QUASI-DEFINITENESS OF GENERALIZED UVAROV TRANSFORMS OF MOMENT FUNCTIONALS

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When σ is a quasi-definite moment functional with the monic orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$, we consider a point masses perturbation τ of σ given by $\tau := \sigma + \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} ((-1)^k u_{lk}/k!) \delta^{(k)}(x - c_l)$, where λ , u_{lk} , and c_l are constants with $c_i \neq c_j$ for $i \neq j$. That is, τ is a generalized Uvarov transform of σ satisfying $A(x)\tau = A(x)\sigma$, where $A(x) = \prod_{l=1}^m (x - c_l)^{m_l+1}$. We find necessary and sufficient conditions for τ to be quasi-definite. We also discuss various properties of monic orthogonal polynomial system $\{R_n(x)\}_{n=0}^{\infty}$ relative to τ including two examples.

1. Introduction

In the study of Padé approximation (see [5, 10, 21]) of Stieltjes type meromorphic functions

$$\int_a^b \frac{d\mu(x)}{z-x} + \sum_{l=1}^m \sum_{k=0}^{m_l} C_{lk} \frac{k!}{(z-c_l)^{k+1}}, \quad (1.1)$$

where $-\infty \leq a < b \leq \infty$, C_{lk} are constants, and $d\mu(x)$ is a positive Stieltjes measure, the denominators $R_n(x)$ of the main diagonal sequence of Padé approximants satisfy the orthogonality

$$\int_a^b R_n(x) \pi(x) d\mu(x) + \sum_{l=1}^m \sum_{k=0}^{m_l} C_{lk} \partial^k [\pi R_n](c_l) = 0, \quad \pi \in \mathbb{P}_{n-1}, \quad (1.2)$$

where $\mathbb{P}_n$ is the space of polynomials of degree $\leq n$ with $\mathbb{P}_{-1} = \{0\}$. That is, $R_n(x)$ ($n \geq 0$) are orthogonal with respect to the measure

$$d\mu + \sum_{l=1}^m \sum_{k=0}^{m_l} (-1)^k C_{lk} \delta^{(k)}(x - c_l), \quad (1.3)$$

which is a point masses perturbation of $d\mu(x)$. Orthogonality to a positive or signed measure perturbed by one or two point masses arises naturally also in orthogonal polynomial eigenfunctions of higher order (≥ 4) ordinary differential equations (see [14, 15, 16, 19]), which generalize Bochner's classification of classical orthogonal polynomials (see [6, 18]). On the other hand, many authors have studied various aspects of orthogonal polynomials with respect to various point masses perturbations of positive-definite (see [1, 2, 8, 14, 27, 28]) and quasi-definite (see [3, 4, 9, 11, 12, 20, 23]) moment functionals. In this work, we consider the most general such situation. That is, we consider a moment functional τ given by

$$\tau := \sigma + \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), \quad (1.4)$$

where σ is a given quasi-definite moment functional, λ , u_{lk} , and c_l are complex numbers with $u_{l,m_l} \neq 0$ and $c_i \neq c_j$ for $i \neq j$, that is, τ is obtained from σ by adding a distribution with finite support. We give necessary and sufficient conditions for τ to be quasi-definite. When τ is also quasi-definite, we discuss various properties of orthogonal polynomials $\{R_n(x)\}_{n=0}^\infty$ relative to τ in connection with orthogonal polynomials $\{P_n(x)\}_{n=0}^\infty$ relative to σ . These generalize many previous works in [4, 9, 11, 12, 20, 23].

2. Necessary and sufficient conditions

For any integer $n \geq 0$, let $\mathbb{P}_n$ be the space of polynomials of degree $\leq n$ and $\mathbb{P} = \bigcup_{n \geq 0} \mathbb{P}_n$. For any $\pi(x)$ in $\mathbb{P}$, let $\deg(\pi)$ be the degree of $\pi(x)$ with the convention that $\deg(0) = -1$. For the moment functionals σ, τ (i.e., linear functionals on $\mathbb{P}$) (see [7]), c in $\mathbb{C}$, and a polynomial $\phi(x) = \sum_{k=0}^n a_k x^k$, let

$$\begin{aligned} \langle \sigma', \pi \rangle &:= -\langle \sigma, \pi' \rangle; & \langle \phi \sigma, \pi \rangle &:= \langle \sigma, \phi \pi \rangle; \\ \langle (x-c)^{-1} \sigma, \pi \rangle &:= \langle \sigma, \theta_c \pi \rangle; & (\theta_c \pi)(x) &:= \frac{\pi(x) - \pi(c)}{x - c}; \\ (\sigma \phi)(x) &:= \sum_{k=0}^n \left(\sum_{j=k}^n a_j \sigma_{jk} \right) x^k; & \langle \sigma \tau, \pi \rangle &= \langle \sigma, \tau \pi \rangle, \quad \pi \in \mathbb{P}. \end{aligned} \quad (2.1)$$

We also let

$$F(\sigma)(z) := \sum_{n=0}^{\infty} \frac{\sigma_n}{z^{n+1}} \quad (2.2)$$

be the (formal) Stieltjes function of σ , where $\sigma_n := \langle \sigma, x^n \rangle$ ($n \geq 0$) are the moments of σ . Following Zhedanov [29], for any polynomials $A(z)$, $B(z)$, $C(z)$, $D(z)$ with no common zero and $|C| + |D| \neq 0$, let

$$S(A, B, C, D)F(\sigma)(z) := \frac{AF(\sigma) + B}{CF(\sigma) + D}. \quad (2.3)$$

If $S(A, B, C, D)F(\sigma) = F(\tau)$ for some moment functional τ , then we call τ a rational (resp., linear) spectral transform of σ (resp., when $C(z) = 0$). Then $S(A, B, C, D)F(\sigma) = F(\tau)$ if and only if

$$\begin{aligned} xA(x)\sigma &= C(x)(\sigma\tau) + xD(x)\tau, \\ \langle \sigma, A \rangle + x(\sigma\theta_0 A)(x) + xB(x) &= (\sigma\tau)(\theta_0 C)(x) + \langle \tau, D \rangle + x(\tau\theta_0 D)(x). \end{aligned} \quad (2.4)$$

In particular, for any c and β in $\mathbb{C}$, let

$$U(c, \beta)F(\sigma) := \frac{(z-c)F(\sigma) + \beta}{z-c} \quad (2.5)$$

be the Uvarov transform (see [28, 29]) of $F(\sigma)$. Then for any $\{c_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ in $\mathbb{C}$,

$$F(\tau) := U(c_k, \beta_k) \cdots U(c_1, \beta_1)F(\sigma) = \frac{A(z)F(\sigma) + B(z)}{A(z)}, \quad (2.6)$$

where $A(z) = \prod_{i=1}^k (z - c_i)$, $B(z) = \sum_{i=1}^k \beta_i \sum_{j=1, j \neq i}^k (z - c_j)$, and by (2.4)

$$A(x)\tau = A(x)\sigma. \quad (2.7)$$

In this case, we say that τ is a generalized Uvarov transform of σ . Conversely, if (2.7) holds for some polynomial $A(x)$ ($\neq 0$), then

$$F(\tau) = \frac{A(z)F(\sigma) + (\tau\theta_0 A)(z) - (\sigma\theta_0 A)(z)}{A(z)} \quad (2.8)$$

and $F(\tau)$ is obtained from $F(\sigma)$ by $\deg(A)$ successive Uvarov transforms (see [29]), that is, τ is a generalized Uvarov transform of σ .

