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*Research Article*

# Vibration Characteristics of Ring-Stiffened Functionally Graded Circular Cylindrical Shells

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The vibration characteristics of ring stiffened cylindrical shells are analyzed. These shells are assumed to be structured from functionally graded materials (FGM) and are stiffened with isotropic rings. The problem is formulated by coupling the expressions for strain and kinetic energies of a circular cylindrical shell with those for rings. The Lagrangian function is framed by taking difference of strain and kinetic energies. The Rayleigh-Ritz approach is employed to obtain shell dynamical equations. The axial model dependence is approximated by characteristic beam functions that satisfy the boundary conditions. The validity and efficiency of the present technique are verified by comparisons of present results with the previous ones determined by other researchers.

## 1. Introduction

Circular cylindrical shells stiffened by rings are widely used in many structural applications such as airplanes, marine crafts, pressure vessels, silos, core barrels of pressurized water reactors, submarine hulls, offshore drilling rings, and construction buildings. Usually cylinders are stiffened by rings or strings to increase the stiffness and strength, reduce the weight structure to be designed. In designing these shells, it is vital to know their resonant frequencies because excessive vibrations can lead to fatigue rupture.

First proper shell theory was proposed by Love [1]. This theory was based on Rayleigh [2] approximations for plates. Arnold and Warburton [3, 4] solved the shell problem for various physical parameters and interpreted the dip in the frequency curve on the basis of shell energy variations. They used Lagrange equations with strain and kinetic energy expressions to derive these equations. Forsberg [5] studied shell equations to scrutinize the effect of boundary conditions on vibration characteristics of circular cylindrical shells. Exponential axial model dependence was measured in this work. Bonger and Archer [6] showed that axisymmetric modes of a general shell revolution are orthogonal for clamped, simply supported, and free end. Tso [7] illustrated that the vibrational modes are orthogonal if the displacements or their corresponding natural forces vanish on the shell by using the Hamilton's principle. Sewall and Naumann [8] studied that stiffened cylindrical shells problem experimentally and analytically. They approximated the shell displacement deformations by the beam functions and used Rayleigh-Ritz method to derive shell frequency equation in the eigenvalue form. The shells are stiffened periodically to increase the stability and efficiency. Wang et al. [9] transform vibration analysis of shell eigen-frequency equation in the general eigenvalue problem. They used Ritz polynomial functions for the axial deformation displacements by considering boundary condition equations. They designed three types of shells by the locations of isotropic rings. Sharma and Johns [10] applied Rayleigh-Ritz method for theoretical analysis of vibrating clamped-free and clamped-stiffened shell using

Flugge's shell theory for a variety of choices of axial model shape. Swaddiwudhipong et al. [11] presented an excellent study on vibrations of cylindrical shells with intermediate supports. They adopted an automated Rayleigh-Ritz method to evaluate the natural frequencies and the mode shapes of shells. Loy and Lam [12] studied the vibration of thin cylindrical shell with ring supports. The study was carried out using Sander's shell theory. The governing equations were obtained using energy functional with the Ritz method.

There is bulk of studies on isotropic homogeneous materials and the studies on new invented composite materials such as functionally graded materials (FGMs) have been carried out by many researchers. In this material compositions and functions are varying continuously from one side to the other side. For example, one side may have high mechanical strength and the other side may have high thermal resistant property; thus, there are "two aspects" in one material. In FGMs, change of compositions is continuous and it does not come out from simply "bonding individual substances." This generates boundaries among the bonded ones. They are all considered by mechanically and chemically. Koizumi [13] and other Japanese materialists gave the idea of fabrication of this material. Benachour et al. [14] gave refined plate theory with four variables for the study of free vibrations of functionally graded plates with arbitrary gradient. Loy et al. [15] have analyzed frequency spectrum of FGM cylindrical shells for simply supported boundary conditions. An FGM cylindrical shell composed of stainless steel and nickel as constituent materials was considered. The material properties are graded in the thickness direction in accordance with a volume fraction law. Ng et al. [16] introduced a formulation for the dynamic stability analysis of functionally graded material (FGM) cylindrical shells under harmonic axial loading. Li and Batra [17] have studied bucking aspect of three layer circular cylindrical shells under axial compressive load by simply supported boundary condition. The middle layer of the cylindrical shell was assumed to be of functionally graded material. Benyoucef et al. [18] investigated the bending behavior of thick functionally graded plates resting on Winkler-Pasternak elastic foundations.

Although an extensive amount of research work has been carried out to study the vibration characteristics of isotropic as well as composite cylindrical shells, there is no evidence of work on vibration of functionally graded cylindrical shells stiffener with ring support. The present study is concerned with analysis of the vibration characteristics of functionally graded cylindrical shell stiffened with isotropic ring on the outer surface of the shell. The Rayleigh-Ritz method is used to formulate the shell eigen-frequency equation.

## 2. Theoretical Consideration

Consider a circular thin shell of uniform thickness $h$, length $L$, radius $R$, mass density $\rho$, modulus of elasticity $E$, Poisson's ratio $\nu$, and shear and effective modulus $G = E/2(1+\nu)$. The shell is reinforced by $q$ ring-stiffeners of either equal or unequal sizes or spacing. The kth stiffener located at $a_kL$ measured from one end of the shell and its rectangular cross-section has a depth $d_k$ and width $b_k$ as shown in Figure 1. The stiffeners may be constructed from materials used for shell or different from shell material, with $\rho_k$ denoting the stiffener mass density, $E_k$ its young's modulus, $G_k$ the shear modulus, and $\nu_k$ the poisson's ratio.

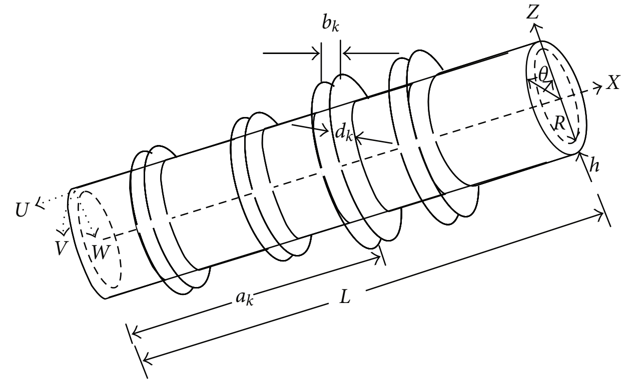


Figure 1: Cylindrical shell with Ring-Stiffeners.

For a thin cylindrical shell, plane stress condition is assumed and the constitutive relation for a thin cylindrical shell is given by

$$\{\sigma\} = [Q]\{e\}, \tag{1}$$

where $\{\sigma\}$ is stress vector, $\{e\}$ is strain vector, and $[Q]$ is reduced stiffness matrix.

Stress vector and strain vector are defined as

$$\begin{aligned}\{\sigma\}^T &= \left\{\sigma_x \quad \sigma_\theta \quad \sigma_{x\theta}\right\},\\ \{e\}^T &= \left\{e_x \quad e_\theta \quad e_{x\theta}\right\},\end{aligned} \tag{2}$$

where $\sigma_x$ and $\sigma_\theta$ are stresses in $x$ and $\theta$ directions $\sigma_{x\theta}$ is shear stress on the $x\theta$ plane, $e_x$, $e_\theta$ are strains in $x$ and $\theta$ directions, and $e_{x\theta}$ is shear strain on $x\theta$ plane. The reduced stiffness matrix is defined as

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}. \tag{3}$$

For the isotropic materials, the reduced stiffness $Q_{ij}(i, j = 1, 2, \text{and } 6)$ are given as

$$\begin{aligned} Q_{11} = Q_{22} &= \frac{E}{1-\nu^2},\\ Q_{12} &= \frac{\nu E}{1-\nu^2},\\ Q_{66} &= \frac{E}{2(1+\nu)}.\end{aligned} \tag{4}$$

