q,k,l}\right)_j = \left\{M \in \left(\mathcal{S}\mathfrak{S}_q\right)_j : \sigma_{a,b,c,k,l}(M) = -M\right\}. \tag{64}$$

Moreover, we denote $\mathcal{S}\mathfrak{S}^{+\pm}_{q,k,l} := \mathcal{S}\mathfrak{S}^{a,1,\pm1}_{q,k,l}$, $\sigma^{+\pm}_{q,k,l} := \sigma_{a,1,\pm1,k,l}$ and $\mathcal{S}\mathfrak{S}^{-,\pm}_{q,k,l} := \mathcal{S}\mathfrak{S}^{-a,1,\pm1}_{q,k,l}$, $\sigma^{-,\pm}_{q,k,l} := \sigma_{-a,1,\pm1,k,l}$, with $a = q^{(k-l)/2}$. The following lemma gives a description of $\mathcal{S}\mathfrak{S}^{a,b,c}_{q,k,l}$. We will need the following notation:

$$\delta_n := \begin{cases} 0, & \text{in } \left(\mathcal{S}\mathfrak{S}^{+,+}_{q,k,l}\right)_{n+1/2}; \\ 1, & \text{in } \left(\mathcal{S}\mathfrak{S}^{+,-}_{q,k,l}\right)_{n+1/2}; \\ n, & \text{in } \left(\mathcal{S}\mathfrak{S}^{-,+}_{q,k,l}\right)_{n+1/2}; \\ n+1, & \text{in } \left(\mathcal{S}\mathfrak{S}^{-,-}_{q,k,l}\right)_{n+1/2}. \end{cases} \tag{65}$$

Lemma 10. *Let a, b, c, k, l be as aforementioned; then, $\mathcal{S}\mathfrak{S}^{a,b,c}_{q,k,l}$ is isomorphic to some of the following algebras: $\mathcal{S}\mathfrak{S}^{+,+}_{q,k,l}$, $\mathcal{S}\mathfrak{S}^{+,-}_{q,k,l}$, $\mathcal{S}\mathfrak{S}^{-,+}_{q,k,l}$, or $\mathcal{S}\mathfrak{S}^{-,-}_{q,k,l}$. And*

$$\begin{aligned} \left(\mathcal{S}\mathfrak{S}^{+\pm}_{q,k,l}\right)_n &= \left\{z^n f\left(T_q\right) E_{11} - (\pm1)^n q^{n(nk-l)/2}\right. \\ &\qquad \left. \times z^n f\left(q^{-n}T_q^{-1}\right) T_q^{nk} E_{22} : f \in \mathbb{C}\left[w^{-1},w\right]\right\}, \\ \left(\mathcal{S}\mathfrak{S}^{+\pm}_{q,k,l}\right)_{n+1/2} &= \left\{z^n f\left(q^{n/2}T_q\right) E_{12} + z^{n+1} g\left(q^{(n+1)/2}T_q\right)\right. \\ &\qquad \times E_{21} : f \in \mathbb{C}\left[w^{-1},w\right]^{(\delta_n, nk+l)}, \\ &\qquad \left. g \in \mathbb{C}\left[w^{-1},w\right]^{(\delta_n+1,(n+1)k-l)}\right\}, \\ \left(\mathcal{S}\mathfrak{S}^{-\pm}_{q,k,l}\right)_n &= \left\{z^n f\left(T_q\right) E_{11} - (\pm1)^n q^{n(nk-l)/2}\right. \\ &\qquad \left. \times z^n f\left(q^{-n}T_q^{-1}\right) T_q^{nk} E_{22} : \ f \in \mathbb{C}\left[w^{-1},w\right]\right\}, \\ \left(\mathcal{S}\mathfrak{S}^{-\pm}_{q,k,l}\right)_{n+1/2} &= \left\{z^n f\left(q^{n/2}T_q\right) E_{12} + z^{n+1} g\left(q^{(n+1)/2}T_q\right)\right. \\ &\qquad \times E_{21} : f \in \mathbb{C}\left[w^{-1},w\right]^{(\delta_n, nk+l)}, \\ &\qquad \left. \in \mathbb{C}\left[w^{-1},w\right]^{(\delta_n,(n+1)k-l)}\right\}, \end{aligned} \tag{66}$$

for all $n \in \mathbb{Z}$.

Proof. It is easy to check that,

$$\Theta_{-s}\sigma_{a,b,c,k,l}\Theta_s = \sigma_{aq^{-sk},bq^{2s},cq^{-sl},k,l}; \tag{67}$$

then; if we take s such that $q^{2s} = b^{-1}$, we obtain the first assertion using (67) and the relations between a,b,c,k,l.

On the other hand, we suppose that $z^n f(T_q)E_{11} + z^n g(T_q)E_{22} \in (\mathcal{S}\mathfrak{S}^{+\pm}_{q,k,l})_n$; then,

$$\begin{aligned} &- z^n f\left(T_q\right) E_{11} - z^n g\left(T_q\right) E_{22} \\ &\quad = \sigma^{+\pm}_{q,k,l}\left(z^n f\left(T_q\right) E_{11} + z^n g\left(T_q\right) E_{22}\right) \\ &\quad = (\pm1)^n q^{n(nk-l)/2} z^n f\left(q^{-n}T_q^{-1}\right) T_q^{nk} E_{22} \\ &\qquad + (\pm1)^n q^{n(nk+l)/2} z^n g\left(q^{-n}T_q^{-1}\right) T_q^{nk} E_{11} \end{aligned} \tag{68}$$

if and only if $g(T_q) = -(\pm1)^n q^{n(nk-l)/2} f(q^{-n}T_q^{-1})T_q^{nk}$. The proof for $(\mathcal{S}\mathfrak{S}^{+\pm}_{q,k,l})_n$ is similar.

Now, by Theorem 1(b) and linearity of $\sigma^{++}_{q,k,l}$, we obtain that $z^n f(T_q)E_{12} + z^{n+1}g(T_q)E_{21} \in (\mathcal{S}\mathfrak{S}^{++}_{q,k,l})_{n+1/2}$ if and only if $z^n f(T_q)E_{12}$ and $z^n g(T_q)E_{21}$ are elements in $(\mathcal{S}\mathfrak{S}^{++}_{q,k,l})_{n+1/2}$. We suppose that $z^{n+1}h(T_q)E_{21} \in (\mathcal{S}\mathfrak{S}^{++}_{q,k,l})_{n+1/2}$, with $h(w) = \sum h_j w^j$; then,

$$\begin{aligned} &- z^{n+1} h\left(T_q\right) E_{21} \\ &\quad = \sigma^{++}_{k,l}\left(z^{n+1} h\left(T_q\right) E_{21}\right) \\ &\quad = -q^{(n+1)((n+1)k-l)/2} z^{n+1} h\left(q^{-(n+1)}T_q^{-1}\right) T_q^{(n+1)k-l} E_{21} \end{aligned} \tag{69}$$

if and only if $h_j q^{-((n+1)/2)j} = q^{-((n+1)/2)((n+1)k-l-j)} h_{(n+1)k-l-j}$, for all $j \in \mathbb{Z}$, which is equivalent to $g(w) = h(q^{-((n+1)/2)}w) \in \mathbb{C}[w^{-1},w]^{(1,(n+1)k-l)}$ (see Remark 8). Therefore,

$$\begin{aligned} z^{n+1} h\left(T_q\right) E_{21} &= z^{n+1} g\left(q^{(n+1)/2}T_q\right) E_{21}, \\ \text{with } g(w) &\in \mathbb{C}\left[w^{-1},w\right]^{(1,(n+1)k-l)}. \end{aligned} \tag{70}$$

