INTEGRAL REPRESENTATIONS FOR PADÉ-TYPE OPERATORS

NICHOLAS J. DARAS

Received 2 October 2000 and in revised form 10 October 2001

The main purpose of this paper is to consider an explicit form of the Padé-type operators. To do so, we consider the representation of Padé-type approximants to the Fourier series of the harmonic functions in the open disk and of the L^p -functions on the circle by means of integral formulas, and, then we define the corresponding Padé-type operators. We are also concerned with the properties of these integral operators and, in this connection, we prove some convergence results.

1. Introduction

Let f be a function analytic in the open unit disk D , with Taylor power series expansion $\sum_{v=0}^{\infty} a_v \cdot z^v$, and let Λ_f be the linear functional on the space of complex polynomials defined by $\Lambda_f(x^v) = a_v$ ($v = 0, 1, 2, \dots$). By Cauchy's integral formula and by a density argument, the functional Λ_f can be extended to the space $A(\bar{D})$ of all functions which are analytic in D and continuous in the open neighborhood of $\bar{D}$ (see [4]). In particular, we have $f(z) = \Lambda_f((1 - x \cdot z)^{-1})$ for any $z \in D$.

Now, let $v_{m+1}(x)$ be an arbitrary polynomial of degree $m+1$, with distinct zeros $\pi_1, \pi_2, \dots, \pi_n$ of respective multiplicities $(m_1+1), (m_2+1), \dots, (m_n+1)$ and $(m_1+1) + (m_2+1) + \dots + (m_n+1) = m+1$.

Let $I(v_{m+1})$ be the linear operator mapping each $g(x) \in A(\bar{D})$ into its Hermite interpolation polynomial G_{m+1} of degree at most m defined by

$$g^{(j)}(\pi_i) = G_{m+1}^{(j)}(\pi_i) \quad \text{for } i = 1, \dots, n, \quad j = 0, 1, \dots, m_i. \quad (1.1)$$

If $g(x, z) = (1 - x \cdot z)^{-1}$, then $\Lambda_f(G_{m+1}(x, z))$ is the so-called Padé-type

approximant to $f(z)$ with generating polynomial $v_{m+1}(x)$. It is a rational function with numerator of degree m and denominator of degree $m+1$, denoted by $(m/(m+1))_f(z)$ and such that

$$f(z) - \left(\frac{m}{m+1} \right)_f(z) = O(z^{m+1}), \quad \text{if } |z| < \min \left\{ \frac{1}{|\pi_1|}, \dots, \frac{1}{|\pi_n|} \right\}. \quad (1.2)$$

If $v_{m+1}(x)$ is identical to the orthogonal polynomial $q_{m+1}(x)$ with respect to Λ_f , that is, the polynomial satisfying the orthogonality conditions $\Lambda_f(x^v \cdot q_{m+1}(x)) = 0$, $v = 0, 1, \dots, m$, then the Padé-type approximant $(m/(m+1))_f(z)$ becomes identical to the classical Padé approximant $[m/(m+1)]_f(z)$ such that

$$f(z) - \left[\frac{m}{m+1} \right]_f(z) = O(z^{2m+2}), \quad \text{if } |z| < \min \left\{ \frac{1}{|\pi_1|}, \dots, \frac{1}{|\pi_n|} \right\}. \quad (1.3)$$

By making use of the notation of duality, we can also write

$$\begin{aligned} \left(\frac{m}{m+1} \right)_f(z) &= \Lambda_f(G_{m+1}(x, z)) \\ &= \langle \Lambda_f, [I(v_{m+1})](1 - x \cdot z)^{-1} \rangle \\ &= \langle [I^*(v_{m+1})](\Lambda_f)(1 - x \cdot z)^{-1} \rangle. \end{aligned} \quad (1.4)$$

In [3], Brezinski showed that the operator which maps f on $(m/(m+1))_f$ can be understood as the mapping of $A^*(\bar{D})$ into itself which maps Λ_f into $[I^*(v_{m+1})](\Lambda_f)$. This mapping, which depends on the generating polynomial $v_{m+1}(x)$, is called the Padé-type operator for the space $O(D)$ of all analytic functions on D and it is exactly the operator $I^*(v_{m+1})$. If $v_{m+1}(x)$ does not depend on Λ_f , then $I^*(v_{m+1})$ is linear. But for Padé approximants, since $v_{m+1}(x)$ is the orthogonal polynomial $q_{m+1}(x)$ of degree $m+1$ with respect to the functional Λ_f , then $v_{m+1}(x)$ depends on Λ_f and the linearity property holds only if the first $2m+2$ moments of both functionals are the same since, then, both orthogonal polynomial of degree $m+1$ will be the same.