In the following, we always assume that τ is a moment functional given by (1.4), where σ is a quasi-definite moment functional. Let $\{P_n(x)\}_{n=0}^{\infty}$ be the monic orthogonal polynomial system (MOPS) relative to σ satisfying the

three term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0, \quad (P_{-1}(x) = 0). \quad (2.9)$$

Since (1.4) implies (2.7) with $A(x) = \prod_{l=1}^m (x - c_l)^{m_l+1}$, τ is a generalized Uvarov transform of σ . Then our main concern is to find conditions under which a generalized Uvarov transform τ , given by (1.4), of σ is also quasi-definite. In other words, we are to solve the division problem (2.7) of the moment functionals.

Let

$$K_n(x, y) := \sum_{j=0}^n \frac{P_j(x)P_j(y)}{\langle \sigma, P_j^2 \rangle}, \quad n \geq 0 \quad (2.10)$$

be the n th kernel polynomial for $\{P_n(x)\}_{n=0}^\infty$ and $K_n^{(i,j)}(x, y) = \partial_x^i \partial_y^j K_n(x, y)$. We need the following lemma which is easy to prove.

Lemma 2.1. *Let $V = (x_1, x_2, \dots, x_n)^t$ and $W = (y_1, y_2, \dots, y_n)^t$ be two vectors in $\mathbb{C}^n$. Then*

$$\det(I_n + VW^t) = 1 + \sum_{j=1}^n x_j y_j, \quad n \geq 1, \quad (2.11)$$

where I_n is the $n \times n$ identity matrix.

Theorem 2.2. *The moment functional τ is quasi-definite if and only if $d_n \neq 0$, $n \geq 0$, where d_n is the determinant of $(\sum_{l=1}^m (m_l + 1)) \times (\sum_{l=1}^m (m_l + 1))$ matrix D_n :*

$$D_n := [A_{tl}(n)]_{t,l=1}^m, \quad n \geq 0, \quad (2.12)$$

where

$$A_{tl}(n) = \left[\delta_{tl} \delta_{ki} + \lambda \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_n^{(k,j)}(c_t, c_l) \right]_{k=0, i=0}^{m_t, m_l}. \quad (2.13)$$

If τ is quasi-definite, then the MOPS $\{R_n(x)\}_{n=0}^\infty$ relative to τ is given by

$$R_n(x) = P_n(x) - \lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_{n-1}^{(0,j)}(x, c_l) R_n^{(i)}(c_l), \quad (2.14)$$

where $\{R_n^{(i)}(c_l)\}_{l=1}^m, i=0^{m_l}$ are given by

$$D_{n-1} \begin{bmatrix} R_n(c_1) \\ R'_n(c_1) \\ \vdots \\ R_n^{(m_1)}(c_1) \\ R_n(c_2) \\ \vdots \\ R_n^{(m_m)}(c_m) \end{bmatrix} = \begin{bmatrix} P_n(c_1) \\ P'_n(c_1) \\ \vdots \\ P_n^{(m_1)}(c_1) \\ P_n(c_2) \\ \vdots \\ P_n^{(m_m)}(c_m) \end{bmatrix}, \quad n \geq 0 \quad (D_{-1} = I). \quad (2.15)$$

Moreover,

$$\langle \tau, R_n^2 \rangle = \frac{d_n}{d_{n-1}} \langle \sigma, P_n^2 \rangle, \quad n \geq 0 \quad (d_{-1} = 1). \quad (2.16)$$

Proof. ($\Rightarrow$). Assume that τ is quasi-definite and expand $R_n(x)$ as

$$R_n(x) = P_n(x) + \sum_{j=0}^{n-1} C_{nj} P_j(x), \quad n \geq 1, \quad (2.17)$$

where $C_{nj} = \langle \sigma, R_n P_j \rangle / \langle \sigma, P_j^2 \rangle$, with $0 \leq j \leq n-1$. Here,

$$\begin{aligned} \langle \sigma, R_n P_j \rangle &= \left\langle \tau - \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), R_n P_j \right\rangle \\ &= -\lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^k \binom{k}{i} R_n^{(i)}(c_l) P_j^{(k-i)}(c_l) \end{aligned} \quad (2.18)$$

so that

$$\begin{aligned} R_n(x) &= P_n(x) - \lambda \sum_{j=0}^{n-1} \frac{P_j(x)}{\langle \sigma, P_j^2 \rangle} \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^k \binom{k}{i} R_n^{(i)}(c_l) P_j^{(k-i)}(c_l) \\ &= P_n(x) - \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^k \binom{k}{i} R_n^{(i)}(c_l) K_{n-1}^{(0, k-i)}(x, c_l) \\ &= P_n(x) - \lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l, i+j}}{i!j!} K_{n-1}^{(0, j)}(x, c_l) R_n^{(i)}(c_l). \end{aligned} \quad (2.19)$$

Hence, we have (2.14). Set the matrices B_l and E_l to be

$$B_l = \begin{bmatrix} R_n(c_l) \\ R'_n(c_l) \\ \vdots \\ R_n^{(m_l)}(c_l) \end{bmatrix}, \quad E_l = \begin{bmatrix} P_n(c_l) \\ P'_n(c_l) \\ \vdots \\ P_n^{(m_l)}(c_l) \end{bmatrix}, \quad 1 \leq l \leq m. \quad (2.20)$$

Then,

$$\begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} = [A_{tl}(n-1)]_{t,l=1}^m \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, \quad (2.21)$$

which gives (2.15). Now,

$$\begin{aligned} D_n &= [A_{tl}(n)]_{t,l=1}^m \\ &= D_{n-1} + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \left[\left[\sum_{j=0}^{m_1-i} \frac{u_{l,i+j}}{i!j!} P_n^{(j)}(c_l) P_n^{(k)}(c_t) \right]_{k=0, i=0}^{m_t \quad m_l} \right]_{t,l=1}^m \\ &= D_{n-1} + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} \left[\sum_{j=0}^{m_1} \frac{u_{1j}}{j!} P_n^{(j)}(c_1), \sum_{j=0}^{m_1-1} \frac{u_{1,j+1}}{j!} P_n^{(j)}(c_1), \dots, \right. \\ &\quad \left. \frac{u_{1,m_1}}{m_1!} P_n(c_1), \sum_{j=0}^{m_2} \frac{u_{2j}}{j!} P_n^{(j)}(c_2), \dots, \right. \\ &\quad \left. \frac{u_{m,m_m}}{m_m!} P_n(c_m) \right] \\ &= D_{n-1} \left[I + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \left[\sum_{j=0}^{m_1} \frac{u_{1j}}{j!} P_n^{(j)}(c_1), \sum_{j=0}^{m_1-1} \frac{u_{1,j+1}}{j!} P_n^{(j)}(c_1), \dots, \right. \right. \\ &\quad \left. \frac{u_{1,m_1}}{m_1!} P_n(c_1), \sum_{j=0}^{m_2} \frac{u_{2j}}{j!} P_n^{(j)}(c_2), \dots, \right. \\ &\quad \left. \left. \frac{u_{m,m_m}}{m_m!} P_n(c_m) \right] \right] \end{aligned} \quad (2.22)$$