Similarly, we prove that $z^n h(T_q)E_{12} \in (\mathcal{S}\mathfrak{S}^{++}_{q,k,l})_{n+1/2}$ if and only if $z^n h(T_q)E_{12} = z^n f(q^{n/2}T_q)E_{12}$, with $f \in \mathbb{C}[w^{-1},w]^{(0,nk+l)}$.

The other proofs are similar. □

Remark 11. From Lemma 9 (resp., Lemma 10), it is clear that an element in $(\mathcal{S}\mathfrak{S}_{q,k})_{n+1/2}$ (resp., $(\mathcal{S}\mathfrak{S}^{\pm,\pm}_{q,k,l})_n$) is totally determined by its value in the position E_{12} (resp., E_{11}), where $n \in \mathbb{Z}$.

Finally, we denote by $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ and $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$ the central extensions of $\mathcal{S}\mathfrak{S}_{q,k}$ and $\mathcal{S}\mathfrak{S}^{\pm\pm}_{q,k,l}$ corresponding to the restrictions of the 2-cocycle ψ, respectively. It is clear that these subalgebras admit a $(1/2)\mathbb{Z}$-graduation compatible with their $\mathbb{Z}_2$-graduation; that is,

$$\widehat{\mathcal{S}\mathfrak{S}}_{q,k} = \left(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}\right)_{\bar{0}} \bigoplus \left(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}\right)_{\bar{1}},$$
$$\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l} = \left(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}\right)_{\bar{0}} \bigoplus \left(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}\right)_{\bar{1}}, \tag{71}$$

where

$$\left(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}\right)_{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} \left(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}\right)_{n},$$
$$\left(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}\right)_{\bar{1}} = \bigoplus_{n\in\mathbb{Z}} \left(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}\right)_{n+1/2},$$
$$\left(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}\right)_{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} \left(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}\right)_{n},$$
$$\left(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}\right)_{\bar{1}} = \bigoplus_{n\in\mathbb{Z}} \left(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}\right)_{n+1/2}, \tag{72}$$

with $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_n = (\mathcal{S}\mathfrak{S}_{q,k})_n + \delta_{n,0}\mathbb{C}C$ and $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_n = (\mathcal{S}\mathfrak{S}^{\pm\pm}_{q,k,l})_n + \delta_{n,0}\mathbb{C}C$, for all $n \in (1/2)\mathbb{Z}$.

3. Quasifinite Highest Weight Modules over $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ and $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$

The goal of this section is to characterize the quasifinite irreducible highest weight modules over $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ and $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$; for this we will apply the general results on quasifinite representations of $(1/2)\mathbb{Z}$-graded Lie superalgebras developed in Section 2 in [6]. Let us recall some general definition and results from [6].

In this section, $\mathfrak{g}$ denote a consistent $(1/2)\mathbb{Z}$-graded Lie superalgebra over $\mathbb{C}$; namely,

$$\mathfrak{g} = \bigoplus_{j\in(1/2)\mathbb{Z}} g_j, \quad \text{where } \left[\mathfrak{g}_i, \mathfrak{g}_j\right] \subset \mathfrak{g}_{i+j} \quad \forall i, j \in \frac{1}{2}\mathbb{Z}, \tag{73}$$

and also

$$\mathfrak{g}_{\bar{0}} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j, \qquad \mathfrak{g}_{\bar{1}} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{j+1/2}. \tag{74}$$

We denote $\mathfrak{g}_\pm = \bigoplus_{j\in(1/2)\mathbb{Z}_+} \mathfrak{g}_{\pm j}$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *parabolic* if

$$\mathfrak{p} = \bigoplus_{j\in(1/2)\mathbb{Z}} \mathfrak{p}_j, \quad \text{where } p_j = g_j \quad \forall j \geqslant 0,$$
$$\mathfrak{p}_{-j} \neq 0, \text{ for some } j \in \mathbb{N}. \tag{75}$$

We assume the following properties on $\mathfrak{g}$:

(SP1) $\mathfrak{g}_0$ is commutative,

(SP2) if $a \in \mathfrak{g}_{-j}$ ($j \in (1/2)\mathbb{N}$) and $[a, \mathfrak{g}_{1/2}] = 0$, then $a = 0$.

Given that $a \in \mathfrak{g}_{-1/2}$ is nonzero, we define $\mathfrak{p}^a = \bigoplus_{j\in(1/2)\mathbb{Z}} \mathfrak{p}^a_j$, where

$$\mathfrak{p}^a_j = \mathfrak{g}_j \quad \forall j \geq 0,$$
$$\mathfrak{p}^a_{-1/2} = \sum \left[\ldots\left[\left[a, \mathfrak{g}_0\right], \mathfrak{g}_0\right], \ldots\right],$$
$$\mathfrak{p}^a_{-k-1/2} = \left[\mathfrak{p}^a_{-1/2}, \mathfrak{p}^a_{-k}\right]. \tag{76}$$

It was proved in [6] that $\mathfrak{p}^a$ is the minimal parabolic subalgebra containing a and also that $[\mathfrak{p}^a, \mathfrak{p}^a] \cap \mathfrak{g}_0 = [a, \mathfrak{g}_{1/2}]$.

Definition 12. (a) A parabolic subalgebra $\mathfrak{p}$ is called nondegenerate if $\mathfrak{p}_{-j}$ has finite codimension in $\mathfrak{g}_{-j}$, for all $j \in (1/2)\mathbb{N}$.

(b) An element $a \in \mathfrak{g}_{-1/2}$ is called nondegenerate if $\mathfrak{p}^a$ is nondegenerate.

We will also require the following condition on $\mathfrak{g}$.

(SP3) If $\mathfrak{p}$ is a nondegenerate parabolic subalgebra of $\mathfrak{g}$, then there exists a nondegenerate element $a \in \mathfrak{p}_{-1/2}$.