The aim of this paper is to consider the explicit form of the Padé-type operator by means of integral representations. [Section 2](#) deals with the definition of integral representations of Padé-type approximants to real-valued L^2 or harmonic functions and, thus, with the expressions of the Padé-type operator for the spaces $L^2_{\mathbb{R}}(C)$ of all real-valued L^2 functions on C , $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ of all real-valued 2π -periodic L^2 functions on $[-\pi, \pi]$, and $H^2_{\mathbb{R}}(D)$ of all real-valued harmonic functions on D . We

also give some examples with applications of these integral representations for the Padé-type operator to the convergence problem of a series of Padé-type approximants and to the problem of finding a sufficient condition permitting the interpretation of any 2π -periodic L^p real-valued function on $[-\pi, \pi]$ as a Padé-type approximant. In [7], by introducing the so-called composed Padé-type approximation, we discussed the general situation of complex-valued harmonic or L^p functions and we showed that any Padé-type approximant in the ordinary sense to a function $f \in O(D)$ is a special case of this composed procedure. It is therefore natural to reflect that any $I^*(v_{m+1})$ can also be viewed as a special case of the operator which maps every $f \in O(D)$ on a composed Padé-type approximant to f . Such a mapping will be called a composed Padé-type operator for $O(D)$. In Section 3, we define and give the explicit form of the composed Padé-type operators for the spaces $L^2_{\mathbb{C}}(C)$ of all complex-valued L^2 functions on C , $L^2_{\mathbb{C}, (2\pi\text{-per})}[-\pi, \pi]$ of all complex-valued 2π -periodic L^2 functions on $[-\pi, \pi]$, and $H_{\mathbb{C}}(D)$ of all complex-valued harmonic functions on D . Since $O(D) \subset H_{\mathbb{C}}(D)$, we thus obtain the desired explicit form of $I^*(v_{m+1})$.

2. Integral representations and Padé-type operators

In [5, 6], we have defined and studied Padé-type approximation to $L^p 2\pi$ -periodic real-valued functions and to harmonic functions in D . In all cases, the development of our theory was analogous to the classical one about analytic functions.

Really, no situation is quite as pleasant as the L^2 case. In this section, we look for another way to introduce Padé-type approximants to L^2 functions and to harmonic functions. Our method is based on integral representation formulas and leads to a number of convergence results.

To begin our discussion, consider any real-valued L^2 function $u(z)$ defined on the unit circle C . Suppose that the Fourier series expansion of $u(e^{it})$ is $\sum_{v=-\infty}^{\infty} \sigma_v \cdot e^{ivt}$. Since u is square integrable, the sequence of partial sums $\{\sum_{v=-n}^n \sigma_v \cdot e^{ivt} : n = 0, 1, 2, \dots\}$ converges to $u(e^{it})$ in the L^2 -norm. Let $P(\mathbb{C})$ be the vector space of all complex-valued analytic polynomials with coefficients in $\mathbb{C}$. For every $p(x) = \sum_{v=0}^m \beta_v \cdot x^v \in P(\mathbb{C})$, we denote by $\bar{p}(x)$ the polynomial $\bar{p}(x) = \sum_{v=0}^m \bar{\beta}_v \cdot x^v \in P(\mathbb{C})$. Define the linear functionals $T_u : P(\mathbb{C}) \rightarrow \mathbb{C}$ and $S_u : P(\mathbb{C}) \rightarrow \mathbb{C}$ associated with u by

$$T_u(x^v) := \sigma_v, \quad S_u(x^v) := \sigma_{-v} \quad (v = 0, 1, 2, \dots). \quad (2.1)$$

As it is well known, the Poisson integral of $u(z) = u(e^{it})$ ($|z| = 1$) extends to a harmonic real-valued function $u(z) = u(r \cdot e^{it})$ in the unit disk D (where $|z| < 1$, $0 \leq r < 1$). This harmonic function being the real part of

54 Integral representations for Padé-type operators

some analytic function in D , we immediately see that $\overline{T_u(x^v)} = \bar{\sigma}_v = \sigma_{-v} = S_u(x^v)$ for any $v \geq 0$. More generally, we have the following proposition.

PROPOSITION 2.1. *For every $p(x) \in P(\mathbb{C})$ there holds*

$$\overline{S_u(p(x))} = T_u(\bar{p}(x)), \quad \overline{S_u(\bar{p}(x))} = T_u(p(x)). \quad (2.2)$$

Proof. Let $p(x) = \sum_{v=0}^m \beta_v x^v \in P(\mathbb{C})$. By linearity, we obtain

$$\begin{aligned} S_u(p(x)) &= S_u\left(\sum_{v=0}^m \beta_v x^v\right) = \sum_{v=0}^m \beta_v S_u(x^v) = \sum_{v=0}^m \beta_v \overline{T_u(x^v)} \\ &= \overline{\sum_{v=0}^m \bar{\beta}_v \cdot T_u(x^v)} = \overline{T_u\left(\sum_{v=0}^m \bar{\beta}_v \cdot x^v\right)} = \overline{T_u(\bar{p}(x))}, \\ \overline{S_u(\bar{p}(x))} &= \overline{S_u\left(\sum_{v=0}^m \bar{\beta}_v x^v\right)} = \overline{\sum_{v=0}^m \bar{\beta}_v S_u(x^v)} = \overline{\sum_{v=0}^m \bar{\beta}_v \overline{T_u(x^v)}} \\ &= \sum_{v=0}^m \beta_v \cdot T_u(x^v) = T_u\left(\sum_{v=0}^m \beta_v \cdot x^v\right) = T_u(p(x)). \end{aligned} \quad (2.3)$$