The components of the strain vector $\{e\}$ are defined as the linear functions of thickness coordinates $z$ and are given as

$$\begin{aligned} e_x &= e_1 + zk_1,\\ e_\theta &= e_2 + zk_2,\\ e_{x\theta} &= \gamma + 2z\tau,\end{aligned} \tag{5}$$

where $e_1$, $e_2$, and $\gamma$ are reference surface strains, and $k_1$, $k_2$, and $\tau$ are the surface curvatures. The strain energy, $S$, of a cylindrical shell can be written as

$$S = \frac{1}{2}\int_0^L\int_0^{2\pi}\{\varepsilon\}^T[T]\{\varepsilon\}Rd\theta dx, \tag{6}$$

where

$$\{\varepsilon\}^T = \{e_1, e_2, \gamma, k_1, k_2, 2\tau\}, \qquad [T] = \begin{bmatrix} A & B \\ B & D \end{bmatrix}, \tag{7}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix},$$
$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}, \tag{8}$$
$$D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix},$$

where $A$, $B$, $D$ are the matrices of the extensional stiffness $A_{ij}$, coupling stiffness $B_{ij}$ and bending stiffness $D_{ij}$ $(i, j = 1, 2, \text{and } 6)$, respectively. Stiffnesses $A_{ij}$, $B_{ij}$, and $D_{ij}$ are defined as

$$\left\{A_{ij}, B_{ij}, D_{ij}\right\} = \int_{-h/2}^{h/2} Q_{ij}\{1, z, z^2\}dz \quad (i, j = 1, 2, \text{and } 6), \tag{9}$$

where the coupling stiffness $B_{ij}$ reduces to zero for isotropic cylindrical shell and nonzero for FGM shells. For FGM shells, the sign of $B_{ij}$ depends upon the order of constituent materials in the FGM. They are positive for a FGM configuration and negative if the order of the constituent material is reversed. This arises because of material properties asymmetry about the mid plane. $Q_{ij}$ are function of $z$ for FGMs. Substituting the expression for $\{\varepsilon\}^T$ and $\{T\}$ in (6)

$$S = \frac{R}{2}\int_0^L\int_0^{2\pi}\left[\begin{matrix} A_{11}e_{11}{}^2 + 2e_1e_2A_{12} + 2e_1k_1B_{11} + 2e_1k_2B_{12} + e_2{}^2A_{22} + 2e_2k_1B_{12} + 2e_2k_2B_{22} \\ +\gamma^2A_{66} + 4\tau\gamma B_{66} + k_1{}^2D_{11} + 2k_1k_2D_{12} + k_2{}^2D_{22} + 4\tau^2D_{66} \end{matrix}\right]Rd\theta dx. \tag{10}$$

At time "$t$" the expression for kinetic energy "$K$" of the vibrating shell is given in the form

$$K = \frac{1}{2}\int_0^L\int_0^{2\pi}\rho_t\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]Rd\theta dx, \tag{11}$$

where $\rho_t$ is the mass density per unit length defined as

$$\rho_t = \int_{-h/2}^{h/2}\rho dz, \tag{12}$$

where $\rho$ is the mass density per unit length. Adopting Sander's thin shell theory for the surface strains [20], the strain energy $S$ is given by

$$S = \frac{R}{2}\int_0^L\int_0^{2\pi}\frac{\partial^2 w}{\partial x^2}\left[\begin{matrix} A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{A_{22}}{R^2}\left(\frac{\partial v}{\partial \theta} - w\right)^2 + \frac{2A_{12}}{R}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial \theta} - w\right) + A_{66}\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial x}\right)^2 \\ +2B_{11}\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{2B_{12}}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)\left(\frac{\partial u}{\partial x}\right) + \frac{2B_{12}}{R}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial v}{\partial \theta} - w\right) \\ +\frac{2B_{22}}{R^3}\left(\frac{\partial v}{\partial \theta} - w\right)\left(\frac{\partial^2 w}{\partial \theta^2} + w\right) + 4B_{66}\left(\frac{\partial^2 w}{\partial x\partial \theta} + \frac{3}{4}\frac{\partial v}{\partial x} - \frac{1}{4R}\frac{\partial u}{\partial \theta}\right)\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right) \\ +D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{D_{22}}{R^4}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)^2 + \frac{2D_{12}}{R^2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial \theta^2} + w\right) \\ +4D_{66}\left(\frac{\partial^2 w}{\partial x\partial \theta} + \frac{3}{4}\frac{\partial v}{\partial x} - \frac{1}{4R}\frac{\partial u}{\partial \theta}\right)^2 \end{matrix}\right] \times Rd\theta dx. \tag{13}$$

The strain energy $S_k$ of the $k$th ring-stiffener is given by Galletly [17] as

$$S_k = \int \left[ \begin{array}{l} \dfrac{E_k I_{zk}}{2(R+e_k)} \left( \dfrac{\partial w_k}{\partial x} + \dfrac{1}{R+e_k} \dfrac{\partial^2 u_k}{\partial \theta^2} \right)^2 + \dfrac{E_k I_{xk}}{2(R+e_k)^3} \left( w_k + \dfrac{\partial^2 w_k}{\partial \theta^2} \right)^2 \\ + \dfrac{E_k A_k}{2(R+e_k)} \left( \dfrac{\partial v_k}{\partial \theta} - w_k \right)^2 + \dfrac{G_k J_k}{2(R+e_k)} \left( -\dfrac{\partial^2 w_k}{\partial x \partial \theta} + \dfrac{1}{R+e_k} \dfrac{\partial u_k}{\partial \theta} \right)^2 \end{array} \right] d\theta, \tag{14}$$

where $e_k$ is the eccentricity of the ring stiffener and is given by

$$\begin{aligned} e_k &= \frac{h+d_k}{2} \quad \text{for externally eccentric stiffener,} \\ e_k &= 0 \quad \text{for concentric stiffener,} \\ e_k &= -\frac{h+d_k}{2} \quad \text{for internally eccentric stiffener.} \end{aligned} \tag{15}$$

The second moment of areas $I_{zk}$, $I_{xk}$, cross-sectional area $A_k$, and the torsional constant $J_k$ are furnished by

$$\begin{aligned} I_{zk} &= \frac{{b_k}^3 d_k}{12}, \qquad I_{xk} = \frac{b_k {d_k}^3}{12}, \qquad A_k = b_k d_k, \\ J_k &= \frac{1}{3}\left[1 - \frac{192 b_k}{\pi^5 d_k} \sum_{n=1,3,5}^{\infty} \frac{1}{n^5} \tanh \frac{n\pi d_k}{2b_k}\right] {b_k}^3 d_k. \end{aligned} \tag{16}$$

The kinetic energy $K_k$ of the $k$th stiffener is given by:

$$\begin{aligned} K_k = \frac{\rho_k}{2} \int_0^{2\pi} \Bigg\{ & A_k \left[ \left(\frac{\partial u_k}{\partial t}\right)^2 + \left(\frac{\partial v_k}{\partial t}\right)^2 + \left(\frac{\partial w_k}{\partial t}\right)^2 \right] \\ & + (I_{xk} + I_{zk}) \left( \frac{\partial^2 w_k}{\partial t \partial x} \right)^2 \Bigg\} (R + e_k) d\theta. \end{aligned} \tag{17}$$

From geometric considerations, the relationships between the displacements $(u_k, v_k, w_k)$ of the kth stiffener and the displacements $(u, v, w)$ of the shell at the position of the stiffener at $x = a_k L$ are given by

$$\begin{aligned} u_k &= u + e_k \frac{\partial w_k}{\partial x}, \\ v_k &= v\left(1 + \frac{e_k}{R}\right) + \frac{e_k}{R} \frac{\partial w}{\partial \theta}, \\ w_k &= w. \end{aligned} \tag{18}$$

Combining the strain and kinetic energies of the cylindrical shell and the $k$th stiffener, the Lagrangian functional is formulated in the following form:

$$\prod = \left[ K + \sum_{k=1}^{q} (K_k) \right] - \left[ S + \sum_{k=1}^{q} (S_k) \right]. \tag{19}$$

## 3. Solution Procedure

The Rayleigh-Ritz approach is employed to analyze vibrational behaviour of stiffened functionally graded circular cylindrical shells. The use of Ritz method provides a rapid convergence and excellent accuracy. The following displacement functions $u, v, w$ are adopted to separate the spatial variables $x, \theta$ and time variable $t$ in the following form:

$$\begin{aligned} u(x,\theta,t) &= A_m U(x) \cos(n\theta) \sin(\omega t), \\ v(x,\theta,t) &= B_m V(x) \sin(n\theta) \sin(\omega t), \\ w(x,\theta,t) &= C_m W(x) \cos(n\theta) \sin(\omega t), \end{aligned} \tag{20}$$

where $n$ is the number of circumferential modes; $\omega$ is the circular frequency of the vibrating shell; and $U(x)$, $V(x)$, and $W(x)$ represents the axial modal dependence in the axial, circumferential, and radial directions, respectively. $A_m$, $B_m$, $C_m$ are the vibrating amplitudes.