A $\mathfrak{g}$-module $V = \bigoplus_{j\in(1/2)\mathbb{Z}} V_j$ is called *quasifinite* if $\dim V_j < \infty$ for all j. Given $\lambda \in \mathfrak{g}_0^*$, a *highest weight module* is a $(1/2)\mathbb{Z}$-graded $\mathfrak{g}$-module $V(\mathfrak{g}, \lambda) = \bigoplus_{j\in(1/2)\mathbb{Z}_+} V_{-j}$, defined by the following properties:

(a) $V_0 = \mathbb{C}v_\lambda$, where v_λ is a nonzero vector;

(b) $hv_\lambda = \lambda(h)v_\lambda$, for all $h \in \mathfrak{g}_0$;

(c) $\mathfrak{g}_+ v_\lambda = 0$;

(d) $\mathcal{U}(\mathfrak{g}_-)v_\lambda = V(\mathfrak{g}, \lambda)$.

A nonzero vector $v \in V(\mathfrak{g}, \lambda)$ is called *singular* if $\mathfrak{g}_+ v = 0$.

The *Verma module* over $\mathfrak{g}$ is defined as usual:

$$M(\mathfrak{g}, \lambda) = \mathcal{U}(\mathfrak{g}) \bigotimes_{\mathcal{U}(\mathfrak{g}_0 \bigoplus \mathfrak{g}_+)} \mathbb{C}_\lambda, \tag{77}$$

where $\mathbb{C}_\lambda := \mathbb{C}c_\lambda$ is the one-dimensional $(\mathfrak{g}_0 \bigoplus \mathfrak{g}_+)$-module given by $hc_\lambda = \lambda(h)c_\lambda$ if $h \in \mathfrak{g}_0$, and $\mathfrak{g}_+ c_\lambda = 0$ and the action of $\mathfrak{g}$ on $M(\mathfrak{g}, \lambda)$ is induced by the left multiplication in $\mathcal{U}(\mathfrak{g})$. Any highest weight module $V(\mathfrak{g}, \lambda)$ is a quotient module of $M(\mathfrak{g}, \lambda)$. The irreducible module $L(\mathfrak{g}, \lambda)$ is the quotient of $M(\mathfrak{g}, \lambda)$ by the maximal proper graded submodule.

Now, let $\mathfrak{p} = \bigoplus_{j\in(1/2)\mathbb{Z}} \mathfrak{p}_j$ be a parabolic subalgebra of $\mathfrak{g}$ and let $\lambda \in \mathfrak{g}_0^*$ be such that $\lambda|_{[\mathfrak{p},\mathfrak{p}]\cap\mathfrak{g}_0} = 0$. Then the $(\mathfrak{g}_0 \bigoplus \mathfrak{g}_+)$-module $\mathbb{C}_\lambda$ extends to a $\mathfrak{p}$-module by letting $\mathfrak{p}_j c_\lambda = 0$ for all $j < 0$, and we may construct the highest weight module

$$M(\mathfrak{p}, \mathfrak{g}, \lambda) = \mathcal{U}(\mathfrak{g}) \bigotimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_\lambda, \tag{78}$$

called the *generalized Verma module*. Clearly all these highest weight modules are graded. The following result gives the characterization of all irreducible quasifinite highest-weight modules.

Theorem 13. *Let $\mathfrak{g} = \bigoplus_{j\in(1/2)\mathbb{Z}}\mathfrak{g}_j$ be a consistent $(1/2)\mathbb{Z}$-graded Lie superalgebra over $\mathbb{C}$ that satisfies conditions (SP1), (SP2), and (SP3). The following conditions on $\lambda \in \mathfrak{g}_0^*$ are equivalent.*

(a) *$M(\mathfrak{g},\lambda)$ contains a singular vector $a\cdot v_\lambda$ in $M(\mathfrak{g},\lambda)_{-1/2}$, where a is nondegenerate.*

(b) *There exists a nondegenerate element $a \in \mathfrak{g}_{-1/2}$, such that $\lambda([\mathfrak{g}_{1/2}, a]) = 0$.*

(c) *$L(\mathfrak{g},\lambda)$ is quasifinite.*

(d) *There exists a nondegenerate element $a \in \mathfrak{g}_{-1/2}$, such that $L(\mathfrak{g},\lambda)$ is the irreducible quotient of the generalized Verma module $M(\mathfrak{g},\mathfrak{p}^a,\lambda)$.*

Proof. See [6]. □

Moreover, we will need the following result. Recall that a quasipolynomial is a combination of functions of the form $p(x)q^{\alpha x}$, where $p(x)$ is a polynomial and $\alpha \in \mathbb{C}$. That is, it satisfies a nontrivial linear differential equation with constant coefficients.

Proposition 14. *Given a quasipolynomial P and a polynomial $B(x) = \prod_i(x - A_i)$, take $b(x) = \prod_i(x - a_i)$ where $a_i = \exp(A_i)$; then, $b(x)(\sum_{n\in\mathbb{Z}} P(n)x^{-n}) = 0$ if and only if $B(d/dx)P(x) = 0$.*

Below, we prove that $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ and $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$ satisfy the properties (SP1), (SP2), and (SP3), which is equivalent to study of its parabolic subalgebras. Then using Theorem 13 and Proposition 14, we obtain two equivalent characterizations of the quasifinite highest weight modules of these algebras. Before studying each particular case, we will consider the following useful result.

Lemma 15. *Let $A = z^n f_1(T_q)E_{11} + z^n f_2(T_q)E_{22}$ and $z^m g(T_q)E_{ij}$ be nonzero elements in $\widehat{\mathcal{S}\mathfrak{S}}_q$, where $n \neq 0$, $i, j = 1, 2$ and let $h \in \mathbb{C}[w^{-1}, w]$ be nonconstant. Then one has the following.*

(a) *$[A, z^m g(T_q)E_{ij}] \neq 0$ or $[A, z^m g(T_q)h(T_q)E_{ij}] \neq 0$, with $i \neq j$.*

(b) *If $f_i \neq 0$, then $[A, z^m g(T_q)E_{ii}] \neq 0$ or $[A, z^m g(T_q)h(T_q)E_{ii}] \neq 0$.*

Proof. (a) We suppose that $[A, z^m g(T_q)E_{12}] = 0$ and $[A, z^m g(T_q)h(T_q)E_{12}] = 0$; then,

$$f_1(q^m T_q)\, g(T_q) = f_2(T_q)\, g(q^n T_q), \tag{79}$$

$$f_1(q^m T_q)\, g(T_q)\, h(T_q) = f_2(T_q)\, g(q^n T_q)\, h(q^n T_q). \tag{80}$$

Using (79) and the hypothesis, we have that f_1 and f_2 are nonzero. Then, if we replace (79) with (80), we obtain $h(T_q) = h(q^n T_q)$ which is a contradiction since h is not constant, $n \neq 0$, and q is not unity root. The other case is similar.