so that

$$d_n = d_{n-1} \left(1 + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} P_n^{(j)}(c_l) R_n^{(i)}(c_l) \right) \quad (2.23)$$

by Lemma 2.1. On the other hand,

$$\begin{aligned} \langle \tau, R_n^2 \rangle &= \langle \tau, R_n P_n \rangle \\ &= \langle \sigma, R_n P_n \rangle + \lambda \left\langle \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), R_n P_n \right\rangle \\ &= \langle \sigma, P_n^2 \rangle + \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{j=0}^k \binom{k}{j} R_n^{(j)}(c_l) P_n^{(k-j)}(c_l) \\ &= \langle \sigma, P_n^2 \rangle + \lambda \sum_{l=1}^m \sum_{j=0}^{m_l} \sum_{k=j}^{m_l} \frac{u_{lk}}{k!} \binom{k}{j} R_n^{(j)}(c_l) P_n^{(k-j)}(c_l) \end{aligned} \quad (2.24)$$

so that

$$\langle \tau, R_n^2 \rangle = \langle \sigma, P_n^2 \rangle + \lambda \sum_{l=1}^m \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) P_n^{(k)}(c_l). \quad (2.25)$$

Hence, from (2.23) and (2.25), we have

$$\langle \sigma, P_n^2 \rangle d_n = d_{n-1} \langle \tau, R_n^2 \rangle, \quad n \geq 0. \quad (2.26)$$

Note that (2.26) also holds for $n = 0$ if we take $d_{-1} = 1$. Hence, $d_n \neq 0$, $n \geq 0$ inductively and we have (2.16).

($\Leftarrow$). Assume that $d_n \neq 0$, with $n \geq 0$ and define $\{R_n(x)\}_{n=0}^\infty$ by (2.14). Then we have, by (2.14) and (2.23),

$$\begin{aligned}
\langle \tau, R_n P_r \rangle &= \langle \sigma, R_n P_r \rangle + \lambda \left\langle \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), R_n P_r \right\rangle \\
&= \langle \sigma, R_n P_r \rangle + \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{j=0}^k \binom{k}{j} R_n^{(j)}(c_l) P_r^{(k-j)}(c_l) \\
&= \langle \sigma, P_n P_r \rangle - \lambda \sum_{l=1}^m \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) \langle \sigma, K_{n-1}^{(0,k)}(x, c_l) P_r(x) \rangle \\
&\quad + \lambda \sum_{l=1}^m \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) P_r^{(k)}(c_l) \\
&= \langle \sigma, P_n P_r \rangle - \lambda \sum_{l=1}^m \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) P_r^{(k)}(c_l) (1 - \delta_{nr}) \\
&\quad + \lambda \sum_{l=1}^m \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) P_r^{(k)}(c_l) \\
&= \begin{cases} 0, & 0 \leq r \leq n-1, \\ \langle \sigma, P_n^2 \rangle + \lambda \sum_{l=1}^m \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} P_n^{(k)}(c_l) R_n^{(j)}(c_l), & r = n, \end{cases} \\
&= \begin{cases} 0, & 0 \leq r \leq n-1, \\ \frac{d_n}{d_{n-1}} \langle \sigma, P_n^2 \rangle \neq 0, & r = n, \end{cases}
\end{aligned} \tag{2.27}$$

since $\langle \sigma, K_{n-1}^{(0,k)}(x, c_l) P_r(x) \rangle = P_r^{(k)}(c_l) (1 - \delta_{nr})$. Hence,

$$\langle \tau, R_n R_m \rangle = \begin{cases} 0, & \text{if } 0 \leq m \leq n-1, \\ \langle \tau, R_n P_n \rangle \neq 0, & \text{if } m = n, \end{cases} \tag{2.28}$$

so that $\{R_n(x)\}_{n=0}^\infty$ is the MOPS relative to τ and so τ is also quasi-definite. $\square$

General division problems of moment functionals

$$D(x)\tau = A(x)\sigma \tag{2.29}$$

is handled in [17], when $D(x)$ and $A(x)$ have no common zero. Theorem 2.2 includes the following as special cases.

- $m = 1$, $m_1 = 0$: Marcellán and Maroni [23],
- $m = 2$, $m_1 = m_2 = 0$: Draïdi and Maroni [9], Kwon and Park [20],

- $m = 1$, $m_1 = 1$: Belmechdi and Marcellán [4],
- $m = 1$: Kim, Kwon, and Park [12].

Some other special cases where σ is a classical moment functional were handled in [2, 1, 3, 8, 11, 14].

From now on, we always assume that $d_n \neq 0$, with $n \geq 0$, so that τ is also quasi-definite.

Theorem 2.3. *For the MOPS $\{R_n(x)\}_{n=0}^\infty$ relative to τ , we have*

(i) *(the three-term recurrence relation)*

$$R_{n+1}(x) = (x - \beta_n)R_n(x) - \gamma_n R_{n-1}(x), \quad n \geq 0, \quad (2.30)$$

where

$$\beta_n = b_n + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} \quad (2.31)$$

$$\times \{P_n^{(j)}(c_l)R_{n+1}^{(i)}(c_l) - P_{n-1}^{(j)}(c_l)R_n^{(i)}(c_l)\} \quad (n \geq 0),$$

$$\gamma_n = \frac{d_n d_{n-2}}{d_{n-1}^2} c_n \quad (n \geq 1). \quad (2.32)$$

(ii) *(the quasi-orthogonality)*

$$\prod_{l=1}^m (x - c_l)^{m_l+1} R_n(x) = \sum_{j=n-r}^{n+r} C_{nj} P_j(x), \quad n \geq r, \quad (2.33)$$

where $r = \sum_{l=1}^m (m_l + 1)$, $C_{n,n-r} \neq 0$, and

$$\begin{aligned} C_{nj} &= \frac{\langle \sigma, \prod_{l=1}^m (x - c_l)^{m_l+1} R_n P_j \rangle}{\langle \sigma, P_j^2 \rangle} \\ &= \frac{\langle \tau, \prod_{l=1}^m (x - c_l)^{m_l+1} R_n P_j \rangle}{\langle \sigma, P_j^2 \rangle}, \quad \text{where } n-r \leq j \leq n+r. \end{aligned} \quad (2.34)$$