Using the expressions for the functions $u$, $v$, and $w$ given in (20) and their partial derivatives in the expressions for $K$, $S$, $S_k$, $K_k$ given in (11), (13), (14), and (17), respectively, and employing the principle of minimum energy, we obtain the following form of the Lagrangian functional:

$$\prod = K_{\max} - S_{\max}, \tag{21}$$

where $K_{\max}$ and $S_{\max}$ are the sum of maximum kinetic and strain energies, respectively, of stiffened functionally graded cylindrical shell. To derive frequency equation, the Lagrangian functional is extremized with respect to the amplitude coefficients $A_m, B_m$, and $C_m$. This yields a set of three homogeneous simultaneous equations

$$\frac{\partial \prod}{\partial A_m} = \frac{\partial \prod}{\partial B_m} = \frac{\partial \prod}{\partial C_m} = 0. \tag{22}$$

Rearranging the terms in (22), we get the following eigenvalue equation to find the shell natural frequencies and mode shapes:

$$[[S] + [S_k]] - \omega^2[[K] + [K_k]]X = [0]. \tag{23}$$

or

$$[[C] - \omega^2[D]]X = [0], \tag{24}$$

where

$$[C] = [[S] + [S_k]], \qquad [D] = [[K] + [K_k]], \tag{25}$$

where $[S]$ and $[K]$ are stiffness and mass matrices of the cylindrical shell, respectively, and $[S_k]$ and $[K_k]$ are the corresponding matrices of the $k$th ring stiffener, and $X = [\, A_m \; B_m \; C_m \,]^T$ The expression for the matrices $[C]$ and $[D]$ are given in the appendix. This is an eigenvalue problem and is solved by using Matlab software package. Equation (24) is true for both isotropic as well as FGM cylindrical shells. For isotropic cylindrical shells the coupling stiffness $B_{ij}$ is zero whereas it is non-zero for FGM cylindrical shells. Eigenvalues correspond to the natural frequencies and eigenvectors to the corresponding mode shapes of shells respectively.

## 4. Functionally Graded Materials

The FGMs are much advanced materials and are used in engineering science and technology. Material properties of an FGM are the functions of the temperature and the position. These properties of a constituent material are managed by a volume fraction. If $P_i$ represents a material property of the $i$th constituent material of an FGM consisting of $k$ constituent materials, then the effective materials property $P$ of the FGM is written as

$$P = \sum_{i=1}^{k} P_i V_i, \tag{26}$$

where $V_i$ is the volume fraction of the $i$th constituent material. Also, the sum of volume fractions of the constituent materials is equal to 1. That is,

$$\sum_{i=1}^{k} V_i = 1. \tag{27}$$

The volume fraction depends upon the thickness variable and is defined as

$$V_i = \left(\frac{Z - R_i}{R_o - R_i}\right)^N \tag{28}$$

for a cylindrical shell. $R_i$ and $R_o$ denote inner and outer radii of the shell, respectively, and $z$ is the thickness variable in the radial direction. $N$ is known as the power law exponent. It is a nonnegative real number and lies between zero and infinity. For a cylindrical shell, the volume fraction is assumed as

$$V_i = \left(\frac{Z + 0.5\,h}{h}\right)^N, \tag{29}$$

where $h$ is the shell uniform thickness. When the shell is considered to consist of two materials, the effective Young's modulus $E$, the poisson ratio $\upsilon$ and the mass density $\rho$ are given by

$$\begin{aligned} E &= (E_1 - E_2)\left(\frac{Z + 0.5\,h}{h}\right)^N + E_2, \\ nv &= (nv_1 - nv_2)\left(\frac{Z + 0.5\,h}{h}\right)^N + nv_2, \\ \rho &= (\rho_1 - \rho_2)\left(\frac{Z + 0.5\,h}{h}\right)^N + \rho_2. \end{aligned} \tag{30}$$

Table 1: Comparison of frequency parameters $\Omega = \omega R\sqrt{(1-\nu^2)\rho/E}$ simply supported isotropic cylindrical shell. ($m = 1$, $L/R = 20$, $h/R = 0.01$, $E = 30e+06 lbf in^{-2}$, $\nu = 0.3$, $\rho = 7.35e^{-04} lbf s^2$).

| $n$ | Loy et al. [15] | Present |
|---|---|---|
| 1 | 0.016102 | 0.016101 |
| 2 | 0.009387 | 0.009381 |
| 3 | 0.022108 | 0.022105 |
| 4 | 0.042096 | 0.042095 |
| 5 | 0.068008 | 0.068008 |
| 6 | 0.099730 | 0.099730 |
| 7 | 0.137239 | 0.137240 |
| 8 | 0.180527 | 0.180529 |
| 9 | 0.229594 | 0.229596 |
| 10 | 0.284435 | 0.284439 |

## 5. Results and Discussion

Numerical technique known as the Rayleigh-Ritz method has been employed to study the vibration characteristics of functionally graded circular cylindrical shells with ring-stiffeners. To confirm the efficiency and validity of the present method, the frequencies of cylindrical shells with and without ring-stiffeners are compared with those values found in the literature.

In Table 1, a comparison of frequency parameters $\Omega = \omega R\sqrt{(1-\nu^2)\rho/E}$ calculated by Loy et al. [15] is done with those values obtained by the present procedure using the Rayleigh-Ritz method for an isotropic cylindrical shell simply supported at both ends. Shell parameters are given in Table 1. Two sets of frequency parameters are very close to each other and a good agreement is observed between them. The minimum frequency parameter occurs at $n = 2$ and is less than, by 0.061%, its corresponding value in Loy et al. [15]. The frequency parameters for $n = 1, 3$, and 4 are 0.006%, 0.01%, and 0.002% lower and the frequency parameters for circumferential wave numbers $n = 7, 8$, 9, and 10 are 0.0007%, 0.001%, 0.0009%, and 0.001% greater than the corresponding results performed by Loy et al. [15]. The frequency parameters for $n = 5, 6$ are the same for both cases. This shows the validity of the present method.

*5.1. Simply Supported Cylindrical Shells with Ring Stiffeners.* The often-cited Galletly [21] S-S shells with 14 evenly spaced eccentric/concentric ring stiffeners of various depth-to-width ratios $d/b$ are analyzed. In Tables 2(a) and 2(b), the nondimensionalized frequency parameters $\Omega$ are listed for isotropic cylindrical shells with ring-stiffeners.

Frequency study is analyzed for 14 evenly spaced eccentric ring stiffeners externally and internally in Tables 2(a) and 2(b), respectively. They are compared with those values evaluated by Swaddiwudhipong et al. [11]. Shell parameters are listed in the tables. The axial wave mode is $m = 1$. The circumferential wave numbers are 2, 3, 4, and 5. It is seen that as the depth-to-width ratio ($d/b$) is increased, the frequency increases. Also the comparison of the results

Table 2

(a) Comparison studies of frequency parameter Ω for simply-supported cylindrical shell with 14 evenly spaced, externally eccentric ring stiffeners. ($L/R$ = 4.5419, $h/R$ = 0.01151, $b/R$ = 0.02107)

| Stiffener's depth-to-width ratio $d/b$ | Circumferential wave no. $n$ | Swaddiwudhipong et al. [11] | Present analysis |
|---|---|---|---|
| 1.3314 | 2 | 0.08472 | 0.08503 |
| 1.3314 | 3 | 0.06711 | 0.07049 |
| 1.3314 | 4 | 0.1072 | 0.11493 |
| 1.3314 | 5 | 0.1714 | 0.18394 |
| 2.6628 | 2 | 0.08372 | 0.08591 |
| 2.6628 | 3 | 0.1095 | 0.12269 |
| 2.6628 | 4 | 0.2037 | 0.2296 |
| 2.6628 | 5 | 0.3271 | 0.37226 |
| 3.9942 | 2 | 0.08788 | 0.09371 |
| 3.9942 | 3 | 0.1592 | 0.18444 |
| 3.9942 | 4 | 0.3049 | 0.35343 |
| 3.9942 | 5 | 0.4598 | 0.57304 |

(b) Comparison studies of frequency parameter Ω for simply-supported cylindrical shell with 14 evenly spaced, internally eccentric ring stiffeners. ($L/R$ = 4.5419, $h/R$ = 0.01151, $b/R$ = 0.02107)

| Stiffener's depth-to-width ratio $d/b$ | Circumferential wave no. $n$ | Swaddiwudhipong et al. [11] | Present analysis |
|---|---|---|---|
| 1.3314 | 2 | 0.08282 | 0.0876 |
| 1.3314 | 3 | 0.05992 | 0.0771 |
| 1.3314 | 4 | 0.08641 | 0.1221 |
| 1.3314 | 5 | 0.1346 | 0.1918 |
| 2.6628 | 2 | 0.08141 | 0.0933 |
| 2.6628 | 3 | 0.08826 | 0.1380 |
| 2.6628 | 4 | 0.1536 | 0.2499 |
| 2.6628 | 5 | 0.2459 | 0.4000 |
| 3.9942 | 2 | 0.08425 | 0.1076 |
| 3.9942 | 3 | 0.1280 | 0.2128 |
| 3.9942 | 4 | 0.2363 | 0.3970 |
| 3.9942 | 5 | 0.3807 | 0.6377 |

shows that the present results are slight higher than those in Swaddiwudhipong et al. [11]. The difference may be due to the axial modal dependence measured by the different sets of functions. In Swaddiwudhipong et al. [11], Ritz polynomials have been used whereas in the present study, characteristics beam functions are utilized.