The proof of (b) is similar to the proof of (a). □

3.1. The Case $\mathfrak{g} = \widehat{\mathcal{S}\mathfrak{S}}_{q,k}$. It is clear that $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ satisfies (SP1). Now, we suppose that $A = z^{-n} f(T_q)E_{11} + z^{-n} g(T_q)E_{22} \in (\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-n}$ (with $n \in \mathbb{N}$) satisfies $[A, (\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{1/2}] = 0$; in particular for all $i \in \mathbb{Z}$

$$\left[A, T_q^i E_{12} - z T_q^{k-i} E_{21}\right] = 0. \tag{81}$$

Then using (81), we obtain that $f(T_q) = q^{-ni} g(T_q)$ for all $i \in \mathbb{Z}$. Hence, $f = g = 0$ (since $n \neq 0$ and q is not a root of unity); therefore, $A = 0$. Now, we suppose that $A = z^{-n} f(T_q)E_{12} - (-1)^{-n} q^{(n-1)nk} z^{-n+1} f(q^n T_q^{-1}) T_q^{(-2n+1)k} E_{21}$ in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-n+1/2}$; (with $n \in \mathbb{N}$) satisfies $[A, (\mathcal{S}\mathfrak{S}_{q,k})_{1/2}] = 0$ then, for all $i \in \mathbb{Z}$ we have that

$$\left[A, T_q^i E_{12} - z T_q^{k-i} E_{21}\right] = 0, \tag{82}$$

which is equivalent to $f(T_q) = q^{nk-i} f(qT_q)$ for all $i \in \mathbb{Z}$. Hence $f = 0$; therefore, $A = 0$. Thus, we prove that $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ satisfies the property (SP2).

Finally, using the following lemma, we will prove that $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ satisfies the property (SP3).

Lemma 16. *Let $\mathfrak{p} = \bigoplus_{j\in(1/2)\mathbb{Z}}\mathfrak{p}_j$ be a $(1/2)\mathbb{Z}$-graded subalgebra of $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$, where $\mathfrak{p}_0 = (\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_0$. Then one has the following.*

(a) *For each $n \in \mathbb{Z}$, $\mathfrak{p}_{n+1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{n+1/2}$ if and only if $\mathfrak{p}_{n+1/2} \neq 0$.*

(b) *For each $n \in \mathbb{Z}$, $\mathfrak{p}_n$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_n$ if and only if there exists $z^n f(q^{n/2}T_q)E_{11} + z^n g(q^{(n-1)/2}T_q)E_{22} \in \mathfrak{p}_n$, such that f and g are nonzero.*

(c) *$\mathfrak{p}_{-n}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-n}$, for all $n \in (1/2)\mathbb{N}$ if and only if $\mathfrak{p}_{-1/2} \neq 0$.*

Proof. (a) We suppose that there exists

$$\begin{aligned} A = {} & z^n f(T_q)\, E_{12} - (-1)^n q^{(n+1)nk} z^{n+1} \\ & \times f(q^{-n} T_q^{-1})\, T_q^{(2n+1)k} E_{21} \in \mathfrak{p}_{n+1/2}, \end{aligned} \tag{83}$$

with $f \neq 0$; then $M_1^j := [A, (T_q^j - T_q^{-j})E_{11}]$ and $M_2^j := [A, (q^{-j/2}T_q^j - q^{j/2}T_q^{-j})E_{22}]$ belong to $\mathfrak{p}_{n+1/2}$, for all $j \in \mathbb{Z}$. Moreover, $A^j = (q^{-(n+1)j} - q^{nj})^{-1}(M_1^j + q^{-(n+1/2)j} M_2^j) \in \mathfrak{p}_{n+1/2}$ and

$$\begin{aligned} A^j = {} & z^n f(T_q)\, T_q^j E_{12} - (-1)^n q^{(n+1)nk-nj} z^{n+1} \\ & \times f(q^{-n} T_q^{-1})\, T_q^{(2n+1)k-j} E_{21}, \end{aligned} \tag{84}$$

for all $j \in \mathbb{Z}^\times$. Therefore, using (84), we have that

$$\begin{aligned} & z^n g(T_q)\, E_{12} - (-1)^n q^{(n+1)nk} z^{n+1} \\ & \times g(q^{-n} T_q^{-1})\, T_q^{(2n+1)k} E_{21} \in \mathfrak{p}_{n+1/2}, \end{aligned} \tag{85}$$

for all $g(w) \in \langle f(w) \rangle$, where $\langle f(w) \rangle$ is the ideal of $\mathbb{C}[w^{-1}, w]$ generated by $f(w)$. Using that $\langle f(w) \rangle$ has finite codimension in $\mathbb{C}[w^{-1}, w]$, we obtain that $\mathfrak{p}_{n+1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{n+1/2}$ (see Remark 11). The proof of the converse is trivial.

(b) We suppose that $A = z^n f(q^{n/2} T_q) E_{11} + z^n g(q^{(n-1)/2} T_q) E_{22} \in \mathfrak{p}_n$, with nonzero f and g in $\mathbb{C}[w^{-1}, w]^{(n,2nk)}$; then similar to the aforementioned argument

$$\begin{aligned} &\left(q^{-nj/2} - q^{nj/2}\right)^{-1} \left[A, \left(T_q^j - T_q^{-j}\right) E_{11}\right] \\ &\quad = z^n f\left(q^{n/2} T_q\right) \left(q^{nj/2} T_q^j + q^{-nj/2} T_q^{-j}\right) E_{11} \in \mathfrak{p}_n, \\ &b\left[A, \left(q^{-j/2} T_q^j - q^{j/2} T_q^{-j}\right) E_{22}\right] \\ &\quad = z^n g\left(q^{(n-1)/2} T_q\right) \\ &\qquad \times \left(q^{(n-1)j/2} T_q^j + q^{-(n-1)j/2} T_q^{-j}\right) E_{22} \in \mathfrak{p}_n, \end{aligned} \tag{86}$$

where $b = (q^{(-n/2)j} - q^{(n/2)j})^{-1}$, and this is true for all $j \in \mathbb{Z}^{\times}$. Therefore

$$\begin{aligned} &z^n h_1\left(q^{n/2} T_q\right) E_{11} \in \mathfrak{p}_n, \quad \forall h_1 \in f(w)\, \mathbb{C}\left[w^{-1}, w\right]^{(1,0)}, \\ &z^n h_2\left(q^{(n-1)/2} T_q\right) E_{22} \in \mathfrak{p}_n, \quad \forall h_2 \in g(w)\, \mathbb{C}\left[w^{-1}, w\right]^{(1,0)}. \end{aligned} \tag{87}$$

Since $f(w)\mathbb{C}[w^{-1}, w]^{(1,0)}$ and $g(w)\mathbb{C}[w^{-1}, w]^{(1,0)}$ have finite codimension in $\mathbb{C}[w^{-1}, w]^{(n,2nk)}$, we obtain that $\mathfrak{p}_n$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_n$. The proof of the converse is trivial.