Proof. For (i), by (2.14), we can rewrite (2.30) as

$$\begin{aligned} &P_{n+1}(x) - \lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_n^{(0,j)}(x, c_l) R_{n+1}^{(i)}(c_l) \\ &= (x - \beta_n) \left\{ P_n(x) - \lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_{n-1}^{(0,j)}(x, c_l) R_n^{(i)}(c_l) \right\} \\ &\quad - \gamma_n \left\{ P_{n-1}(x) - \lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_{n-2}^{(0,j)}(x, c_l) R_{n-1}^{(i)}(c_l) \right\}. \end{aligned} \quad (2.35)$$

After multiplying (2.35) by $P_n(x)$ and applying σ , we have

$$\begin{aligned} & -\lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} P_n^{(j)}(c_l) R_{n+1}^{(i)}(c_l) \\ & = (b_n - \beta_n) \langle \sigma, P_n^2 \rangle - \lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} P_{n-1}^{(j)}(c_l) R_n^{(i)}(c_l). \end{aligned} \quad (2.36)$$

Hence, we have (2.31) and by (2.16)

$$\gamma_n = \frac{\langle \tau, R_n^2 \rangle}{\langle \tau, R_{n-1}^2 \rangle} = \frac{d_n d_{n-2}}{d_{n-1}^2} c_n \quad (n \geq 1). \quad (2.37)$$

For (ii), $\prod_{l=1}^m (x - c_l)^{m_l+1} R_n(x) = \sum_{j=0}^{n+r} C_{nj} P_j(x)$, where $r = \sum_{l=1}^m (m_l+1)$ and

$$\begin{aligned} C_{nk} \langle \sigma, P_k^2 \rangle &= \left\langle \sigma, \prod_{l=1}^m (x - c_l)^{m_l+1} R_n(x) P_k(x) \right\rangle \\ &= \left\langle \prod_{l=1}^m (x - c_l)^{m_l+1} \tau, R_n(x) P_k(x) \right\rangle \\ &= \left\langle \tau, \prod_{l=1}^m (x - c_l)^{m_l+1} R_n(x) P_k(x) \right\rangle = 0, \quad \text{if } r+k < n. \end{aligned} \quad (2.38)$$

Hence, $C_{nk} = 0$, $0 \leq k \leq n - r - 1$ and $C_{n,n-r} \neq 0$ so that we have (2.33) and (2.34). $\square$

Corollary 2.4. *Assume that σ is positive-definite and let $[\xi, \eta]$ be the true interval of the orthogonality of σ . Then*

(i) $\prod_{l=1}^m (x - c_l)^{m_l+1} R_n(x)$ *has at least* $n - r$ *distinct nodal zeros (i.e., zeros of odd multiplicities) in* (ξ, η) .

(ii) $R_n(x)$ *has at least* $n - r - m$ *distinct nodal zeros in* (ξ, η) .

If furthermore m_l ($1 \leq l \leq m$) *are odd or* $\xi \geq c_l$ ($1 \leq l \leq m$), *then*

(iii) $R_n(x)$ *has at least* $n - r$ *distinct nodal zeros in* (ξ, η) .

Proof. (i) and (ii) are trivial by (2.33).

For (iii), assume that m_l ($1 \leq l \leq m$) are odd. Then $\tilde{\sigma} := \prod_{l=1}^m (x - c_l)^{m_l+1} \sigma$ is also positive-definite on $[\xi, \eta]$. Let $\{\tilde{P}_n(x)\}_{n=0}^\infty$ be the MOPS relative to $\tilde{\sigma}$. Then we may write $R_n(x) = \sum_{j=0}^n \tilde{C}_{nj} \tilde{P}_j(x)$, where

$$\begin{aligned} \tilde{C}_{nk} \langle \tilde{\sigma}, \tilde{P}_k^2 \rangle &= \langle \tilde{\sigma}, R_n \tilde{P}_k \rangle = \left\langle \prod_{l=1}^m (x - c_l)^{m_l+1} \tau, R_n \tilde{P}_k \right\rangle \\ &= \left\langle \tau, \prod_{l=1}^m (x - c_l)^{m_l+1} R_n \tilde{P}_k \right\rangle. \end{aligned} \quad (2.39)$$

Hence, $\tilde{C}_{nk} = 0$, $0 \leq k \leq n-r-1$ so that $R_n(x) = \sum_{j=n-r}^n \tilde{C}_{nj} \tilde{P}_j(x)$. Hence, $R_n(x)$ has at least $n-r$ distinct nodal zeros in (ξ, η) . In case $\xi \geq c_1$ ($1 \leq l \leq m$), $\tilde{\sigma} = \prod_{l=1}^m (x - c_l)^{m_l+1} \sigma$ is also positive-definite on $[\xi, \eta]$ so that by the same reasoning as above $R_n(x)$ has at least $n-r$ distinct nodal zeros in (ξ, η) . $\square$

Theorem 2.5. *For any polynomial $p(x)$ of degree at most n , we have*

$$\langle \tau, L_n^{(0,k)}(x, y) p(x) \rangle = p^{(k)}(y), \quad (2.40)$$

where $L_n(x, y) = \sum_{i=0}^n R_i(x) R_i(y) / \langle \tau, R_i^2 \rangle$, $n \geq 0$, is the n th kernel polynomial for $\{R_n(x)\}_{n=0}^\infty$ and

$$L_n(x, y) = K_n(x, y) - \frac{\lambda}{d_n} \sum_{l=1}^m \sum_{i=0}^{m_l} |D_n^u| \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_n^{(0,j)}(x, c_l), \quad (2.41)$$

where $u = \sum_{k=1}^{l-1} (m_k + 1) + (i + 1)$ and D_n^i is the matrix obtained from D_n by replacing the i th column of D_n by

$$\begin{aligned} & [K_n(c_1, y), K_n^{(1,0)}(c_1, y), \dots, K_n^{(m_1,0)}(c_1, y), \\ & K_n(c_2, y), K_n^{(1,0)}(c_2, y), \dots, K_n^{(m_m,0)}(c_m, y)]^T. \end{aligned} \quad (2.42)$$

Proof. If $\deg(p) \leq n$, then $p(x) = \sum_{i=0}^n (\langle \tau, p R_i \rangle / \langle \tau, R_i^2 \rangle) R_i(x)$ so that

$$\begin{aligned} \langle \tau, L_n^{(0,k)}(x, y) p(x) \rangle &= \sum_{i=0}^n \frac{\langle \tau, p R_i \rangle}{\langle \tau, R_i^2 \rangle} \langle \tau, L_n^{(0,k)}(x, y) R_i(x) \rangle \\ &= \sum_{i=0}^n \frac{\langle \tau, p R_i \rangle}{\langle \tau, R_i^2 \rangle} \sum_{j=0}^n \frac{R_j^{(k)}(y)}{\langle \tau, R_j^2 \rangle} \langle \tau, R_j(x) R_i(x) \rangle \\ &= \sum_{i=0}^n \frac{\langle \tau, p R_i \rangle}{\langle \tau, R_i^2 \rangle} R_i^{(k)}(y) = p^{(k)}(y). \end{aligned} \quad (2.43)$$