*5.2. Comparisons of Results of FGM Cylindrical Shells without Ring Stiffeners.* The functionally graded material considered here is composed of stainless steel and nickel. The variations of natural frequencies of a functionally graded cylindrical shell are compared with those determined by Naeem et al. [19]. The influence of the constituent volume fractions is studied by varying the volume fractions of the Stainless Steel and Nickel. This is carried out by varying the value of power law exponent $N$. The effects of the FGM configuration are studied by studying the frequencies of two functionally graded cylindrical shells. Type-I functionally graded cylindrical shell has Nickel on its inner surface and Stainless Steel on its outer surface and Type-II functionally graded cylindrical shell has Stainless Steel on its inner surface and Nickel on its outer surface. The material properties for Stainless Steel and Nickel, calculated at $T = 300\,\mathrm{K}$, are presented in Table 3.

Natural frequencies (Hz) determined by the present method for a simply supported functionally graded cylindrical shell are presented in Tables 4(a) and 4(b) and are compared with the corresponding results evaluated by Naeem et al. [19]. In Table 4(a), a comparison of natural frequencies for the shell is given for the power law exponent $N = 0.5$, 1, and 15. The respective frequencies determined by two techniques are the same for the circumferential wave number $n = 1$ and for $n$ greater than 4 against power law exponent $N = 1$. The frequencies determined by the present method are slightly greater than those in the study of Naeem et al. [19] for the circumferential wave number $n$ between 2 and 10 against the power law exponent $N = 0.5$ and for $n = 2$, 3, and 4 against power law exponent $N = 1$. The frequencies determined by the present method are slightly lower for the circumferential wave number $n = 3$ to $n = 10$ against the power law exponent $N = 15$. The minimum frequency corresponds to $n = 4$ and decreases with $N = 15$ and corresponding decrement is 0.0218 per cent.

In Table 4(a), a comparison of of natural frequencies for the shell is given for the power law exponent $N = 0.5$, 1, and 15. The respective frequencies determined by two techniques are the same for the circumferential wae number $n = 1$ and for $n$ greater than 4 against power law exponent $N = 1$. The frequencies determined by the present method are slightly lower than those in the study of Naeem et al. [19] for the circumferential wave number $n$ between 2 and 10 against the power law exponent $N = 0.5$ and for $n = 2$, 3, and 4 against power law exponent $N = 1$. The minimum frequency corresponds to $n = 4$ and decreases with $N = 0.5$ and corresponding decrement is 0.0143 per cent.

*5.3. Functionally Graded Circular Cylindrical Shells with Ring-Stiffeners*

*5.3.1. Variations of Natural Frequencies with Circumferential Wave Number (n).* Tables 5 and 6 list the variations of natural frequencies (Hz) with the circumferential wave number "$n$" for a type I FG cylindrical shell with the Nickel ring-stiffeners. The axial mode "$m$" is taken to be unity. $L/R$ is kept equal to 20 where $R/h = 500$, 20 in Tables 5 and 6, respectively. The columns $N^{SS}$ and $N^{N}$ show the natural frequencies (Hz) for a Stainless Steel cylindrical shell and Nickel cylindrical shell, respectively. The influence of the value of $N$, which affects the constituent volume fractions, is analyzed. It is seen that when the value of $N$ is increased, the natural frequencies decreased. The decrease in the natural frequencies in Table 5, from $N = 1$ to $N = 15$ is about 1.6% at $n = 1$ and about 1.5% at $n = 10$. When $N$is small,

Table 3: Properties of materials.

| Coefficients | Stainless steel | | | Nickel | | |
|---|---|---|---|---|---|---|
| | $E$ (N/m$^2$) | $\nu$ | $\rho$ (kg/m$^3$) | $E$(N/m$^2$) | $\nu$ | $E$(N/m$^2$) |
| $p_0$ | $201.04 \times 10^9$ | 0.3262 | 8166 | $223.95 \times 10^9$ | 0.3100 | 8900 |
| $p_{-1}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $p_1$ | $3.079 \times 10^{-4}$ | $-2.002 \times 10^{-4}$ | 0 | $-2.794 \times 10^{-4}$ | 0 | 0 |
| $p_2$ | $-6.534 \times 10^{-7}$ | $3.797 \times 10^{-7}$ | 0 | $-3.998 \times 10^{-9}$ | 0 | 0 |
| $p_3$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | $2.07788 \times 10^{11}$ | 0.317756 | 8166 | $2.05098 \times 10^{11}$ | 0.3100 | 8900 |

Table 4

(a) Comparison of natural frequencies (Hz) for type I FG cylindrical shell with simply-supported boundary conditions. ($m = 1$, $L/R = 20$, $R/h = 500$)

| $n$ | Naeem et al. [19] | | | Present | | |
|---|---|---|---|---|---|---|
| | $N = 0.5$ | $N = 1$ | $N = 15$ | $N = 0.5$ | $N = 1$ | $N = 15$ |
| 1 | 13.321 | 13.211 | 12.933 | 13.321 | 13.211 | 12.932 |
| 2 | 4.5162 | 4.4794 | 4.3829 | 4.5164 | 4.4796 | 4.3830 |
| 3 | 4.1903 | 4.1562 | 4.0646 | 4.1909 | 4.1564 | 4.0641 |
| 4 | 7.0967 | 7.0379 | 6.8851 | 7.0976 | 7.0381 | 6.8836 |
| 5 | 11.335 | 11.241 | 10.998 | 11.336 | 11.241 | 10.996 |
| 6 | 16.594 | 16.455 | 16.101 | 16.595 | 16.455 | 16.097 |
| 7 | 22.826 | 22.635 | 22.148 | 22.828 | 22.635 | 22.143 |
| 8 | 30.023 | 29.771 | 29.132 | 30.025 | 29.771 | 29.125 |
| 9 | 38.181 | 37.862 | 37.048 | 38.185 | 37.862 | 37.039 |
| 10 | 47.301 | 46.905 | 45.897 | 47.305 | 46.905 | 45.886 |

(b) Comparison of natural frequencies (Hz) for type II FG cylindrical shell with simply-supported boundary conditions. ($m = 1$, $L/R = 20$, $R/h = 500$)

| $n$ | Naeem et al. [19] | | | Present | | |
|---|---|---|---|---|---|---|
| | $N = 0.5$ | $N = 1$ | $N = 15$ | $N = 0.5$ | $N = 1$ | $N = 15$ |
| 1 | 13.103 | 13.211 | 13.505 | 13.103 | 13.211 | 13.505 |
| 2 | 4.4382 | 4.4742 | 4.5759 | 4.4378 | 4.4739 | 4.5757 |
| 3 | 4.1152 | 4.1486 | 4.2451 | 4.1144 | 4.1481 | 4.2454 |
| 4 | 6.9754 | 7.0330 | 7.1943 | 6.9744 | 7.0327 | 7.1956 |
| 5 | 11.145 | 11.238 | 11.494 | 11.143 | 11.237 | 11.496 |
| 6 | 16.317 | 16.453 | 16.453 | 16.315 | 16.452 | 16.830 |
| 7 | 22.447 | 22.633 | 23.147 | 22.444 | 22.633 | 23.152 |
| 8 | 29.524 | 29.770 | 30.446 | 29.521 | 29.770 | 30.453 |
| 9 | 37.548 | 37.861 | 38.720 | 37.544 | 37.861 | 38.729 |
| 10 | 46.517 | 46.904 | 47.968 | 46.511 | 46.904 | 47.979 |

Table 5: Variation of natural frequencies against circumferential wave number $n$, type I FG cylindrical shell with nickel ring stiffeners ($m = 1$, $h/R = 0.002$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 26.831 | 25.864 | 26.497 | 26.421 | 26.335 | 26.176 | 26.018 | 25.922 | 25.894 |
| 2 | 30.397 | 29.289 | 30.014 | 29.926 | 29.828 | 29.645 | 29.466 | 29.355 | 29.323 |
| 3 | 47.192 | 45.494 | 46.603 | 46.468 | 46.317 | 46.037 | 45.762 | 45.594 | 45.545 |
| 4 | 82.416 | 79.471 | 81.393 | 81.158 | 80.896 | 80.411 | 79.935 | 79.643 | 79.559 |
| 5 | 131.10 | 126.42 | 129.47 | 129.10 | 128.68 | 127.91 | 127.16 | 126.69 | 126.56 |
| 6 | 191.52 | 184.69 | 189.14 | 188.60 | 187.99 | 186.86 | 185.76 | 185.09 | 184.89 |
| 7 | 263.21 | 253.82 | 259.59 | 259.20 | 258.37 | 256.82 | 255.30 | 254.37 | 254.11 |
| 8 | 346.02 | 333.69 | 341.74 | 340.75 | 339.66 | 337.62 | 335.63 | 334.41 | 334.06 |
| 9 | 439.88 | 424.20 | 434.43 | 433.18 | 431.79 | 429.21 | 426.68 | 425.12 | 424.68 |
| 10 | 544.74 | 525.34 | 538.01 | 536.46 | 534.73 | 531.53 | 528.40 | 526.48 | 525.93 |