(c) We suppose that $\mathfrak{p}_{-1/2} \neq 0$; then, in order to prove that $\mathfrak{p}_{-n}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-n}$, for all $n \in (1/2)\mathbb{N}$, we only need to see that this is true for all $n \in \mathbb{N}$, since by using (SP2), we obtain that $\mathfrak{p}_{-n+1/2} \neq 0$ for all $n \in \mathbb{N}$; then, from (a) we have that $\mathfrak{p}_{-n+1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-n+1/2}$ for all $n \in \mathbb{N}$. By hypothesis, there exists $A = z^{-1} f(T_q) E_{12} + f(q T_q^{-1}) T_q^{-k} E_{21} \in \mathfrak{p}_{-1/2}$ with $f \neq 0$; then, using (84), we have that $B = z^{-1} f(T_q) T_q E_{12} + q f(q T_q^{-1}) T_q^{-k-1} E_{21} \in \mathfrak{p}_{-1/2}$, and

$$[A, B] := z^{-1} f_1\left(q^{-1/2} T_q\right) E_{11} + z^{-1} g_1\left(q^{-1} T_q\right) E_{22} \in \mathfrak{p}_{-1}, \tag{88}$$

where f_1 and g_1 are nonzero. Hence, by (b), $\mathfrak{p}_{-1}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-1}$. Moreover, by (87) we obtain that

$$z^{-1} f_1\left(q^{-1/2} T_q\right) h\left(q^{-1/2} T_q\right) E_{11} \in \mathfrak{p}_{-1}, \tag{89}$$

$$z^{-1} g_1\left(q^{-1} T_q\right) h\left(q^{-1} T_q\right) E_{22} \in \mathfrak{p}_{-1}, \tag{90}$$

for all $h(w) \in \mathbb{C}[w^{-1}, w]^{(1,0)}$. Now, by induction we suppose that $\mathfrak{p}_{-n}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-n}$; then, there exists $A = z^{-n} f(q^{-n/2} T_q) E_{11} + z^{-n} g(q^{-(n+1)/2} T_q) E_{22} \in \mathfrak{p}_{-n}$ where f and g are nonzero, and by (89) and Lemma 15(b),

$$\begin{aligned} &\left[A, z^{-1} f_1\left(q^{-1/2} T_q\right) E_{11}\right] \neq 0 \\ &\text{or } \left[A, z^{-1} f_1\left(q^{-1/2} T_q\right) h\left(q^{-1/2} T_q\right) E_{11}\right] \neq 0, \end{aligned} \tag{91}$$

for some no constant $h(w)$ in $\mathbb{C}[w^{-1}, w]^{(1,0)}$; therefore, there exists a nonzero element $z^{-n-1} \widetilde{f}(T_q) E_{11}$ in $\mathfrak{p}_{-n-1}$. Similarly, using (90) and Lemma 15(b), we see that there exists a nonzero element $z^{-n-1} \widetilde{g}(T_q) E_{22}$ in $\mathfrak{p}_{-n-1}$; then, using (b), $\mathfrak{p}_{-n-1}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-n-1}$, finishing the induction. The proof of the converse is trivial. □

Corollary 17. *(a) Any parabolic subalgebra of $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ is nondegenerate.*

(b) Any nonzero element of $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-1/2}$ is nondegenerate.

(c) $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$ satisfies (SP3).

Proof. Let $\mathfrak{p}$ be a parabolic subalgebra of $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$; by definition there exists $j \in (1/2)\mathbb{N}$ such that $\mathfrak{p}_{-j} \neq 0$, and then by (SP2) $\mathfrak{p}_{-1/2} \neq 0$; the proof of (a) follows from Lemma 16(c). Finally, (b) follows from (a), and (c) follows from (b). □

A functional $\lambda \in (\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_0^*$ is described by its labels $\Delta_{n,1} = -\lambda((T_q^n - T_q^{-n}) E_{11})$, $\Delta_{n,2} = -\lambda((q^{-n} T_q^n - T_q^{-n}) E_{22})$ with $n \in \mathbb{Z}$, and the central charge $\lambda(C) = c$. We will consider the generating series

$$\Delta_{\lambda,i}(x) = \sum_{n \in \mathbb{Z}} \Delta_{n,i} x^{-n}, \quad \text{with } i = 1, 2. \tag{92}$$

Theorem 18. *An irreducible highest weight $\widehat{\mathcal{S}\mathfrak{S}}_{q,k}$-module $L(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}, \lambda)$ is quasifinite if and only if one of the following equivalent conditions holds.*

(a) *There exists a Laurent polynomial $b(x)$ such that*

$$b(qx)\left(\Delta_{\lambda,1}(x) + \Delta_{\lambda,2}(x) + c\right) = 0. \tag{93}$$

(b) *There exists a quasipolynomial $P(x)$ such that*

$$P(n) = \Delta_{n,1} + \Delta_{n,2} + \delta_{n,0} c, \tag{94}$$

for all $n \in \mathbb{Z}$.

Proof. By Theorem 13, $L(\widehat{\mathcal{S}\mathfrak{S}}_{q,k}, \lambda)$ is quasifinite if and only if there exist a nondegenerate element a in $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-1/2}$, such that $\lambda([(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{1/2}, a]) = 0$.

Now, let $a = z^{-1} b(T_q) E_{12} + b(q T_q^{-1}) T_q^{-k} E_{21}$ be a nonzero element of $(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{-1/2}$ (with $b(w) = \sum_j b_j w^j$); then, by Corollary 17(b), a is a nondegenerate element. Then, $\lambda([(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{1/2}, a]) = 0$ if and only if, for all $i \in \mathbb{Z}$,

$$\begin{aligned} 0 &= \lambda\left(\left[\left(T_q^i E_{12} - z T_q^{k-i} E_{21}, a\right)\right]\right) \\ &= \sum_j q^j b_j \left(\Delta_{k-i+j,1} + \Delta_{k-i+j,2}\right) + q^{-k+i} b_{-k+i} c. \end{aligned} \tag{95}$$

Multiplying (95) by x^{-k+i} and adding over $i \in \mathbb{Z}$, we obtain that

$$0 = \sum_{j,i} q^j b_j \left(\Delta_{k-i+j,1} + \Delta_{k-i+j,2}\right) x^{-k+i} + \sum_i q^{-k+i} b_{-k+i} x^{-k+i} c = b(qx)\left(\Delta_{\lambda,1}(x) + \Delta_{\lambda,2}(x) + c\right). \tag{96}$$

The equivalence between (a) and (b) follows from Proposition 14. □

3.2. The Case $\mathfrak{g} = \widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$. It is clear that $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$ satisfies (SP1). Now, we prove that $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$ satisfies (SP2). We suppose that $A = z^{-n} f(q^{-n/2} T_q) E_{12} + z^{-n+1} g(q^{(-n+1)/2} T_q) E_{21}$ in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n+1/2}$ (with $n \in \mathbb{N}$) satisfies $[A, (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{1/2}] = 0$; in particular if we take nonzero elements $h_{12}(T_q) E_{12}$ and $z h_{21}(q^{1/2} T_q) E_{21}$ in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{1/2}$, we have that

$$\left[A, h_{12}\left(T_q\right) E_{12}\right] = 0; \tag{97}$$

that is, $g(q^{(-n+1)/2} T_q) h_{12}(T_q) = 0$; therefore, $g(w) = 0$. Similarly,

$$\left[A, z h_{21}\left(q^{1/2} T_q\right) E_{21}\right] = 0, \tag{98}$$

which is equivalent to $f(q^{-n/2+1} T_q) h_{21}(q^{1/2} T_q) = 0$; therefore, $f(w) = 0$, and then $A = 0$.