□

COROLLARY 2.2. *For every $p(x) \in P(\mathbb{C})$ there holds*

$$\operatorname{Re} T_u(\bar{p}(x)) = \operatorname{Re} S_u(p(x)), \quad \operatorname{Re} T_u(p(x)) = \operatorname{Re} S_u(\bar{p}(x)). \quad (2.4)$$

Now, observe that the linear functional S_u can be extended continuously on the space $L^2_{\mathbb{R}}(C)$ of all real-valued square integrable functions on the unit circle C . In fact, if $p(x) = \sum_{v=0}^m \beta_v x^v \in P(\mathbb{C})$ then, by Hölder's inequality, we get

$$\begin{aligned} |S_u(p(x))|^2 &= \left| \sum_{v=0}^m \beta_v \sigma_{-v} \right|^2 \\ &= \left| \sum_{v=0}^m \bar{\beta}_v \sigma_v \right|^2 \\ &= \left| \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{it}) \cdot \left(\sum_{v=0}^m \bar{\beta}_v \cdot e^{-ivt} \right) dt \right|^2 \\ &= \left| \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{it}) \cdot \overline{p(e^{it})} dt \right|^2 \\ &\leq c_u \cdot \|p(x)\|_2^2, \end{aligned} \quad (2.5)$$

for some positive constant c_u depending only on u , and hence, by the Hahn-Banach theorem, there is a continuous linear extension of S_u on $L^2_{\mathbb{R}}(C)$. It follows, from the Riesz representation theorem, that there exists a unique $F_u \in L^2_{\mathbb{R}}(C)$ such that

$$S_u(g) = \int_C g(\zeta) \cdot \overline{F_u(\zeta)} d\zeta = i \cdot \int_{-\pi}^{\pi} g(e^{i\theta}) \cdot \overline{F_u(e^{i\theta})} \cdot e^{i\theta} d\theta \quad (2.6)$$

for all $g \in L^2_{\mathbb{R}}(C)$. In particular, if $g(\zeta) = \zeta^v$ then

$$S_u(\zeta^v) = \int_C \zeta^v \overline{F_u(\zeta)} d\zeta = i \cdot \int_{-\pi}^{\pi} e^{iv\theta} \cdot \overline{F_u(e^{i\theta})} \cdot e^{i\theta} d\theta. \quad (2.7)$$

But

$$S_u(\zeta^v) = \sigma_{-v} = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \cdot e^{iv\theta} d\theta, \quad (2.8)$$

and therefore

$$\overline{F_u(e^{i\theta})} = -i \cdot u(e^{i\theta}) \cdot e^{-i\theta}, \quad (2.9)$$

which implies that

$$S_u(g) = \int_{-\pi}^{\pi} g(e^{i\theta}) \cdot u(e^{i\theta}) d\theta \quad (2.10)$$

for all $g \in L^2_{\mathbb{R}}(C)$. In view of [Corollary 2.2](#), we have thus obtained the following theorem.

THEOREM 2.3. *Let $M = (\pi_{m,k})_{m \geq 0, 0 \leq k \leq m}$ be an infinite triangular interpolation matrix with complex entries and, for any $m \geq 0$, let $G_m(x, z)$ be the unique polynomial of degree at most m which interpolates the function $(1 - x \cdot z)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \dots, \pi_{m,m}$ (where z is fixed and $|\pi_{m,k}| < 1$).*

(a) *For any real-valued function $u \in L^2_{\mathbb{R}}(C)$, the Padé-type approximant $\text{Re}(m/(m+1))_u(z)$ to $u(z)$ has the following integral representation:*

$$\text{Re} \left(\frac{m}{m+1} \right)_u(z) = \frac{1}{2\pi} \cdot \int_C u(\zeta) \text{Re} \left\{ \frac{4\pi \bar{G}_m(\zeta, z) - 1}{i\zeta} \right\} d\zeta \quad (|z| = 1), \quad (2.11)$$

or equivalently

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right)_u (e^{it}) &= \int_{-\pi}^{\pi} u(e^{i\theta}) \cdot 2 \operatorname{Re} \left\{ \bar{G}_m(e^{i\theta}, e^{it}) - \frac{1}{4\pi} \right\} d\theta \\ &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \cdot \operatorname{Re} \{ 4\pi \cdot \bar{G}_m(e^{i\theta}, e^{it}) - 1 \} d\theta, \end{aligned} \quad (2.12)$$

where $-\pi \leq t \leq \pi$.