Table 6: Variation of natural frequencies against circumferential wave number $n$, type I FG cylindrical shell with nickel ring stiffeners ($m = 1$, $h/R = 0.05$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 14.715 | 14.025 | 14.436 | 14.421 | 14.360 | 14.246 | 14.134 | 14.066 | 14.046 |
| 2 | 35.517 | 33.764 | 34.836 | 34.768 | 34.618 | 34.342 | 34.068 | 33.885 | 33.828 |
| 3 | 96.256 | 91.453 | 94.364 | 94.184 | 93.772 | 93.017 | 92.275 | 91.782 | 91.626 |
| 4 | 183.17 | 174.01 | 179.56 | 179.21 | 178.43 | 176.99 | 175.57 | 174.64 | 174.34 |
| 5 | 295.50 | 280.72 | 289.66 | 289.11 | 287.84 | 285.52 | 283.24 | 281.73 | 281.25 |
| 6 | 433.06 | 411.40 | 424.51 | 423.70 | 421.84 | 418.44 | 415.10 | 412.88 | 412.18 |
| 7 | 595.76 | 565.96 | 584.00 | 582.89 | 580.33 | 575.65 | 571.05 | 568.00 | 567.04 |
| 8 | 783.53 | 744.35 | 768.06 | 766.60 | 763.24 | 757.08 | 751.04 | 747.03 | 745.76 |
| 9 | 996.33 | 946.51 | 976.66 | 974.80 | 970.53 | 962.70 | 955.02 | 949.92 | 948.31 |
| 10 | 1234.1 | 1172.4 | 1209.7 | 1207.46 | 1202.17 | 1192.5 | 1182.9 | 1176.6 | 1174.6 |

Table 7: Variation of natural frequencies against circumferential wave number $n$, type II FG cylindrical shell with nickel ring stiffeners ($m = 1$, $h/R = 0.002$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 26.831 | 25.864 | 26.176 | 26.251 | 26.335 | 26.498 | 26.663 | 26.768 | 26.798 |
| 2 | 30.397 | 29.289 | 29.647 | 29.732 | 29.830 | 30.016 | 30.204 | 30.324 | 30.359 |
| 3 | 47.192 | 45.494 | 46.044 | 46.176 | 46.325 | 46.611 | 46.897 | 47.082 | 47.135 |
| 4 | 82.416 | 79.471 | 80.426 | 80.655 | 80.914 | 81.408 | 81.904 | 82.225 | 82.317 |
| 5 | 131.10 | 126.42 | 127.94 | 128.30 | 128.71 | 129.50 | 130.28 | 130.80 | 130.94 |
| 6 | 191.52 | 184.69 | 186.90 | 187.43 | 188.03 | 189.18 | 190.33 | 191.08 | 191.29 |
| 7 | 263.21 | 253.82 | 256.87 | 257.60 | 258.42 | 260.00 | 261.58 | 262.60 | 262.90 |
| 8 | 346.02 | 333.69 | 337.69 | 338.65 | 339.73 | 341.80 | 343.88 | 345.22 | 345.61 |
| 9 | 439.88 | 424.20 | 429.29 | 430.51 | 431.89 | 434.52 | 437.15 | 438.87 | 439.36 |
| 10 | 544.74 | 525.34 | 531.64 | 533.15 | 534.85 | 538.11 | 541.37 | 543.49 | 544.10 |

the natural frequencies approached those of $N^{SS}$ and when $N$ is large they approached those of $N^{N}$. Hence the natural frequencies for $N$greater than zero fell between those of $N^{SS}$ and $N^{N}$ for a given circumferential wave number $n$.

Tables 7 and 8 show the variation of the natural frequencies (Hz) versus the circumferential wave number "$n$" for a Type II FGM cylindrical shell with Nickel ring-stiffeners. The influence of $N$ or constituent volume fraction on the natural frequencies is the opposite of a Type I FGM cylindrical shell. Unlike a Type I FGM cylindrical shell where the natural frequencies (Hz) decreased with $N$, the natural frequencies (Hz) for a Type II FGM cylindrical shell increased with $N$. The increase in the natural frequencies (Hz) from $N = 1$ to $N = 15$ is about 1.64% at $n = 1$ and about 1.61% at $n = 10$. Thus the influence of the constituent volume fractions for a Type II FGM cylindrical shell is different from that of a Type I FGM cylindrical shell.

Comparing the frequencies in Tables 5 and 6 with those in Tables 7 and 8, it can be seen that for $N > 1$, the natural frequencies of a Type II FG cylindrical shell with Nickel ring-stiffeners are higher than those a Type I FG cylindrical shell with Nickel ring-stiffeners.

On the other hand, for $N < 1$, the frequencies for a Type I FGM cylindrical shell with Nickel ring-stiffeners are higher than a Type II FG cylindrical shell with Nickel ring-stiffeners. For example, for $N = 15$ at $n = 10$ and $h/R = 0.002$, the natural frequencies (Hz) for a Type II FG cylindrical shell with Nickel ring-stiffeners are about 3.23% higher than a Type I FG cylindrical shell with Nickel ring-stiffeners. For $N = 0.5$ at $n = 10$ and $h/R = 0.002$, the frequencies for a Type I FGM cylindrical shell with Nickel ring-stiffeners are 1.26% higher than a Type II FG cylindrical shell with Nickel ring-stiffeners. Thus the natural frequencies are affected by the configuration of the constituent materials in the functionally graded cylindrical shells.

Tables 9 and 10 show the variation of natural frequencies against circumferential wave number $n$, for Type I FG cylindrical shell with Stainless Steel ring-stiffeners. The influence of the value of $N$, which affects the constituent volume fraction, can be seen from the tables. It is seen that when the value of $N$ is increased, the natural frequencies decreased. The decrease in the natural frequencies in Table 9, from $N = 1$ to $N = 15$ is about 1.60% at $n = 1$ and about 1.56% at $n = 10$. When $N$ is small, the natural frequencies approached those of $N^{SS}$ and when $N$is large, they approached those of $N^{N}$. Hence the natural frequencies for $N$ greater than zero fell between those of $N^{SS}$ and $N^{N}$ for a given circumferential wave number $n$.

Tables 11 and 12 show the variation of natural frequencies against circumferential wave number $n$, for Type II FG cylindrical shell with Stainless Steel ring-stiffeners. The influence of $N$ or the constituent volume fraction

Table 8: Variation of natural frequencies against circumferential wave number $n$, type II FG cylindrical shell with nickel ring stiffeners ($m = 1$, $h/R = 0.05$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 14.714 | 14.025 | 14.245 | 14.298 | 14.359 | 14.475 | 14.594 | 14.669 | 14.691 |
| 2 | 35.517 | 33.764 | 34.329 | 34.461 | 34.610 | 34.891 | 35.178 | 35.383 | 35.446 |
| 3 | 96.256 | 91.453 | 93.018 | 93.383 | 93.790 | 94.559 | 95.344 | 95.892 | 96.063 |
| 4 | 183.17 | 174.01 | 177.00 | 177.70 | 178.47 | 179.94 | 181.43 | 182.48 | 182.80 |
| 5 | 295.50 | 280.72 | 285.54 | 286.66 | 287.92 | 290.28 | 292.70 | 294.38 | 294.90 |
| 6 | 433.06 | 411.40 | 418.47 | 420.12 | 421.96 | 425.42 | 428.96 | 431.42 | 432.19 |
| 7 | 595.76 | 565.96 | 575.69 | 577.96 | 580.49 | 585.25 | 590.12 | 593.50 | 594.57 |
| 8 | 783.53 | 744.35 | 757.14 | 760.12 | 763.44 | 769.71 | 776.11 | 780.56 | 781.96 |
| 9 | 996.33 | 946.51 | 962.78 | 966.56 | 970.79 | 978.76 | 986.89 | 992.55 | 994.33 |
| 10 | 1234.1 | 1172.4 | 1192.5 | 1197.2 | 1202.5 | 1212.3 | 1222.4 | 1229.4 | 1231.6 |

Table 9: Variation of natural frequencies against circumferential wave number $n$, type I FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $h/R = 0.002$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 25.274 | 24.348 | 24.955 | 24.881 | 24.799 | 24.646 | 24.496 | 24.403 | 24.377 |
| 2 | 29.526 | 28.439 | 29.151 | 29.064 | 28.968 | 28.788 | 28.613 | 28.504 | 28.473 |
| 3 | 45.740 | 44.073 | 45.161 | 45.029 | 44.881 | 44.605 | 44.337 | 44.171 | 44.124 |
| 4 | 79.445 | 76.570 | 78.446 | 78.217 | 77.962 | 77.487 | 77.024 | 76.739 | 76.658 |
| 5 | 126.21 | 121.65 | 124.63 | 124.26 | 123.86 | 123.10 | 122.37 | 121.92 | 121.79 |
| 6 | 184.32 | 177.66 | 182.00 | 181.47 | 180.88 | 179.78 | 178.71 | 178.05 | 177.86 |
| 7 | 253.29 | 244.15 | 250.11 | 249.38 | 248.57 | 247.06 | 245.59 | 244.68 | 244.42 |
| 8 | 332.97 | 320.96 | 328.79 | 327.83 | 326.77 | 324.79 | 322.85 | 321.66 | 321.32 |
| 9 | 423.29 | 408.03 | 417.98 | 416.77 | 415.41 | 412.90 | 410.43 | 408.92 | 408.49 |
| 10 | 524.21 | 505.31 | 517.64 | 516.14 | 514.46 | 511.34 | 508.29 | 506.42 | 505.88 |

on the natural frequencies is opposite of a Type I FGM cylindrical shell. Here, the natural frequencies for a Type II FGM cylindrical shell increase with increasing the value of $N$. The increase in the natural frequencies from $N = 1$ to $N = 15$ is about 1.67% at $n = 1$ and about 1.63% at $n = 10$. Thus the influence of the constituent volume fractions for a Type II FGM cylindrical shell is different from that of a Type I FGM cylindrical shell.