Remark 19. Given $B = z^n f(q^{n/2} T_q) E_{12} + z^{n+1} g(q^{(n+1)/2} T_q) E_{21} \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{n+1/2}$ and $A = h(T_q) E_{11} - h(T_q^{-1}) E_{22} \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_0$ with $h \in \mathbb{C}[w^{-1}, w]$, then $M := [A, B] \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{n+1/2}$, where $M_{ii} = 0$ $(i = 1, 2)$ and

$$M_{12} = -z^n f\left(q^{n/2} T_q\right)\left(h\left(q^n T_q\right) + h\left(T_q^{-1}\right)\right), \tag{99}$$

$$M_{21} = z^{n+1} g\left(q^{(n+1)/2} T_q\right)\left(h\left(T_q\right) + h\left(q^{-n-1} T_q^{-1}\right)\right). \tag{100}$$

Now, let $A = z^{-n} f_1(T_q) E_{11} + z^{-n} f_2(T_q) E_{22} \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n}$ (with $n \in \mathbb{N}$) be such that A is nonzero; then, if we take nonzero $B = f(T_q) E_{12}$ in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{1/2}$ by Remark 19, there exists $f(T_q) h(T_q) E_{12} \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{1/2}$ such that h is not constant; then by Lemma 15(a), we have that

$$\left[A, f\left(T_q\right) E_{12}\right] \neq 0 \quad \text{or} \quad \left[A, f\left(T_q\right) h\left(T_q\right) E_{12}\right] \neq 0. \tag{101}$$

Therefore $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$ satisfies (SP2).

In order to prove that $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$ satisfies (SP3), we will need the following result. We denote

$$\delta'_n := \begin{cases} \delta_n + 1, & \text{in } \left(\mathcal{S}\mathfrak{S}^{+,\pm}_{q,k,l}\right)_{n+1/2}, \\ \delta_n, & \text{in } \left(\mathcal{S}\mathfrak{S}^{-,\pm}_{q,k,l}\right)_{n+1/2}. \end{cases} \tag{102}$$

Lemma 20. *Let $\mathfrak{p} = \bigoplus_{j \in (1/2)\mathbb{Z}} \mathfrak{p}_j$ be a $(1/2)\mathbb{Z}$-graded subalgebra of $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$, with $\mathfrak{p}_0 = (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_0$. Then one has the following:*

(a) *for each $n \in \mathbb{Z}$, $\mathfrak{p}_n$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_n$ if and only if $\mathfrak{p}_n \neq 0$;*

(b) *for each $n \in \mathbb{Z}$, $\mathfrak{p}_{n+1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{n+1/2}$ if and only if there exists $z^n f(q^n T_q) E_{12} + z^{n+1} g(q^{(n+1)/2} T_q) E_{21} \in \mathfrak{p}_{n+1/2}$, such that f and g are nonzero;*

(c) *$\mathfrak{p}_{-n}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n}$ for all $n \in (1/2)\mathbb{N}$, if and only if $\mathfrak{p}_{-1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-1/2}$.*

Proof. (a) Let $A = z^n f(T_q) E_{11} - (\pm 1)^n q^{n(nk-l)/2} z^n f(q^{-n} T_q^{-1}) T_q^{nk} E_{22} \in \mathfrak{p}_n$, with $f \neq 0$; then, $B_j := (1 - q^{nj})^{-1} [A, T_q^j E_{11} - T_q^{-j} E_{22}] \in \mathfrak{p}_n$ and

$$B_j = z^n f\left(T_q\right) T_q^j E_{11} - (\pm 1)^n q^{(n(nk-l)/2) - nj} \times z^n f\left(q^{-n} T_q^{-1}\right) T_q^{nk-j} E_{22}, \tag{103}$$

for all $j \in \mathbb{Z}^{\times}$; therefore, by (103) we have that

$$z^n g\left(T_q\right) E_{11} - (\pm 1)^n q^{n(nk-l)/2} z^n g\left(q^{-n} T_q^{-1}\right) T_q^{nk} E_{22} \in \mathfrak{p}_n, \tag{104}$$

for all $g(w) \in \langle f(w) \rangle$, and since $\langle f(w) \rangle$ has finite codimension in $\mathbb{C}[w^{-1}, w]$, we have that $\mathfrak{p}_n$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_n$ (see Remark 11). The proof of the converse is trivial.

(b) We suppose that there exist nonzero elements

$$z^n f_1\left(T_q\right) E_{12}, \qquad z^{n+1} f_2\left(T_q\right) E_{21} \quad \text{in } \mathfrak{p}_{n+1/2}; \tag{105}$$

then, using Remark 19, we obtain that $z^n g_1(T_q) E_{12} \in \mathfrak{p}_{n+1/2}$ for all $g_1(q^{-n/2} w) \in f_1(w) \mathbb{C}[w^{-1}, w]^{(1,0)}$ (see (99)) and $z^{n+1} g_2(T_q) E_{21} \in \mathfrak{p}_{n+1/2}$ for all $g_2(q^{(-n+1)/2} w) \in f_2(w) \mathbb{C}[w^{-1}, w]^{(1,0)}$ (see (100)). Then using that $f_1(w) \mathbb{C}[w^{-1}, w]^{(1,0)}$ has finite codimension in $\mathbb{C}[w^{-1}, w]^{(\delta_n, nk+l)}$ and $f_2(w) \mathbb{C}[w^{-1}, w]^{(1,0)}$ has finite codimension in $\mathbb{C}[w^{-1}, w]^{(\delta'_n, (n+1)k-l)}$, we obtain that $\mathfrak{p}_{n+1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{n+1/2}$. Therefore in order to prove that $\mathfrak{p}_{n+1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{n+1/2}$, we only need to see that there exist nonzero elements as aforementioned in $\mathfrak{p}_{n+1/2}$. Let $B = z^n f(q^{n/2} T_q) E_{12} + z^{n+1} g(q^{(n+1)/2} T_q) E_{21} \in \mathfrak{p}_{n+1/2}$ with nonzero f and g, and let A and M be as in Remark 19. Then, taking $h(w) = w - q^n w^{-1}$ (observe that $h(T_q^{-1}) = -h(q^n T_q)$), we obtain that

$$M = z^{n+1} g\left(q^{(n+1)/2} T_q\right) \times \left(h\left(T_q\right) + h\left(q^{-(n+1)} T_q^{-1}\right)\right) \times E_{21} \in \mathfrak{p}_{n+1/2}, \quad \text{with } M \neq 0. \tag{106}$$