(b) Let $f \in L^2[-\pi, \pi]$ be a 2π -periodic real-valued function, with Fourier coefficients $\{c_v : v = \pm 0, \pm 1, \pm 2, \dots\}$. Since $f(t) = \sum_{v=-\infty}^{\infty} c_v \cdot e^{ivt}$ in the L^2 -norm, the function $f(t)$ can be viewed as a function of the unit circle, and therefore the Padé-type approximant $\operatorname{Re}(m/(m+1))_f(t)$ to $f(t)$ has the following integral representation:

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right)_f (t) &= \int_{-\pi}^{\pi} f(\theta) \cdot 2 \operatorname{Re} \left\{ \bar{G}_m(e^{i\theta}, e^{it}) - \frac{1}{4\pi} \right\} d\theta \\ &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(\theta) \cdot \operatorname{Re} \{ 4\pi \cdot \bar{G}_m(e^{i\theta}, e^{it}) - 1 \} d\theta \quad (-\pi \leq t \leq \pi). \end{aligned} \quad (2.13)$$

Proof. We have

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right)_u (e^{it}) &= 2 \operatorname{Re} T_u(G_m(x, e^{it})) - u(0) \\ &= 2 \operatorname{Re} S_u(\bar{G}_m(x, e^{it})) - u(0) \\ &= 2 \operatorname{Re} \int_{-\pi}^{\pi} \bar{G}_m(e^{i\theta}, e^{it}) u(e^{i\theta}) d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) \operatorname{Re} [4\pi \bar{G}_m(e^{i\theta}, e^{it})] d\theta \\ &\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \cdot \operatorname{Re} \{ 4\pi \bar{G}_m(e^{i\theta}, e^{it}) - 1 \} d\theta \\ &= \int_{-\pi}^{\pi} u(e^{i\theta}) \cdot 2 \operatorname{Re} \left\{ \bar{G}_m(e^{i\theta}, e^{it}) - \frac{1}{4\pi} \right\} d\theta \quad (-\pi \leq t \leq \pi). \end{aligned} \quad (2.14)$$

Setting $z = e^{it}$ and $\zeta = e^{i\theta}$, we also have

$$\begin{aligned}
 \operatorname{Re} \left(\frac{m}{m+1} \right)_u (z) &= \operatorname{Re} \left(\frac{m}{m+1} \right)_u (e^{it}) \\
 &= \operatorname{Re} \left[\frac{1}{2\pi i} \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \left\{ \frac{4\pi \bar{G}_m(e^{i\theta}, e^{it}) - 1}{e^{i\theta}} \right\} i e^{i\theta} d\theta \right] \\
 &= \operatorname{Re} \left[\frac{1}{2\pi i} \cdot \int_C u(\zeta) \left\{ \frac{4\pi \bar{G}_m(\zeta, z) - 1}{\zeta} \right\} d\zeta \right] \\
 &= \frac{1}{2\pi} \cdot \int_C u(\zeta) \operatorname{Re} \left\{ \frac{4\pi \bar{G}_m(\zeta, z) - 1}{i\zeta} \right\} d\zeta \quad (|z| = 1).
 \end{aligned} \tag{2.15}$$

This completes the proof of (a). The proof of (b) is exactly similar and is based on the identification between $L^2_{\mathbb{R}}(C)$ and the space of all 2π -periodic L^2 real-valued functions on $[-\pi, \pi]$ (every $u(z) = u(e^{it}) \in L^2_{\mathbb{R}}(C)$ can be interpreted as a 2π -periodic real-valued function $f(t) \in L^2[-\pi, \pi]$ and conversely). $\square$

In order to simplify the formalism, we make use of the notation

$$\begin{aligned}
 \operatorname{Re} \left\{ \frac{B_m(\zeta, z)}{i\zeta} \right\} &:= \operatorname{Re} \left\{ \frac{4\pi \cdot \bar{G}_m(\zeta, z) - 1}{i\zeta} \right\}, \\
 \operatorname{Re} B_m(e^{i\theta}, e^{it}) &:= \operatorname{Re} \{ 4\pi \cdot \bar{G}_m(e^{i\theta}, e^{it}) - 1 \}.
 \end{aligned} \tag{2.16}$$

As it is well known, the function $\operatorname{Re}(m/(m+1))_u(z)$ ($|z| = 1$) is continuous (see [6]). Hence, the integral operator $\operatorname{Re}(m/(m+1))$ maps $L^2_{\mathbb{R}}(C)$ into $L^2_{\mathbb{R}}(C)$ and therefore, by the closed graph theorem, it is continuous (of course, under the assumption that $|\pi_{m,k}| < 1$ for all $k \leq m$). The integral operator

$$\begin{aligned}
 \operatorname{Re} \left(\frac{m}{m+1} \right) : L^2_{\mathbb{R}}(C) &\longrightarrow L^2_{\mathbb{R}}(C); \\
 u(z) &\longrightarrow \operatorname{Re} \left(\frac{m}{m+1} \right)_u (z) = \frac{1}{2\pi} \cdot \int_C u(\zeta) \cdot \operatorname{Re} \frac{B_m(\zeta, z)}{i\zeta} d\zeta
 \end{aligned} \tag{2.17}$$

is called the Padé-type operator for $L^2_{\mathbb{R}}(C)$. Its adjoint operator is given by

$$\begin{aligned}
 \operatorname{Re} \left(\frac{m}{m+1} \right)^* : L^2_{\mathbb{R}}(C) &\longrightarrow L^2_{\mathbb{R}}(C); \\
 u(z) &\longrightarrow \operatorname{Re} \left(\frac{m}{m+1} \right)^*_u (z) = \frac{1}{2\pi} \cdot \int_C u(\zeta) \cdot \operatorname{Re} \frac{B_m(z, \zeta)}{iz} d\zeta.
 \end{aligned} \tag{2.18}$$