Comparing the frequencies in Tables 9 and 10 with those in Tables 11 and 12, it can be seen that for $N > 1$, the natural frequencies of a Type II FGM cylindrical shell with Stainless Steel ring-stiffeners are higher than those of a Type I FGM cylindrical shell with Stainless Steel ring-stiffeners. On the other hand, for $N < 1$, the frequencies for a Type I FGM cylindrical shell with Stainless Steel ring-stiffeners are higher than those for a Type II FGM cylindrical shell with Stainless Steel ring-stiffeners. For example, for $N = 15$ at $n = 10$ and $h/R = 0.002$, the natural frequencies for a Type II FGM with Stainless Steel ring-stiffeners is about 3.27% higher than a Type I FGM cylindrical shell with Stainless Steel ring-stiffeners. For $N = 0.5$ at $n = 10$ and $h/R = 0.002$, the natural frequency for a Type I FGM cylindrical shell with Stainless Steel ring-stiffeners is 1.2% higher than a Type II FGM cylindrical shell with Stainless Steel ring-stiffeners. Thus the natural frequencies are affected by the configuration of the constituent materials in the functionally graded cylindrical shells.

*5.3.2. Variation of Minimum Frequency with L/R.* Tables 13 and 14 show the variations of the fundamental natural frequencies (Hz) with the $L/R$ ratio for a Type I and Type II FGM cylindrical shells with Stainless Steel ring-stiffeners. For a Type I FGM cylindrical shell with Stainless Steel ring-stiffeners, the fundamental frequencies decreased with $N$, and for a Type II FGM cylindrical shell with Stainless Steel ring-stiffeners, the fundamental frequencies increased with $N$. The difference in the fundamental frequencies between $N = 1$ and $N = 15$ is about 1.78% for Type I and 1.87% for Type II FGM cylindrical shells with Stainless steel ring-stiffeners. The fundamental natural frequencies for Type I and Type II FGM cylindrical shells with Stainless Steel ring-stiffeners occur at the same circumferential wave numbers. For all values of $N$, the fundamental natural frequencies fall between those for $N^{SS}$ and $N^{N}$.

*5.3.3. Variation of Minimum Frequency with h/R.* Tables 15 and 16 show the variations of the fundamental natural frequencies (Hz) with the $h/R$ ratio for a Type I and Type II FGM cylindrical shells with Stainless Steel ring-stiffeners. For a Type I FGM cylindrical shell with Stainless Steel ring-stiffeners, the fundamental frequencies decreased with $N$,

Table 10: Variation of natural frequencies against circumferential wave number $n$, type I FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $h/R = 0.05$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 14.482 | 13.800 | 14.206 | 14.192 | 14.131 | 14.019 | 13.908 | 13.841 | 13.821 |
| 2 | 35.176 | 33.434 | 34.499 | 34.433 | 34.283 | 34.009 | 33.737 | 33.555 | 33.498 |
| 3 | 95.883 | 91.090 | 93.995 | 93.817 | 93.405 | 92.652 | 91.911 | 91.419 | 91.263 |
| 4 | 182.65 | 173.50 | 179.04 | 178.70 | 177.91 | 176.48 | 175.07 | 174.13 | 173.83 |
| 5 | 294.72 | 279.96 | 288.89 | 288.35 | 287.08 | 284.76 | 282.48 | 280.97 | 280.49 |
| 6 | 431.95 | 410.31 | 423.41 | 422.60 | 420.75 | 417.35 | 414.01 | 411.79 | 411.09 |
| 7 | 594.24 | 564.47 | 582.49 | 581.38 | 578.83 | 574.15 | 569.56 | 566.51 | 565.55 |
| 8 | 781.54 | 742.40 | 766.08 | 764.63 | 761.27 | 755.12 | 749.08 | 745.07 | 743.81 |
| 9 | 993.81 | 944.04 | 974.15 | 972.31 | 968.04 | 960.22 | 952.54 | 947.45 | 945.84 |
| 10 | 1231.0 | 1169.4 | 1206.6 | 1204.4 | 1199.1 | 1189.4 | 1179.9 | 1173.6 | 1171.6 |

Table 11: Variation of natural frequencies against circumferential wave number $n$, type II FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $h/R = 0.002$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 25.274 | 24.348 | 24.646 | 24.718 | 24.799 | 24.955 | 25.113 | 25.214 | 25.243 |
| 2 | 29.526 | 28.439 | 28.790 | 28.874 | 28.970 | 29.152 | 29.338 | 29.455 | 29.490 |
| 3 | 45.740 | 44.073 | 44.613 | 44.742 | 44.888 | 45.168 | 45.451 | 45.632 | 45.684 |
| 4 | 79.445 | 76.570 | 77.502 | 77.726 | 77.978 | 78.461 | 78.945 | 79.259 | 79.349 |
| 5 | 126.21 | 121.65 | 123.12 | 123.48 | 123.88 | 124.65 | 125.42 | 125.91 | 126.06 |
| 6 | 184.32 | 177.66 | 179.81 | 180.34 | 180.92 | 182.04 | 183.16 | 183.89 | 184.09 |
| 7 | 253.29 | 244.15 | 247.11 | 247.82 | 248.63 | 250.16 | 251.70 | 252.70 | 252.99 |
| 8 | 332.97 | 320.96 | 324.85 | 325.79 | 326.85 | 328.86 | 330.88 | 332.20 | 332.57 |
| 9 | 423.29 | 408.03 | 412.97 | 414.16 | 415.51 | 418.07 | 420.64 | 422.30 | 422.78 |
| 10 | 524.21 | 505.31 | 511.44 | 512.91 | 514.57 | 517.74 | 520.92 | 522.99 | 523.58 |

and for a Type II FGM cylindrical shell with Stainless Steel ring-stiffeners, the fundamental frequencies increased with $N$. For all values of $N$, the fundamental natural frequencies lie between those for $N^{SS}$ and $N^{N}$.

The frequency characteristics of FGM cylindrical shells with ring-stiffeners are similar to those for homogeneous isotropic cylindrical shells. Other interesting frequency characteristics are also observed in the FGM cylindrical shells. These characteristics arise when the constituent volume fractions and the configurations of the constituent materials in the functionally graded cylindrical shells are varied in the thickness direction.

## 6. Conclusion

In this study, the Rayleigh-Ritz approach has been employed to analyze the vibration characteristics of functionally graded circular cylindrical shells with ring-stiffeners of different materials. The axial model dependence has been approximated by the characteristic beam functions. Sander's thin shell theory of first order has been used to perform the vibration analysis. From the vibration results of cylindrical shells with identical and evenly spaced ring-stiffeners, it is found that stiffeners placed eccentrically are more effective than concentric ones. The study is carried out for isotropic as well as two types of functionally graded cylindrical shell with and without ring stiffeners where the configurations of the constituent materials in the functionally graded cylindrical shells are varied by the volume fraction law. One is termed as Type I FG cylindrical shell and has properties that vary continuously from Nickel on its inner surface and Stainless Steel on its outer surface. The other is termed as a Type II FG cylindrical shell and has properties that vary continuously from Stainless Steel on its inner surface and Nickel on its outer surface. A validation of the analysis has been carried out by comparing results with those found in literature and a good agreement has been observed among the results evaluated by different shell theories and numerical approaches. It is seen that the variations of natural frequency of FGM circular cylinders are similar to that of isotropic ones. The frequency is influenced by the volume fraction law exponents. It decreases or increases with N depending upon the order of constituent material in FGM shells. For the Type I and Type II FG cylindrical shells the natural frequencies for all values of N lie between those for a Stainless Steel and Nickel cylindrical shells. For $N < 1$ the natural frequencies for Type I FG cylindrical shells are higher than those for Type II FG cylindrical shells and for $N > 1$ the natural frequencies for Type II FG cylindrical shells are higher than those for Type I FG cylindrical shells. Thus the constituent volume fractions and the configurations of the constituent materials