Similarly, taking $h(w) = w - q^{-n-1}w^{-1}$ (observe that $h(T_q) = -h(q^{-n-1}T_q^{-1})$), we obtain

$$M = -z^n f\left(q^{n/2}T_q\right) \times \left(h\left(q^n T_q\right) + h\left(T_q^{-1}\right)\right) E_{12} \in \mathfrak{p}_{n+1/2}, \quad \text{with } M \neq 0, \tag{107}$$

proving the existence of nonzero elements of those forms. The proof of the converse is trivial.

(c) We suppose that $\mathfrak{p}_{-1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-1/2}$; then, in order to prove that $\mathfrak{p}_{-n}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n}$ for all $n \in (1/2)\mathbb{N}$, we only need to see that this is true for all $-n+1/2$ with $n \in \mathbb{N}$, since in this case by using (SP2), we obtain that $\mathfrak{p}_{-n} \neq 0$ for all $n \in \mathbb{N}$; then, by (a), we have that $\mathfrak{p}_{-n}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n}$, for all $n \in \mathbb{N}$.

By hypothesis, there exist nonzero elements $A = z^{-1}f(q^{-1}T_q)E_{12}$ and $B = g(T_q)E_{21}$ in $\mathfrak{p}_{-1/2}$; therefore, $[A, B] \in \mathfrak{p}_{-1}$ is nonzero. By induction, we suppose that $\mathfrak{p}_{-n+1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n+1/2}$; then, there exist nonzero elements $z^{-n}f_{12}(q^nT_q)E_{12}$ and $z^{-n+1}f_{21}(q^{(-n+1)/2}T_q)E_{21}$ in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n+1/2}$, and by Remark 19, we also have $z^{-n}f_{12}(q^nT_q)g_{12}(T_q)E_{12}, z^{-n+1}f_{21}(q^{(-n+1)/2}T_q)g_{21}(T_q)E_{21}$ in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n+1/2}$ with no constant g_{ij} for all $i \neq j$. Then by Lemma 15(a), we have that

$$\begin{aligned} &\left[z^{-n}f_{12}\left(q^nT_q\right)E_{12}, [A, B]\right] \neq 0 \\ \text{or } &\left[z^{-n}f_{12}\left(q^nT_q\right)g_{12}\left(T_q\right)E_{12}, [A, B]\right] \neq 0. \end{aligned} \tag{108}$$

Therefore from (108), there exists a nonzero element $z^{-n-1}\widetilde{f}_{12}(T_q)E_{12}$ in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n-1/2}$. Similarly, we prove that there exists nonzero $z^{-n}\widetilde{f}_{21}(T_q)E_{21}$ in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-n-1/2}$; finally the proof follows from (b). The proof of the converse is trivial. □

Corollary 21. *(a) $a = z^{-1}f(q^{-1/2}T_q)E_{12} + g(T_q)E_{21} \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-1/2}$ is nondegenerate if and only if f and g are nonzero.*

(b) $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$ satisfies (SP3).

Proof. (a) Let $a = z^{-1}f(q^{-1/2}T_q)E_{12} + g(T_q)E_{21} \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-1/2}$. Then, if f and g are nonzero by Lemma 20(b) and $\mathfrak{p}^a_{-1/2}$ has finite codimension in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-1/2}$, then by Lemma 20(c), a is a nondegenerate element. Reciprocally, if $f = 0$ or $g = 0$, then by definition $\mathfrak{p}^a_{-1} = 0$.

(b) Let $\mathfrak{p}$ be a nondegenerate parabolic subalgebra of $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$; then there exists $a = z^{-1}f(q^{-1}T_q)E_{12} + g(T_q)E_{21} \in \mathfrak{p}_{-1/2}$ where f and g are nonzero. Then the proof follows from (a). □

A functional $\lambda \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})^*_0$ is described by its labels, $\Delta_n = \lambda(T_q^nE_{11} - T_q^{-n}E_{22})$ with $n \in \mathbb{Z}$ and the central charge $\lambda(C) = c$. Moreover, we define $\Delta^0_n = (\Delta_n - \Delta_{-n}) = \lambda((T_q^n - T_q^{-n})E_{11} + (T_q^n - T_q^{-n})E_{22})$. We consider the generating series

$$\begin{aligned} \Delta_\lambda(x) &= \sum_{n\in\mathbb{Z}} \Delta_n x^{-n}, \\ \Delta^0_\lambda(x) &= \Delta_\lambda(x) - \Delta_\lambda\left(x^{-1}\right) = \sum_{n\in\mathbb{Z}} \Delta^0_n x^{-n}. \end{aligned} \tag{109}$$

Theorem 22. *An irreducible highest weight $\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}$-module $L(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}, \lambda)$ is quasifinite if and only if one of the following equivalent conditions holds.*

(a) *There exist Laurent polynomials $f(w)$ and $g(w)$ such that*

$$\begin{aligned} g(x)\Delta^0_\lambda(x) &= 0, \\ f(x)\Delta^0_\lambda\left(q^{-1/2}x\right) &= 0. \end{aligned} \tag{110}$$

(b) *There exist quasipolynomials $P_{12}(x)$ and $P_{21}(x)$ such that*

$$\begin{aligned} P_{12}(n) &= q^{-n/2}\Delta^0_n, \\ P_{21}(n) &= \Delta^0_n, \end{aligned} \tag{111}$$

for all $n \in \mathbb{Z}$.

Proof. By Theorem 13, $L(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l}, \lambda)$ is quasifinite if and only if there exists a nondegenerate element a in $(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-1/2}$, such that $\lambda([(\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{1/2}, a]) = 0$.