In fact, to $\text{Re}(m/(m+1))$ there corresponds a unique operator $\text{Re}(m/(m+1))^* : L^2_{\mathbb{R}}(C) \rightarrow L^2_{\mathbb{R}}(C)$ satisfying $\langle \text{Re}(m/(m+1))_u, w \rangle = \langle u, \text{Re}(m/(m+1))^*_w \rangle$, that is,

$$\int_C \text{Re} \left(\frac{m}{m+1} \right)_u (\zeta) \cdot w(\zeta) d\zeta = \int_C u(z) \cdot \text{Re} \left(\frac{m}{m+1} \right)^*_w (z) dz \quad (2.19)$$

for all $u, w \in L^2_{\mathbb{R}}(C)$; since, by Fubini's theorem,

$$\begin{aligned} & \int_C \text{Re} \left(\frac{m}{m+1} \right)_u (\zeta) \cdot w(\zeta) d\zeta \\ &= \int_C \frac{1}{2\pi} \int_C u(z) \cdot \text{Re} \frac{B_m(z, \zeta)}{iz} dz w(\zeta) d\zeta \\ &= \int_C u(z) \cdot \left(\frac{1}{2\pi} \int_C w(\zeta) \cdot \text{Re} \frac{B_m(z, \zeta)}{iz} d\zeta \right) dz, \end{aligned} \quad (2.20)$$

we conclude that

$$\text{Re} \left(\frac{m}{m+1} \right)^*_w (z) = \frac{1}{2\pi} \int_C w(\zeta) \cdot \text{Re} \frac{B_m(z, \zeta)}{iz} d\zeta \quad (w \in L^2_{\mathbb{R}}(C)). \quad (2.21)$$

Similarly, as it is pointed out in [6], for any real-valued 2π -periodic function $f \in L^2[-\pi, \pi]$, the Padé-type approximant $\text{Re}(m/(m+1))_f(t)$ is continuous, and, by construction, 2π -periodic. It follows that the integral operator $\text{Re}(m/(m+1))$ maps the space $L^2_{\mathbb{R}, (2\pi\text{-per})}[-\pi, \pi]$ of real-valued 2π -periodic functions of $L^2[-\pi, \pi]$ into itself. Hence, by the closed graph theorem, the operator

$$\begin{aligned} & \text{Re} \left(\frac{m}{m+1} \right) : L^2_{\mathbb{R}, (2\pi\text{-per})}[-\pi, \pi] \longrightarrow L^2_{\mathbb{R}, (2\pi\text{-per})}[-\pi, \pi]; \\ & f(t) \longrightarrow \text{Re} \left(\frac{m}{m+1} \right)_f(t) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(\theta) \cdot \text{Re} B_m(e^{i\theta}, e^{it}) d\theta \end{aligned} \quad (2.22)$$

is continuous and is called the *Padé-type operator* for $L^2_{\mathbb{R}, (2\pi\text{-per})}[-\pi, \pi]$. Its adjoint operator is then given by

$$\begin{aligned} & \text{Re} \left(\frac{m}{m+1} \right)^* : L^2_{\mathbb{R}, (2\pi\text{-per})}[-\pi, \pi] \longrightarrow L^2_{\mathbb{R}, (2\pi\text{-per})}[-\pi, \pi]; \\ & f(t) \longrightarrow \text{Re} \left(\frac{m}{m+1} \right)^*_f(t) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(\theta) \cdot \text{Re} B_m(e^{it}, e^{i\theta}) d\theta. \end{aligned} \quad (2.23)$$

In fact, to $\text{Re}(m/(m+1))$ we associate the unique operator $\text{Re}(m/(m+1))^* : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi] \rightarrow L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ satisfying

$$\left\langle \text{Re}\left(\frac{m}{m+1}\right)_f, g \right\rangle = \left\langle f, \text{Re}\left(\frac{m}{m+1}\right)_g^* \right\rangle, \quad (2.24)$$

that is,

$$\int_{-\pi}^{\pi} \text{Re}\left(\frac{m}{m+1}\right)_f(t) \cdot g(t) dt = \int_{-\pi}^{\pi} f(\theta) \cdot \text{Re}\left(\frac{m}{m+1}\right)_g^*(\theta) d\theta \quad (2.25)$$

for all $f, g \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$; it follows, from Fubini's theorem, that

$$\begin{aligned} \int_{-\pi}^{\pi} \text{Re}\left(\frac{m}{m+1}\right)_f(t) \cdot g(t) dt \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(\theta) \cdot \text{Re} B_m(e^{i\theta}, e^{it}) d\theta g(t) dt \\ &= \int_{-\pi}^{\pi} f(\theta) \cdot \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \cdot \text{Re} B_m(e^{i\theta}, e^{it}) dt \right) d\theta, \end{aligned} \quad (2.26)$$

and consequently

$$\begin{aligned} \text{Re}\left(\frac{m}{m+1}\right)_g^*(\theta) \\ &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} g(t) \cdot \text{Re} B_m(e^{i\theta}, e^{it}) dt \quad (g \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]). \end{aligned} \quad (2.27)$$

Summarizing, we have the following theorem.