Expand $L_n(x, y)$ as $L_n(x, y) = \sum_{j=0}^n a_{nj}(y) P_j(x)$, where

$$\begin{aligned} a_{nj}(y) &= \frac{\langle \sigma, L_n(x, y) P_j(x) \rangle}{\langle \sigma, P_j^2 \rangle} \\ &= \frac{\langle \tau, L_n(x, y) P_j(x) \rangle}{\langle \sigma, P_j^2 \rangle} - \frac{\lambda}{\langle \sigma, P_j^2 \rangle} \\ &\quad \times \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \langle \delta^{(k)}(x - c_l), L_n(x, y) P_j(x) \rangle \\ &= \frac{P_j(y)}{\langle \sigma, P_j^2 \rangle} - \frac{\lambda}{\langle \sigma, P_j^2 \rangle} \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^k \binom{k}{i} L_n^{(i,0)}(c_l, y) P_j^{(k-i)}(c_l) \end{aligned} \quad (2.44)$$

by (2.40). Hence

$$\begin{aligned}
 L_n(x, y) &= K_n(x, y) - \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^k \binom{k}{i} L_n^{(i,0)}(c_l, y) K_n^{(0,k-i)}(x, c_l) \\
 &= K_n(x, y) - \lambda \sum_{l=1}^m \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_n^{(0,j)}(x, c_l) L_n^{(i,0)}(c_l, y),
 \end{aligned} \tag{2.45}$$

and so

$$\begin{aligned}
 &D_n [L_n(c_1, y), L_n^{(1,0)}(c_1, y), \dots, L_n^{(m_1,0)}(c_1, y), \\
 &\quad L_n(c_2, y), \dots, L_n^{(m_m,0)}(c_m, y)]^T \\
 &= [K_n(c_1, y), K_n^{(1,0)}(c_1, y), \dots, K_n^{(m_1,0)}(c_1, y), \\
 &\quad K_n(c_2, y), \dots, K_n^{(m_m,0)}(c_m, y)]^T.
 \end{aligned} \tag{2.46}$$

Hence, we have (2.41) from (2.45) and (2.46). $\square$

3. Semi-classical case

Since τ is a linear spectral transform (see [29]) of σ , if σ is a Laguerre-Hahn form (see [22]) or a semi-classical form (see [24]) or a second degree form (see [26]), then so is τ . Here, we consider the semi-classical case more closely.

Definition 3.1 (see Maroni [24]). A moment functional σ is said to be semi-classical if σ is quasi-definite and satisfies a Pearson type functional equation

$$(\phi(x)\sigma)' - \psi(x)\sigma = 0 \tag{3.1}$$

for some polynomials $\phi(x)$ and $\psi(x)$ with $\deg(\phi) \geq 0$ and $\deg(\psi) \geq 1$.

For a semi-classical moment functional σ , we call

$$s := \min \max (\deg(\phi) - 2, \deg(\psi) - 1) \tag{3.2}$$

the class number of σ , where the minimum is taken over all pairs (ϕ, ψ) of polynomials satisfying (3.1). We then call the MOPS $\{P_n(x)\}_{n=0}^\infty$ relative to σ a semi-classical OPS (SCOPS) of class s .

From now on, we assume that σ is a semi-classical moment functional satisfying (3.1) and set $s := \max(\deg(\phi) - 2, \deg(\psi) - 1)$. Then τ satisfies the functional equation

$$(T(x)\phi(x)\tau)' = (T'(x)\phi(x) + T(x)\psi(x))\tau, \tag{3.3}$$

where

$$T(x) = \prod_{l=1}^m (x - c_l)^{m_l+2}. \quad (3.4)$$

We now determine the class number of τ . By (3.3), if σ is of class s , then τ is of class $\leq s + \sum_{l=1}^m (m_l + 2)$.

Lemma 3.2 (see [25]). *The semi-classical moment functional σ satisfying (3.1) is of class s if and only if for any zero c of $\phi(x)$,*

$$\mathbb{N}(\sigma; c) := |r_c| + |\langle \sigma, q_c(x) \rangle| \neq 0, \quad (3.5)$$

where $\phi(x) = (x - c)\phi_c(x)$ and $\phi_c(x) - \psi(x) = (x - c)q_c(x) + r_c$.

Theorem 3.3. *Assume that σ is of class $s = \max(\deg(\phi) - 2, \deg(\psi) - 1)$. Then τ is of class $s + \sum_{l=1}^m (m_l + 2)$ if $\phi(c_l) \neq 0$, $1 \leq l \leq m$.*

Proof. Assume $\phi(c_l) \neq 0$, $1 \leq l \leq m$. Let $\tilde{\phi}(x) = T(x)\phi(x)$ and $\tilde{\psi}(x) = T'(x)\phi(x) + T(x)\psi(x)$. For any zero c of $\tilde{\phi}(x)$, let $\tilde{\phi}(x) = (x - c)\tilde{\phi}_c(x)$ and $\tilde{\phi}_c(x) - \tilde{\psi}(x) = (x - c)\tilde{q}_c(x) + \tilde{r}_c$. Then either $c = c_t$ ($1 \leq t \leq m$) or $\phi(c) = 0$.

If $c = c_t$ ($1 \leq t \leq m$), then

$$\tilde{\phi}_c(x) - \tilde{\psi}(x) = \frac{T(x)\phi(x)}{x - c_t} - T'(x)\phi(x) - T(x)\psi(x) = (x - c_t)\tilde{q}_c(x). \quad (3.6)$$

Hence, $\tilde{r}_c = 0$ but

$$\begin{aligned} \langle \tau, \tilde{q}_c(x) \rangle &= \left\langle \sigma + \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), \tilde{q}_c(x) \right\rangle \\ &= \left\langle \sigma + \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), \right. \\ &\quad \left. \frac{T(x)\phi(x)}{(x - c_t)^2} - \frac{T'(x)\phi(x) + T(x)\psi(x)}{x - c_t} \right\rangle \\ &= \left\langle (\phi\sigma)' - \psi\sigma, \frac{T(x)}{x - c_t} \right\rangle \\ &\quad + \left\langle \lambda \sum_{l=1}^m \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), \right. \\ &\quad \left. \frac{T(x)\phi(x)}{(x - c_t)^2} - \frac{T'(x)\phi(x) + T(x)\psi(x)}{x - c_t} \right\rangle \\ &= -\lambda u_{t, m_t} (m_t + 1) \prod_{\substack{l=1 \\ l \neq t}}^m (c_t - c_l) \phi(c_t) \neq 0, \end{aligned} \quad (3.7)$$

so that $\mathbb{N}(\tau, c) \neq 0$.

If $c \neq c_t$ ($1 \leq t \leq m$), then $\phi(c) = 0$, $\tilde{\phi}_c(x) = T(x)\phi_c(x)$, and

$$\tilde{\phi}_c(x) - \tilde{\psi}(x) = T(x)\phi_c(x) - T'(x)\phi(x) - T(x)\psi(x). \quad (3.8)$$

Hence, $\tilde{r}_c = T(c)\phi_c(c) - T(c)\psi(c) = T(c)r_c$. If $r_c \neq 0$, then $\tilde{r}_c \neq 0$ so that $\mathbb{N}(\tau; c) \neq 0$.