Now, let $a = z^{-1}f(q^{-1/2}T_q)E_{12} + g(T_q)E_{21} \in (\widehat{\mathcal{S}\mathfrak{S}}^{\pm\pm}_{q,k,l})_{-1/2}$, with $f(w) = \sum_j f_jw^j \in \mathbb{C}[w^{-1}, w]^{(\delta_{-1}, l-k)}$ and $g(w) = \sum_j g_jw^j \in \mathbb{C}[w^{-1}, w]^{(\delta'_{-1}, -l)}$ such that f and g are nonzero. Then by Corollary 21(a), a is a nondegenerate element. Then, $\lambda([(\widehat{\mathcal{S}\mathfrak{S}}_{q,k})_{1/2}, a]) = 0$ if and only if, for all $i \in \mathbb{Z}$,

$$\lambda\left(\left[\left(T_q^i - (-1)^{\delta_0}T_q^{l-i}\right)E_{12}, a\right]\right) = 0, \tag{112}$$

$$\lambda\left(\left[z\left(q^{i/2}T_q^i - (-1)^{\delta'_0}q^{(k-l-i)/2}T_q^{k-l-i}\right)E_{21}, a\right]\right) = 0. \tag{113}$$

From (112), we obtain that

$$\begin{aligned} 0 &= \lambda\left(g\left(T_q\right)\left(T_q^i - (-1)^{\delta_0}T_q^{l-i}\right)Id\right) \\ &= \sum_j g_j\Delta^0_{i+j}. \end{aligned} \tag{114}$$

Multiplying (114) by x^{-i} and adding over $i \in \mathbb{Z}$, we obtain that

$$\begin{aligned} 0 &= \sum_{i,j} g_j\Delta^0_{i+j}x^{-i} \\ &= g(x)\Delta^0_\lambda(x). \end{aligned} \tag{115}$$

Then, by (113) and using Remark 8, we obtain that

$$\begin{aligned}0 &= \lambda\Bigg(\Bigg(\sum_j f_j q^{(i+j)/2} T_q^{i+j} - \sum_j f_j q^{-(i+j)/2} T_q^{-i-j}\Bigg) E_{11} \\ &\qquad + \Bigg(\sum_j f_j q^{-(i+j)/2} T_q^{i+j} - \sum_j f_j q^{(i+j)/2} T_q^{-i-j}\Bigg) E_{22}\Bigg) \\ &\quad + \Bigg(\sum_j f_j q^{-(i+j)/2} T_q^{i+j} - \sum_j f_j q^{(i+j)/2} T_q^{-i-j}\Bigg)_0 c \\ &= \sum_j f_j q^{(i+j)/2} \Delta_{i+j} - \sum_j f_j q^{-(i+j)/2} \Delta_{-i-j}.\end{aligned} \tag{116}$$

Multiplying (116) by x^{-i} and adding over $i \in \mathbb{Z}$, we obtain that

$$\begin{aligned}0 &= \sum_{i,j} f_j q^{(i+j)/2} \Delta_{i+j} x^{-i-j} x^j - \sum_{i,j} f_j q^{-(i+j)/2} \Delta_{-i-j} x^{-i-j} x^j \\ &= \sum_j f_j \Delta_\lambda \left(q^{-1/2} x\right) x^j - \sum_j f_j \Delta_\lambda \left(q^{1/2} x^{-1}\right) x^j \\ &= f(x)\, \Delta_\lambda^0 \left(q^{-1/2} x\right).\end{aligned} \tag{117}$$

The equivalence between (a) and (b) follows from Proposition 14. □

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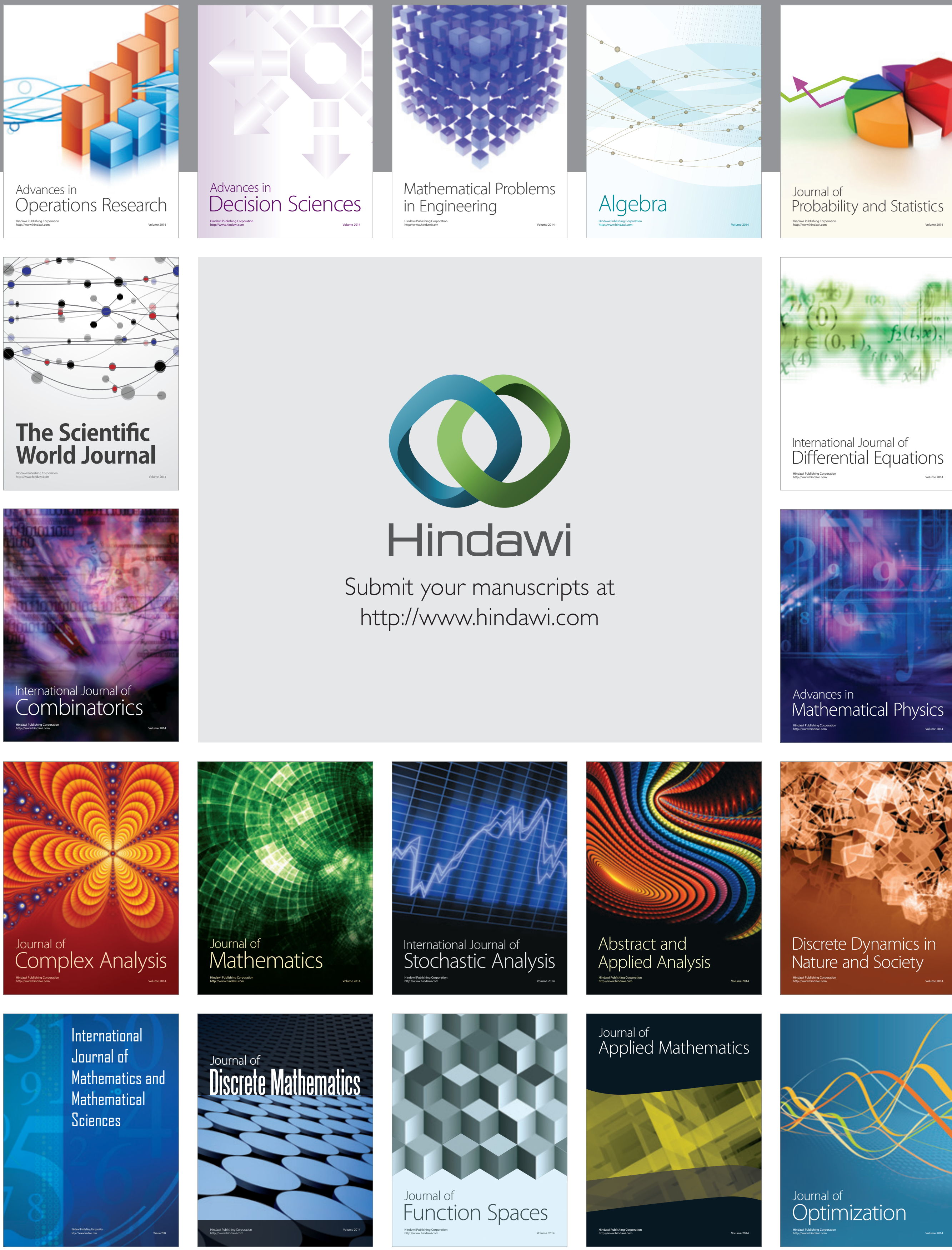
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