THEOREM 2.4. *If $m \geq 0$, then for any $u(z) \in L^2_{\mathbb{R}}(C)$ and any $f(t) \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$, there holds*

$$\begin{aligned} \text{Re}\left(\frac{m}{m+1}\right)_u^*(z) &= \frac{1}{2\pi} \cdot \int_C u(\zeta) \cdot \text{Re} \frac{B_m(z, \zeta)}{iz} d\zeta, \\ \text{Re}\left(\frac{m}{m+1}\right)_f^*(t) &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(\theta) \cdot \text{Re} B_m(e^{it}, e^{i\theta}) d\theta. \end{aligned} \quad (2.28)$$

The continuity of the Padé-type operators $\text{Re}(m/(m+1))$ leads immediately to the following convergence results which can be considered as a first example of their application.

THEOREM 2.5. (a) If the sequence $\{u_n \in L^2_{\mathbb{R}}(C) : n = 0, 1, 2, \dots\}$ converges to $u \in L^2_{\mathbb{R}}(C)$ in the L^2 -norm, then there holds $\lim_{n \rightarrow \infty} \text{Re}(m/(m+1))_{u_n}(z) = \text{Re}(m/(m+1))_u(z)$ in the L^2 -norm.

(b) If the sequence $\{f_n \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi] : n = 0, 1, 2, \dots\}$ converges to $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ in the L^2 -norm, then there holds $\lim_{n \rightarrow \infty} \text{Re}(m/(m+1))_{f_n}(t) = \text{Re}(m/(m+1))_f(t)$ in the L^2 -norm.

COROLLARY 2.6. (a) If the series of functions $u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z)$ (where $a_n \in \mathbb{R}$, $u_n \in L^2_{\mathbb{R}}(C)$) converges in the L^2 -norm, then $\text{Re}(m/(m+1))_u(z) = \sum_{n=0}^{\infty} a_n \text{Re}(m/(m+1))_{u_n}(z)$ in the L^2 -norm.

(b) If the series of functions $f(t) = \sum_{n=0}^{\infty} a_n \cdot f_n(t)$ (where $a_n \in \mathbb{R}$, $f_n \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$) converges in the L^2 -norm then $\text{Re}(m/(m+1))_f(t) = \sum_{n=0}^{\infty} a_n \text{Re}(m/(m+1))_{f_n}(t)$ in the L^2 -norm.

Now we determine the conditions under which the integral operator $\text{Re}(m/(m+1))$ is compact onto $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$. Since, for each fixed $t \in [-\pi, \pi]$, the kernel function $\text{Re}B_m(e^{i\theta}, e^{it})$ is bounded in θ , it follows, from Tonelli's theorem, that the following theorem holds true.

THEOREM 2.7. If there is a constant $c_* < \infty$ such that

$$\int_{-\pi}^{\pi} |\text{Re}B_m(e^{i\theta}, e^{it})|^2 d\theta \leq (2\pi)^2 \cdot c_* \quad (2.29)$$

for almost all $t \in [-\pi, \pi]$, then the Padé-type operator $\text{Re}(m/(m+1)) : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi] \rightarrow L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ is compact. Moreover,

$$\left\| \text{Re} \left(\frac{m}{m+1} \right) \right\| \leq (2\pi)^{5/2} \cdot c_* \quad (2.30)$$

and $\text{Re}(m/(m+1))^*$ is also compact.

It is readily seen that if the Padé-type operator $\text{Re}(m/(m+1)) : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi] \rightarrow L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ is compact, then it is not one-to-one. This follows from the fact that $\dim L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi] = \infty$ and therefore 0 must be an eigenvalue of $\text{Re}(m/(m+1))$. However, it would be interesting to know a necessary and sufficient condition under which for any $h \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ there is an $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ such that $\text{Re}(m/(m+1))_f = h$. Of course, such a general condition is the inequality

$$\left\| \operatorname{Re} \left(\frac{m}{m+1} \right)_f^* \right\|_2 \geq c \cdot \|f\|_2 \quad (2.31)$$

or alternatively,

$$\int_{-\pi}^{\pi} |f(t)|^2 dt \leq c \cdot \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f(\theta) \cdot \operatorname{Re} B_m(e^{it}, e^{i\theta}) d\theta \right|^2 dt \quad (2.32)$$

for some constant $c > 0$ and for every $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$. Obviously, this inequality is true if and only if

$$|f(t)| \leq c \cdot \left| \int_{-\pi}^{\pi} f(\theta) \cdot \operatorname{Re} B_m(e^{it}, e^{i\theta}) d\theta \right| \quad (2.33)$$

for almost all $t \in [-\pi, \pi]$, and thus we have proved the following theorem describing a sufficient condition under which every function in $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ is a Padé-type approximant.