If $r_c = 0$, then $\langle \sigma, q_c(x) \rangle \neq 0$ and $\tilde{r}_c = 0$ so that

$$\tilde{q}_c(x) = T(x)q_c(x) - T'(x)\phi_c(x). \quad (3.9)$$

Then

$$\langle \tau, \tilde{q}_c(x) \rangle = \langle \sigma, \tilde{q}_c(x) \rangle = \langle \sigma, T(x)q_c(x) - T'(x)\phi_c(x) \rangle. \quad (3.10)$$

Set $Q_1(x)$, $Q_2(x)$, $Q_3(x)$, and r_1 , r_2 , r_3 to be

$$\begin{aligned} T(x) &= (x-c)Q_1(x) + r_1; \\ T'(x) &= (x-c)Q_2(x) + r_2; \\ Q_1(x) &= (x-c)Q_3(x) + r_3. \end{aligned} \quad (3.11)$$

Then $Q_2(x) = Q_1'(x) + Q_3(x)$ and $r_2 = r_3 = Q_1(c)$.

Hence,

$$\begin{aligned} \langle \tau, \tilde{q}_c(x) \rangle &= \langle \sigma, Q_1(x)(\phi_c(x) - \psi(x)) \rangle + \langle \sigma, r_1 q_c(x) \rangle \\ &\quad - \langle \sigma, Q_2(x)\phi(x) \rangle - \langle \sigma, r_2 \phi_c(x) \rangle \\ &= \langle \sigma, Q_3(x)\phi(x) \rangle + \langle \sigma, r_3 \phi_c(x) \rangle \\ &\quad - \langle \sigma, Q_1(x)\psi(x) \rangle + \langle \sigma, r_1 q_c(x) \rangle \\ &\quad - \langle \sigma, Q_2(x)\phi(x) \rangle - \langle \sigma, r_2 \phi_c(x) \rangle \\ &= \langle \phi(x)\sigma, Q_3(x) \rangle + \langle \phi(x)\sigma, Q_1'(x) \rangle \\ &\quad - \langle \phi(x)\sigma, Q_2(x) \rangle + r_1 \langle \sigma, q_c(x) \rangle \\ &= r_1 \langle \sigma, q_c(x) \rangle = \prod_{l=1}^m (c - c_l)^{m_l+2} \langle \sigma, q_c(x) \rangle \neq 0. \end{aligned} \quad (3.12)$$

Hence $\mathbb{N}(\tau; c) \neq 0$. □

4. Examples

As illustrating examples, we consider the following example.

Example 4.1. Let

$$\tau := \sigma + \lambda(u_{10}\delta(x-1) + u_{20}\delta(x+1) + u_{11}\delta'(x-1) + u_{21}\delta'(x+1)), \quad (4.1)$$

where $\lambda \neq 0$, $|u_{10}| + |u_{20}| + |u_{11}| + |u_{21}| \neq 0$, and σ is the Jacobi moment functional defined by

$$\langle \sigma, \pi \rangle = \int_{-1}^1 (1-x)^\alpha (1+x)^\beta \pi(x) dx \quad (\alpha > -1, \beta > -1), \quad \pi \in \mathbb{P}. \quad (4.2)$$

Then

$$\begin{aligned} P_n(x) &= P_n^{(\alpha, \beta)}(x) \\ &= \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}, \quad n \geq 0 \end{aligned} \quad (4.3)$$

are the Jacobi polynomials satisfying

$$\begin{aligned} (1-x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + n(n + \alpha + \beta + 1)y(x) &= 0, \\ \langle \sigma, P_n^{(\alpha, \beta)}(x)^2 \rangle &:= k_n \\ &= \frac{2^{\alpha+\beta+2n+1} n! \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1) (2n+\alpha+\beta+1) (n+\alpha+\beta+1)_n^2}, \quad n \geq 0, \end{aligned} \quad (4.4)$$

where

$$(a)_k = \begin{cases} 1, & \text{if } k = 0 \\ a(a+1) \cdots (a+k-1), & \text{if } k \geq 1. \end{cases} \quad (4.5)$$

In this case, using the differential-difference relation

$$(P_n^{(\alpha, \beta)}(x))^{(\nu)} = \frac{n!}{(n-\nu)!} P_{n-\nu}^{(\alpha+\nu, \beta+\nu)}(x), \quad \nu = 0, 1, 2, \dots, n \geq \nu, \quad (4.6)$$

the structure relation

$$(1-x^2)P_n^{(\alpha, \beta)}(x)' = \tilde{\alpha}_n P_{n+1}^{(\alpha, \beta)}(x) + \tilde{\beta}_n P_n^{(\alpha, \beta)}(x) + \tilde{\gamma}_n P_{n-1}^{(\alpha, \beta)}(x), \quad n \geq 0, \quad (4.7)$$

where

$$\begin{aligned} \tilde{\alpha}_n &= -n, \\ \tilde{\beta}_n &= \frac{2(\alpha - \beta)n(n + \alpha + \beta + 1)}{(2n + \alpha + \beta)(2n + 2 + \alpha + \beta)}, \\ \tilde{\gamma}_n &= \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}, \end{aligned} \quad (4.8)$$

and the three term recurrence relation

$$P_{n+1}^{(\alpha, \beta)}(x) = (x - \beta_n)P_n^{(\alpha, \beta)}(x) - \gamma_n P_{n-1}^{(\alpha, \beta)}(x), \quad (4.9)$$

where

$$\begin{aligned}\beta_n &= \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + 2 + \alpha + \beta)}, \\ \gamma_n &= \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)},\end{aligned}\tag{4.10}$$

we can easily obtain (see [1, equations (30)–(32)]):

$$\begin{aligned}K_{n-1}^{(0,0)}(1,1) &= \frac{(P_n^{(\alpha,\beta)}(1))^2 n(n + \beta)}{k_{n-1}(2n + \alpha + \beta + 1)\gamma_n(\alpha + 1)}, \\ K_{n-1}^{(0,0)}(1,-1) &= -\frac{n P_n^{(\alpha,\beta)}(-1) P_n^{(\alpha,\beta)}(1)}{k_{n-1}(2n + \alpha + \beta + 1)\gamma_n}, \\ K_{n-1}^{(0,1)}(1,1) &= \frac{(P_n^{(\alpha,\beta)})'(1) P_n^{(\alpha,\beta)}(1)(n + \beta)(n - 1)}{k_{n-1}(2n + \alpha + \beta + 1)\gamma_n(\alpha + 2)}, \\ K_{n-1}^{(0,1)}(1,-1) &= -\frac{(P_n^{(\alpha,\beta)})'(-1) P_n^{(\alpha,\beta)}(1)(n - 1)}{k_{n-1}(2n + \alpha + \beta + 1)\gamma_n}, \\ K_{n-1}^{(1,1)}(1,1) &= P_n^{(\alpha,\beta)}(1) (P_n^{(\alpha,\beta)})'(1)(n - 1)(n + \beta) \\ &\quad \times \frac{[(\alpha + 2)(n^2 + n\alpha + n\beta) - (\alpha + 1)(\alpha + \beta + 2)]}{2k_{n-1}(2n + \alpha + \beta + 1)\gamma_n(\alpha + 1)(\alpha + 2)(\alpha + 3)}, \\ K_{n-1}^{(0,0)}(1,1) &= -\frac{P_n^{(\alpha,\beta)}(1) (P_n^{(\alpha,\beta)})'(-1)(n - 1)[n^2 + n\alpha + n\beta - \alpha - \beta - 2]}{2k_{n-1}(2n + \alpha + \beta + 1)\gamma_n(\alpha + 1)},\end{aligned}\tag{4.11}$$