Table 12: Variation of natural frequencies against circumferential wave number $n$, type II FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $h/R = 0.05$, $L/R = 20$, $d/b = 1.334$).

| $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 14.482 | 13.800 | 14.018 | 14.071 | 14.130 | 14.245 | 14.362 | 14.436 | 14.458 |
| 2 | 35.176 | 33.434 | 33.995 | 34.126 | 34.274 | 34.553 | 34.839 | 35.043 | 35.105 |
| 3 | 95.883 | 91.090 | 92.651 | 93.015 | 93.421 | 94.188 | 94.972 | 95.519 | 95.691 |
| 4 | 182.65 | 173.50 | 176.49 | 177.18 | 177.96 | 179.42 | 180.91 | 181.95 | 182.28 |
| 5 | 294.72 | 279.96 | 284.78 | 285.90 | 287.15 | 289.51 | 291.92 | 293.60 | 294.13 |
| 6 | 431.95 | 410.31 | 417.37 | 419.02 | 420.85 | 424.31 | 427.85 | 430.31 | 431.08 |
| 7 | 594.24 | 564.47 | 574.19 | 576.45 | 578.98 | 583.74 | 588.60 | 591.98 | 593.05 |
| 8 | 781.54 | 742.40 | 755.17 | 758.14 | 761.47 | 767.73 | 774.12 | 778.57 | 779.97 |
| 9 | 993.81 | 944.04 | 960.28 | 964.06 | 968.29 | 976.24 | 984.38 | 990.03 | 991.81 |
| 10 | 1231.0 | 1169.4 | 1189.5 | 1194.2 | 1199.4 | 1209.3 | 1219.3 | 1226.3 | 1228.5 |

Table 13: Variation of natural frequencies against $L/R$, type I FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $h/R = 0.002$, $d/b = 1.334$).

| $L/R$ | $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 20 | 2875.5 | 2866.8 | 2872.5 | 2871.8 | 2871.1 | 2869.6 | 2868.2 | 2867.3 | 2867.1 |
| 0.5 | 15 | 1614.0 | 1602.9 | 1610.2 | 1609.4 | 1608.4 | 1606.5 | 1604.7 | 1603.5 | 1603.3 |
| 1.0 | 11 | 900.77 | 890.59 | 897.30 | 896.49 | 895.59 | 893.89 | 892.22 | 891.20 | 890.90 |
| 2.0 | 8 | 488.92 | 480.57 | 486.06 | 485.40 | 484.66 | 483.27 | 481.91 | 481.07 | 480.82 |
| 5.0 | 5 | 181.84 | 177.15 | 180.23 | 179.85 | 179.44 | 178.66 | 177.90 | 177.43 | 177.29 |
| 10 | 4 | 100.74 | 97.538 | 99.630 | 99.380 | 99.095 | 98.566 | 98.046 | 97.727 | 97.636 |
| 20 | 3 | 45.740 | 44.073 | 45.162 | 45.029 | 44.881 | 44.605 | 44.336 | 44.171 | 44.124 |
| 50 | 2 | 13.408 | 12.874 | 13.222 | 13.179 | 13.132 | 13.044 | 12.958 | 12.905 | 12.890 |
| 100 | 1 | 2.5497 | 2.4455 | 2.5135 | 2.5052 | 2.4960 | 2.4788 | 2.4620 | 2.4516 | 2.4486 |

Table 14: Variation of natural frequencies against $L/R$, type II FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $h/R = 0.002$, $d/b = 1.334$).

| $L/R$ | $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ | $N = 30$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 20 | 2875.5 | 2866.8 | 2869.3 | 2870.5 | 2871.2 | 2872.7 | 2874.1 | 2875.0 | 2875.2 |
| 0.5 | 15 | 1614.0 | 1602.9 | 1606.3 | 1607.6 | 1608.5 | 1610.4 | 1612.2 | 1613.4 | 1613.7 |
| 1.0 | 11 | 900.77 | 890.59 | 893.66 | 894.81 | 895.71 | 897.41 | 899.07 | 900.15 | 900.45 |
| 2.0 | 8 | 488.92 | 480.57 | 483.07 | 484.00 | 484.74 | 486.14 | 487.51 | 488.40 | 488.65 |
| 5.0 | 5 | 181.84 | 177.15 | 178.52 | 179.06 | 179.47 | 180.26 | 181.04 | 181.54 | 181.68 |
| 10 | 4 | 100.74 | 97.538 | 98.430 | 98.833 | 99.115 | 99.652 | 100.19 | 100.54 | 100.64 |
| 20 | 3 | 45.740 | 44.073 | 44.505 | 44.742 | 44.888 | 45.168 | 45.450 | 45.631 | 45.684 |
| 50 | 2 | 13.408 | 12.874 | 12.994 | 13.086 | 13.133 | 13.223 | 13.314 | 13.373 | 13.390 |
| 100 | 1 | 2.5497 | 2.4455 | 2.4788 | 2.4868 | 2.4959 | 2.5135 | 2.5314 | 2.5427 | 2.5461 |

Table 15: Variation of fundamental natural frequencies against $h/R$, type I FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $L/R = 20$, $d/b = 1.334$).

| $h/R$ | $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.001 | 3 | 52.567 | 50.896 | 51.992 | 51.859 | 51.711 | 51.434 | 51.163 | 50.995 |
| 0.005 | 2 | 24.993 | 23.985 | 24.643 | 24.563 | 24.473 | 24.307 | 24.145 | 24.045 |
| 0.007 | 2 | 23.114 | 22.162 | 22.783 | 22.707 | 22.623 | 22.466 | 22.313 | 22.218 |
| 0.01 | 2 | 21.393 | 20.493 | 21.081 | 21.000 | 20.929 | 20.782 | 20.637 | 20.547 |
| 0.02 | 1 | 15.812 | 15.090 | 15.561 | 15.504 | 15.440 | 15.321 | 15.204 | 15.133 |
| 0.03 | 1 | 15.096 | 14.396 | 15.060 | 14.798 | 14.736 | 14.621 | 14.507 | 14.438 |
| 0.04 | 1 | 14.717 | 14.028 | 14.472 | 14.424 | 14.362 | 14.249 | 14.137 | 14.069 |
| 0.05 | 1 | 14.482 | 13.800 | 14.245 | 14.191 | 14.131 | 14.019 | 13.908 | 13.841 |

Table 16: Variation of fundamental natural frequencies against $h/R$, type II FG cylindrical shell with stainless steel ring stiffeners ($m = 1$, $L/R = 20$, $d/b = 1.334$).

| $h/R$ | $n$ | $N^{SS} = 0$ | $N^{N} = 0$ | $N = 0.5$ | $N = 0.7$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 15$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.001 | 3 | 52.567 | 50.896 | 51.439 | 51.569 | 51.716 | 51.996 | 52.279 | 52.459 |
| 0.005 | 2 | 24.993 | 23.985 | 24.309 | 24.387 | 24.475 | 24.645 | 24.816 | 24.927 |
| 0.007 | 2 | 23.114 | 22.162 | 22.467 | 22.541 | 22.624 | 22.784 | 22.945 | 23.050 |
| 0.01 | 2 | 21.393 | 20.493 | 20.782 | 20.851 | 20.930 | 21.081 | 21.232 | 21.333 |
| 0.02 | 1 | 15.812 | 15.090 | 15.321 | 15.376 | 15.440 | 15.561 | 15.685 | 15.764 |
| 0.03 | 1 | 15.096 | 14.396 | 14.620 | 14.674 | 14.735 | 14.853 | 14.974 | 15.050 |
| 0.04 | 1 | 14.717 | 14.028 | 14.248 | 14.301 | 14.361 | 14.477 | 14.596 | 14.671 |
| 0.05 | 1 | 14.482 | 13.800 | 14.018 | 14.070 | 14.130 | 14.245 | 14.362 | 14.436 |

affect the natural frequencies. This work can be extended to analyze vibrations of cylindrical shells by varying material composition by the interchange of isotropic and functionally graded layers in the radial direction.