THEOREM 2.8. *If there is a constant $c > 0$ such that*

$$|f(t)| \leq c \cdot \left| \int_{-\pi}^{\pi} f(\theta) \cdot \operatorname{Re} B_m(e^{it}, e^{i\theta}) d\theta \right| \quad (2.34)$$

almost everywhere on $[-\pi, \pi]$, for every $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$, then the range of $\operatorname{Re}(m/(m+1))$ equals $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$.

Finally, we turn to integral representation formulas in the harmonic case. If u is harmonic and real-valued in the unit disk, then, for any $0 \leq r < 1$, the restriction $u_r(t) = u(r \cdot e^{it})$ ($-\pi \leq t \leq \pi$) of $u(z)$ to the circle of radius r can be interpreted as a real-valued, 2π -periodic function in $L^2[-\pi, \pi]$. According to [Theorem 2.3](#), the Padé-type approximant $\operatorname{Re}(m/(m+1))_{u_r}(t)$ to $u_r(t)$ is given by the integral representation formula

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right)_{u_r}(t) &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u_r(\theta) \cdot \operatorname{Re} \{ 4\pi \cdot \bar{G}_m(r \cdot e^{i\theta}, r \cdot e^{it}) - 1 \} d\theta \\ &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u_r(r \cdot e^{i\theta}) \cdot \operatorname{Re} \{ 4\pi \cdot \bar{G}_m(r \cdot e^{i\theta}, r \cdot e^{it}) - 1 \} d\theta. \end{aligned} \quad (2.35)$$

62 Integral representations for Padé-type operators

After a simple change of variables $z = r \cdot e^{it}$ and $\zeta = r \cdot e^{i\theta}$, we obtain

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) &= \frac{1}{2\pi} \cdot \int_{|\zeta|=r} u(\zeta) \cdot \operatorname{Re} \left\{ \frac{4\pi \cdot \bar{G}_m(\zeta, z) - 1}{\zeta i} \right\} d\zeta \\ &= \frac{1}{2i} \cdot \int_{|\zeta|=r} u(\zeta) \cdot \operatorname{Re} \left\{ \frac{B_m(\zeta, z)}{\zeta i} \right\} d\zeta, \end{aligned} \quad (2.36)$$

and hence we can state the following theorem.

THEOREM 2.9. *Let $M = (\pi_{m,k})_{m \geq 0, 0 \leq k \leq m}$ be an infinite triangular interpolation matrix with complex entries and, for any $m \geq 0$, let $G_m(x, z)$ be the unique polynomial of degree at most m which interpolates the function $(1 - x \cdot z)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \dots, \pi_{m,m}$ (where z is fixed and $|\pi_{m,k}| < 1$ for each $k \leq m$).*

The Padé-type approximant $\operatorname{Re}(m/(m+1))_u(z)$ to the harmonic real-valued function $u(z)$ in the disk is given by the following integral representation formula:

$$\operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} u(\zeta) \cdot \operatorname{Re} \left\{ \frac{B_m(\zeta, z)}{i\zeta} \right\} d\zeta \quad (z \in D). \quad (2.37)$$

As it is mentioned in [5], the function $\operatorname{Re}(m/(m+1))_u(z)$ is the real part of an analytic function in the unit disk, and therefore, it is a harmonic real-valued function in D (of course, under the assumption that $|\pi_{m,k}| < 1$ for all $k \leq m$). If $H_{\mathbb{R}}(D)$ is the space of all harmonic real-valued functions in D , the integral operator

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right) : H_{\mathbb{R}}(D) &\longrightarrow H_{\mathbb{R}}(D); \\ u(z) &\longrightarrow \operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} u(\zeta) \cdot \operatorname{Re} \left\{ \frac{B_m(\zeta, z)}{i\zeta} \right\} d\zeta \end{aligned} \quad (2.38)$$

is said to be a *Padé-type operator* of $H_{\mathbb{R}}(D)$. It is easily seen that a Padé-type operator of $H_{\mathbb{R}}(D)$ is continuous. For, if $\{u_n \in H_{\mathbb{R}}(D) : n = 0, 1, 2, \dots\}$ and if $\lim_{n \rightarrow \infty} u_n = u \in H_{\mathbb{R}}(D)$ compactly in the disk D , then, by the