where

$$K_n(x, y) = \sum_{k=0}^n \frac{P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y)}{k_n}\tag{4.12}$$

is the n th kernel polynomial of $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ and $K_n^{(i,j)}(x, y) = \partial_x^i \partial_y^j K_n(x, y)$. Using the symmetry of the Jacobi kernels, we obtain that the moment functional τ in (4.1) is quasi-definite if and only if

$$d_n = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \neq 0, \quad n \geq 0,\tag{4.13}$$

where

$$\begin{aligned}
 A_{11} &= \begin{pmatrix} 1 + \lambda u_{10} K_n^{(0,0)}(1,1) + \lambda u_{11} K_n^{(0,1)}(1,1) & \lambda u_{11} K_n^{(0,0)}(1,1) \\ \lambda u_{10} K_n^{(1,0)}(1,1) + \lambda u_{11} K_n^{(1,1)}(1,1) & 1 + \lambda u_{11} K_n^{(1,0)}(1,1) \end{pmatrix} \\
 A_{12} &= \begin{pmatrix} \lambda u_{20} K_n^{(0,0)}(1,-1) + \lambda u_{21} K_n^{(0,1)}(1,-1) & \lambda u_{21} K_n^{(0,0)}(1,-1) \\ \lambda u_{20} K_n^{(1,0)}(1,-1) + \lambda u_{21} K_n^{(1,1)}(1,-1) & \lambda u_{21} K_n^{(1,0)}(1,-1) \end{pmatrix} \\
 A_{21} &= \begin{pmatrix} \lambda u_{10} K_n^{(0,0)}(-1,1) + \lambda u_{11} K_n^{(0,1)}(-1,1) & \lambda u_{11} K_n^{(0,0)}(-1,1) \\ \lambda u_{10} K_n^{(1,0)}(-1,1) + \lambda u_{11} K_n^{(1,1)}(-1,1) & \lambda u_{11} K_n^{(1,0)}(-1,1) \end{pmatrix} \\
 A_{22} &= \begin{pmatrix} 1 + \lambda u_{20} K_n^{(0,0)}(-1,-1) + \lambda u_{21} K_n^{(0,1)}(-1,-1) & \lambda u_{21} K_n^{(0,0)}(-1,-1) \\ \lambda u_{20} K_n^{(1,0)}(-1,-1) + \lambda u_{21} K_n^{(1,1)}(-1,-1) & 1 + \lambda u_{21} K_n^{(1,0)}(-1,-1) \end{pmatrix}.
 \end{aligned} \tag{4.14}$$

Álvarez-Nodarse, J. Arvesú, and F. Marcellán [1] showed that for any values of λ and $u_{10}, u_{20}, u_{11}, u_{21}, d_n \neq 0$ for large n so that $R_n(x)$ exists for large n . Moreover, they express $R_n(x)$ as

$$\begin{aligned}
 R_n(x) &= (1 + n\zeta_n + n\eta_n) P_n^{(\alpha, \beta)}(x) + [\zeta_n(1-x) - \eta_n(1+x) + \theta_n] (P_n^{(\alpha, \beta)}(x))' \\
 &\quad + [\chi_n(1+x) - \omega_n(1-x)] (P_n^{(\alpha, \beta)}(x))'',
 \end{aligned} \tag{4.15}$$

where $\zeta_n, \eta_n, \theta_n, \chi_n$, and ω_n are constants depending on $n, \lambda, u_{10}, u_{20}$, and u_{11} , (see [1, equations (47)–(50)]). They also express $R_n(x)$ as a generalized hypergeometric series ${}_6F_5$ (see [1, Proposition 2]).

Example 4.2. Consider a moment functional τ given by

$$\tau := \sigma + \lambda \sum_{k=0}^N \frac{(-1)^k u_k}{k!} \delta^{(k)}(x), \tag{4.16}$$

where $\lambda \neq 0, u_k \in \mathbb{C}, N \in \{0, 1, 2, \dots\}$ and σ is the Laguerre moment functional defined by

$$\langle \sigma, p \rangle = \int_0^\infty x^\alpha e^{-x} \pi(x) dx \quad (\alpha > -1), \quad \pi \in \mathbb{P}. \tag{4.17}$$

Then

$$\begin{aligned}
 P_n(x) &= L_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \\
 &= (-1)^n (\alpha+1)_n {}_1F_1(-n; \alpha+1; x), \quad n \geq 0
 \end{aligned} \tag{4.18}$$

are the monic Laguerre polynomials satisfying

$$\begin{aligned} xy''(x) + (1 + \alpha - x)y'(x) + ny(x) &= 0, \\ \langle \sigma, L_n^{(\alpha)}(x)^2 \rangle &= n! \Gamma(n + \alpha + 1), \quad n \geq 0. \end{aligned} \quad (4.19)$$

Hence

$$L_n^{(\alpha)}(0) = \frac{(-1)^n \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)} = (-1)^n \binom{n + \alpha}{n} n!. \quad (4.20)$$

Hence, by Theorem 2.2, the moment functional τ in (4.16) is quasi-definite if and only if $d_n \neq 0$, where d_n is the determinant of the $(N + 1) \times (N + 1)$ matrix D_n ,

$$D_n := \left[\delta_{ij} + \lambda \sum_{k=0}^{N-j} \frac{u_{j+k}}{j!k!} K_n^{(i,k)}(0,0) \right]_{i,j=0}^N, \quad n \geq 0, \quad (4.21)$$

where $K_n(x, y) = \sum_{k=0}^n L_k^{(\alpha)}(x) L_k^{(\alpha)}(y) / \langle \sigma, L_k^{(\alpha)}(x)^2 \rangle$.