## Appendix

## Stiffness and Mass Matrices

One has

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \qquad [D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}, \tag{A.1}$$

where

$$\begin{aligned} c_{11} = {} & R\Bigg(A_{11}\int_0^L \left(\frac{dU}{dx}\right)^2 dx \\ & + \frac{n^2}{R^2}\left(A_{66} - B_{66} + \frac{1}{4}B_{66}\right)\int_0^L U^2 dx\Bigg) \\ & + \left(\frac{n^2}{(R+e_k)^3}\left(G_k J_k + n^2 E_k I_{zk}\right)U^2\right), \\ c_{12} = {} & R\Bigg[\frac{n}{R}A_{12}\int_0^L V\frac{dU}{dx}dx \\ & + \frac{n}{R}\left(-A_{66} - B_{66} + \frac{3}{4}D_{66}\right)\int_0^L U\frac{dV}{dx}dx\Bigg], \\ c_{13} = {} & \Bigg[R\Bigg(\left(-\frac{A_{12}}{R} + \frac{1-n^2}{R^2}B_{12}\right)\int_0^L W\frac{dU}{dx}dx \\ & + B_{11}\int_0^L \frac{d^2W}{dx^2}\frac{dU}{dx}dx \\ & + \frac{n^2}{R}\left(2B_{66} - D_{66}\right)\int_0^L U\frac{dW}{dx}dx \\ & + \left(\frac{n^2 E_k I_{zk}}{(R+e_k)^2}\left(\frac{n^2 e_k}{(R+e_k)} - 1\right)\right)U\frac{dW}{dx} \\ & + \left(\frac{n^2 G_k J_k}{(R+e_k)^2}\left(\frac{e_k}{(R+e_k)} - 1\right)\right)U\frac{dW}{dx}\Bigg], \end{aligned}$$

$$\begin{aligned} c_{22} = {} & R\Bigg(\frac{n^2}{R^2}A_{22}\int_0^L V^2 dx \\ & + \left(A_{66} + 3B_{66} + \frac{9}{4}D_{66}\right)\int_0^L \left(\frac{dV}{dx}\right)^2 dx\Bigg) \\ & + \frac{n^2 E_k A_k}{R+e_k}\left(1 + \frac{e_k}{R}\right)^2 V^2, \\ c_{23} = {} & R\Bigg(\frac{n}{R^2}\left(-A_{22} + \frac{1-n^2}{R}B_{22}\right)\int_0^L VW dx \\ & - n\left(2B_{66} + 3D_{66}\right)\int_0^L \frac{dV}{dx}\frac{dW}{dx}dx \\ & + \frac{n}{R}B_{12}\int_0^L V\frac{d^2W}{dx^2}dx\Bigg) \\ & - \frac{nE_k A_k}{(R+e_k)}\left(1 + \frac{e_k}{R}\right)\left(1 + \frac{n^2 e_k}{R}\right)VW, \\ c_{33} = {} & R\Bigg\{\frac{1}{R^2}\left(A_{22} - \frac{2(1-n^2)}{R}B_{22} + \frac{(1-n^2)^2}{R^2}D_{22}\right) \\ & \times \int_0^L W^2 dx + D_{11}\int_0^L \left(\frac{d^2W}{dx^2}\right)^2 dx \\ & \times \frac{2}{R^2}\left(-RB_{12} + (1-n^2)D_{12}\right)\int_0^L W\frac{d^2W}{dx^2}dx \\ & + 4n^2 D_{66}\int_0^L \frac{dW}{dx}dx\Bigg\} \\ & + \left(\frac{E_k I_{zk}}{(R+e_k)}\left(1 - \frac{n^2 e_k}{R+e_k}\right)^2 \right. \\ & \left. + \frac{n^2 G_k J_k}{(R+e_k)}\left(1 - \frac{e_k}{R+e_k}\right)^2\right)\left(\frac{dW}{dx}\right)^2 \\ & + \left(\frac{E_k I_{xk}}{(R+e_k)^3}\left(1-n^2\right)^2 \right. \\ & \left. + \left(1 + \frac{n^2 e_k}{R}\right)^2 \frac{E_k A_k}{(R+e_k)}\right)W^2, \end{aligned}$$

$$
\begin{aligned}
d_{11} &= Rh\rho \int_0^L U^2 dx + \rho_k A_k (R+e_k) U^2, \\
d_{12} &= 0, \\
d_{13} &= \rho_k A_k e_k (R+e_k) U \frac{dw}{dx}, \\
d_{22} &= Rh\rho \int_0^L V^2 dx + \rho_k A_k (A+e_k) \left(1+\frac{e_k}{R}\right)^2 V^2, \\
d_{23} &= -n e_k \rho_k A_k (R+e_k) \left(1+\frac{e_k}{R}\right) \frac{1}{R} V W, \\
d_{33} &= R\rho h \int_0^L W^2 dx \\
&\quad + \rho_k (R+e_k)\left({e_k}^2 A_k + I_{xk} + I_{zk}\right)\left(\frac{dw}{dx}\right)^2 \\
&\quad + \rho_k A_k (R+e_k)\left(1+\frac{n^2 {e_k}^2}{R^2}\right) W^2.
\end{aligned}
\tag{A.2}
$$

Here

$$
\begin{aligned}
U &= U(x) = \frac{dV(x)}{dx}, \\
V &= V(x) = \sin\left(\frac{m\pi x}{L}\right), \\
W &= W(x) = V(x), \\
\int_0^L V^2(x) dx &= \frac{L}{2}, \\
\int_0^L U^2(x) dx &= \frac{(m\pi)^2}{2L}, \\
\int_0^L \left(\frac{dU(x)}{dx}\right)^2 dx &= \frac{(m\pi)^4}{2L^3}, \\
\int_0^L \frac{d^2 V(x)}{dx^2} V(x) dx &= -\frac{(m\pi)^2}{2L}.
\end{aligned}
\tag{A.3}
$$

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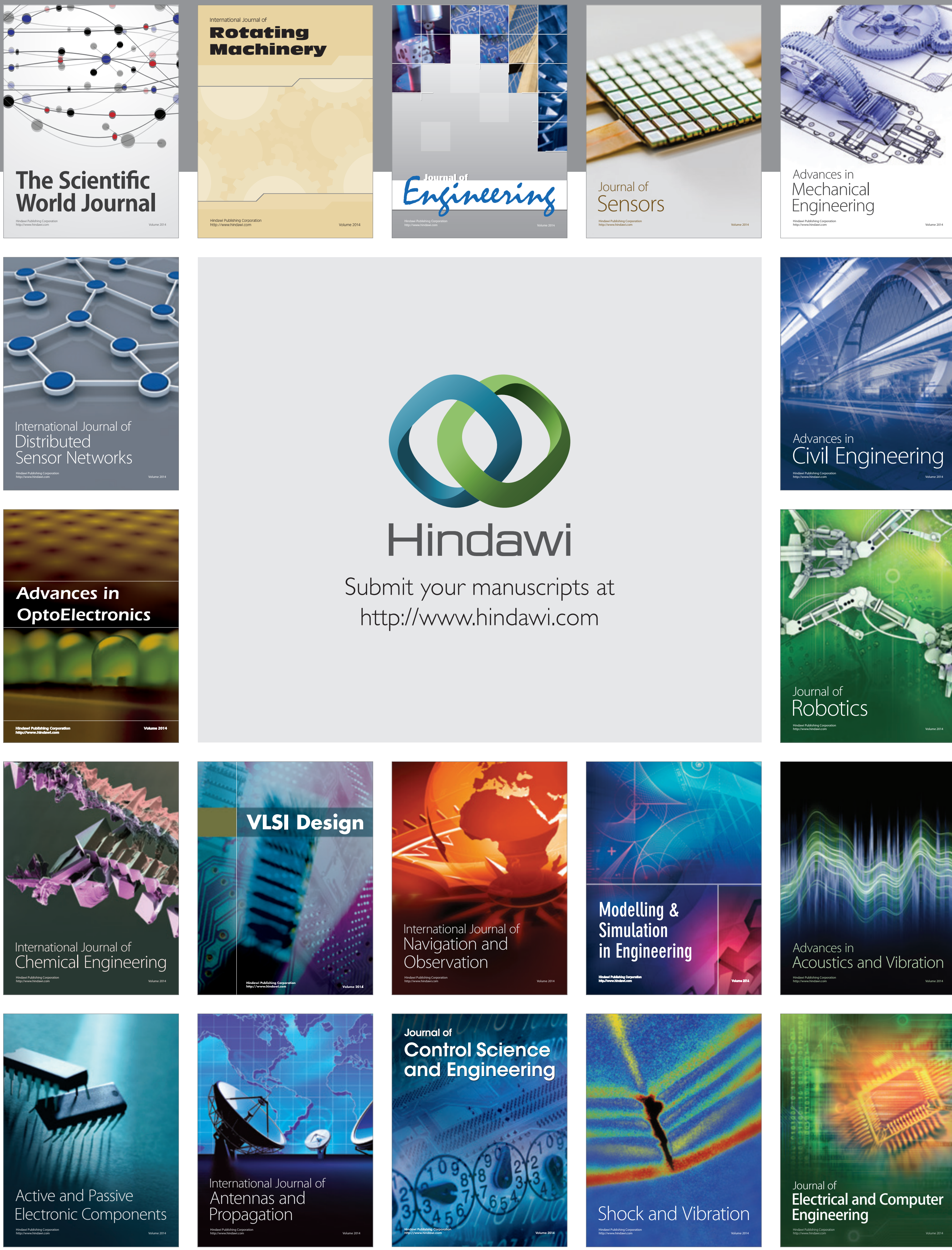
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