When $d_n \neq 0$ for $n \geq 0$, we now claim that the MOPS $\{R_n(x)\}_{n=0}^\infty$ relative to τ can be given as

$$R_n(x) = \sum_{k=0}^{N+1} A_k^{(n)} \partial_x^k L_n^{(\alpha)}(x), \quad n \geq 0 \quad (4.22)$$

for suitable constants $A_k^{(n)}$ ($0 \leq k \leq N + 1$) with $A_0^{(n)} = 1$. For any fixed $n \geq 1$, set

$$\tilde{R}_n(x) := \sum_{k=0}^{N+1} A_k \partial_x^k L_n^{(\alpha)}(x), \quad (4.23)$$

where $\{A_k\}_{k=0}^{N+1}$ are constants to be determined. Note here that if $0 \leq n \leq N$, then $\partial_x^k L_n^{(\alpha)}(x) = 0$ for $n + 1 \leq k \leq N + 1$ so that we may take A_k for $n + 1 \leq k \leq N + 1$ to be 0. Since $(L_n^{(\alpha)}(x))' = n L_{n-1}^{(\alpha+1)}(x)$, $n \geq 1$, we have

$$\tilde{R}_n(x) = \sum_{k=0}^{N+1} (n - k + 1)_k A_k L_{n-k}^{(\alpha+k)}(x), \quad (4.24)$$

where $L_n^{(\alpha)}(x) = 0$ for $n < 0$. We now show that the coefficients $\{A_k\}_{k=0}^{N+1}$ can

be chosen so that

$$\langle \tau, x^m \tilde{R}_n(x) \rangle = 0, \quad 0 \leq m \leq n-1. \quad (4.25)$$

If $N+1 \leq m < n$, then by (4.24)

$$\begin{aligned} \langle \tau, x^m \tilde{R}_n(x) \rangle &= \int_0^\infty x^m \tilde{R}_n(x) x^\alpha e^{-x} dx + \lambda \sum_{k=0}^N \frac{u_k}{k!} (x^m \tilde{R}_n(x))^{(k)}(0) \\ &= \sum_{k=0}^{N+1} \frac{n!}{(n-k)!} A_k \int_0^\infty x^m L_{n-k}^{(\alpha+k)}(x) x^\alpha e^{-x} dx \\ &= \sum_{k=0}^{N+1} \frac{n!}{(n-k)!} A_k \int_0^\infty x^{m-k} L_{n-k}^{(\alpha+k)}(x) x^{\alpha+k} e^{-x} dx \\ &= 0. \end{aligned} \quad (4.26)$$

We now assume that $0 \leq m \leq \min(N, n-1)$. Then

$$\begin{aligned} \langle \sigma, x^m L_{n-k}^{(\alpha+k)}(x) \rangle &= \int_0^\infty x^m L_{n-k}^{(\alpha+k)}(x) x^\alpha e^{-x} dx \\ &= \int_0^\infty x^{m-k} L_{n-k}^{(\alpha+k)}(x) x^{\alpha+k} e^{-x} dx = 0, \end{aligned} \quad (4.27)$$

for $0 \leq k \leq m$. For $m+1 \leq k \leq n$,

$$\begin{aligned} \langle \sigma, x^m L_{n-k}^{(\alpha+k)}(x) \rangle &= (-1)^{n-k} (n-k)! \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \binom{n+\alpha}{n-k-j} \int_0^\infty x^{m+\alpha+j} e^{-x} dx \\ &= (-1)^{n-k} (n-k)! \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \binom{n+\alpha}{n-k-j} \Gamma(m+\alpha+j+1) \\ &= (-1)^{n-k} (n-k)! \binom{n+\alpha}{n-k} \Gamma(m+\alpha+1) \\ &\quad \times {}_2F_1(-n+k, m+\alpha+1; k+\alpha+1; 1) \\ &= (-1)^{n-k} (n-k)! \binom{n-m-1}{n-k} \Gamma(m+\alpha+1) \end{aligned} \quad (4.28)$$

by (4.18) and ${}_2F_1(-n, b; c; 1) = (c-b)_n / (c)_n$. Hence by (4.20)

$$\begin{aligned}
 \langle \tau, x^m \tilde{R}_n(x) \rangle &= \sum_{k=0}^{N+1} (n-k+1)_k A_k \langle \sigma, x^m L_{n-k}^{(\alpha+k)}(x) \rangle \\
 &\quad + \lambda \sum_{l=0}^N \frac{u_l}{l!} \sum_{k=0}^{N+1} (n-k+1)_k A_k (x^m L_{n-k}^{(\alpha+k)}(x))^{(l)}(0) \\
 &= n! \Gamma(m+\alpha+1) \sum_{k=m+1}^{N+1} (-1)^{n-k} \binom{n-m-1}{n-k} A_k \\
 &\quad + \lambda n! \sum_{l=m}^N \frac{u_l}{(l-m)!} \sum_{k=0}^{N+1} (-1)^{n-k-l+m} \binom{n+\alpha}{n-k-l+m} A_k, \\
 &\qquad\qquad\qquad 0 \leq m \leq \min(N, n-1), \tag{4.29}
 \end{aligned}$$

where $\binom{n}{k} = 0$, for $k < 0$. Hence $\langle \tau, x^m \tilde{R}_n(x) \rangle = 0$, with $0 \leq m \leq n-1$ if and only if

$$\begin{aligned}
 &\Gamma(m+\alpha+1) \sum_{k=m+1}^{N+1} (-1)^{n-k} \binom{n-m-1}{n-k} A_k \\
 &\quad + \lambda \sum_{l=m}^N \frac{u_l}{(l-m)!} \sum_{k=0}^{N+1} (-1)^{n-k-l+m} \binom{n+\alpha}{n-k-l+m} A_k = 0, \tag{4.30} \\
 &\qquad\qquad\qquad 0 \leq m \leq \min(N, n-1).
 \end{aligned}$$

Since (4.30) is a homogeneous system of $N+1$ (resp., n) equations for $N+2$ (resp., $n+1$) unknowns $\{A_k\}_{k=0}^{N+1}$ (resp., $\{A_k\}_{k=0}^n$) when $n \geq N+1$ (resp., $n \leq N$), there always exists a nontrivial solution $\{A_k\}_{k=0}^{N+1}$. With this choice of $\{A_k\}_{k=0}^{N+1}$, $\tilde{R}_n(x)$ is a nonzero polynomial of degree $\leq n$ and $\langle \tau, x^m \tilde{R}_n(x) \rangle = 0$ for $0 \leq m \leq n-1$ so that $\deg(\tilde{R}_n) = n$, that is, $A_0 \neq 0$. Then $A_0^{-1} \tilde{R}_n(x) = R_n(x)$ so that we have (4.22).

Now we can express $R_n(x)$ as a hypergeometric series (see [13]);

$$\begin{aligned}
 R_n(x) &= \frac{\beta_0 \beta_1 \cdots \beta_N}{(\alpha+1)_{N+1}} (-1)^n (\alpha+1)_n (A_0 + A_1 + \cdots + A_{N+1}) \\
 &\quad \times {}_{N+2}F_{N+2} \left(\begin{matrix} -n, \beta_0+1, \beta_1+1, \dots, \beta_N+1 \\ \alpha+N+2, \beta_0, \beta_1, \dots, \beta_N \end{matrix} \middle| x \right) \tag{4.31}
 \end{aligned}$$

for suitable constants $\{\beta_j\}_{j=0}^N$.

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