maximum principle for harmonic functions, we have

$$\begin{aligned}
 & \sup_{|z| \leq r} \left| \operatorname{Re} \left(\frac{m}{m+1} \right)_{u_n}(z) - \operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) \right| \\
 &= \sup_{|z|=r} \left| \operatorname{Re} \left(\frac{m}{m+1} \right)_{u_n}(z) - \operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) \right| \\
 &= \frac{1}{2\pi} \cdot \sup_{|z|=r} \left| \int_{|\zeta|=r} [u_n(\zeta) - u(\zeta)] \cdot \operatorname{Re} \left\{ \frac{B_m(\zeta, z)}{\zeta i} \right\} d\zeta \right| \quad (2.39) \\
 &\leq \frac{1}{2\pi r} \cdot 2\pi r \cdot \left\{ \sup_{|z|=r, |\zeta|=r} |\operatorname{Re} B_m(\zeta, z)| \right\} \cdot \left\{ \sup_{|\zeta|=r} |u_n(\zeta) - u(\zeta)| \right\} \\
 &\leq L(r, m) \cdot \left\{ \sup_{|\zeta|=r} |u_n(\zeta) - u(\zeta)| \right\}
 \end{aligned}$$

for any $r < 1$, and the continuity of $\operatorname{Re}(m/(m+1)) : H_{\mathbb{R}}(D) \rightarrow H_{\mathbb{R}}(D)$ follows.

As for the L^2 -case, the continuity of the Padé-type operator for $H_{\mathbb{R}}(D)$ leads to the following convergence results.

THEOREM 2.10. *If the sequence $\{u_n : n = 0, 1, 2, \dots\}$ of harmonic real-valued functions in the open unit disk converges compactly to $u \in H_{\mathbb{R}}(D)$, then there holds*

$$\lim_{n \rightarrow \infty} \operatorname{Re} \left(\frac{m}{m+1} \right)_{u_n}(z) = \operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) \quad (2.40)$$

compactly in D .

COROLLARY 2.11. *If the series of harmonic real-valued functions*

$$u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z) \quad (a_n \in \mathbb{R}, u_n \in H_{\mathbb{R}}(D)) \quad (2.41)$$

converges compactly in the disk, then

$$\operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) = \sum_{n=0}^{\infty} a_n \operatorname{Re} \left(\frac{m}{m+1} \right)_{u_n}(z), \quad (2.42)$$

the convergence of the series being compact in D .

Remark 2.12. In [2], Brezinski showed that the (Hermite) interpolation polynomial $G_m(x, z)$ of $(1 - xz)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \dots, \pi_{m,m}$ is given by

$$G_m(x, z) = \frac{1}{1 - x \cdot z} \cdot \left(1 - \frac{v_{m+1}(x)}{v_{m+1}(z^{-1})} \right) \quad (z \neq \pi_{m,k}^{-1}, k = 0, 1, \dots, m), \quad (2.43)$$

where $v_{m+1}(x)$ is any generating polynomial $v_{m+1}(x) = \gamma \cdot \prod_{k=0}^m (x - \pi_{m,k})$ ($\gamma \neq 0$). We thus obtain the following expressions for the kernels $\operatorname{Re}\{B_m(\zeta, z)/\zeta i\}$ and $\operatorname{Re} B_m(e^{i\theta}, e^{it})$:

$$\begin{aligned} \operatorname{Re} \left\{ \frac{B_m(\zeta, z)}{\zeta i} \right\} &= \operatorname{Re} \left\{ \frac{-4i\zeta^{-1}}{1 - \zeta \cdot \bar{z}} \left(1 - \bar{z}^{m+1} \cdot \prod_{k=0}^m \frac{\zeta - \overline{\pi_{m,k}}}{1 - \bar{z} \cdot \overline{\pi_{m,k}}} \right) - \zeta^{-1} \right\}, \\ \operatorname{Re} B_m(e^{i\theta}, e^{it}) &= \operatorname{Re} \left\{ \frac{4\pi}{1 - e^{i(\theta-t)}} \left(1 - \prod_{k=0}^m \frac{e^{i\theta} - \overline{\pi_{m,k}}}{e^{it} - \overline{\pi_{m,k}}} \right) - 1 \right\}. \end{aligned} \quad (2.44)$$

If, for example, $\pi_{m,0} = \dots = \pi_{m,m} = 0$, then for any $u \in L^2_{\mathbb{R}}(C)$, we have

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right)_u(z) &= \operatorname{Re} \left\{ \frac{2}{\pi i} \cdot \sum_{v=0}^m \bar{z}^v \int_C u(\zeta) \cdot \zeta^{v-1} d\zeta - \frac{2}{\pi i} \cdot \int_C u(\zeta) \cdot \zeta^{-1} d\zeta \right\} \quad (z \in C) \\ &\quad (2.45) \end{aligned}$$

or

$$\begin{aligned} \operatorname{Re} \left(\frac{m}{m+1} \right)_u(e^{it}) &= 2 \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \cos(\theta - t) d\theta - 2 \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \cos[m - (\theta - t)] d\theta \\ &= 4 \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \cdot \sin \left[\frac{(m+1)\theta - (m+1)t}{2} \right] \\ &\quad \cdot \sin \left[\frac{(m-1)\theta - (m-1)t}{2} \right] d\theta \quad (-\pi \leq t \leq \pi). \end{aligned} \quad (2.46)$$

3. Integral representations and composed Padé-type approximation

We are now in a position to generalize the definitions and results of [Section 2](#) to the context of composed Padé-type approximation. Recall that a composed Padé-type approximant to a harmonic complex-valued function $u = u_1 + iu_2$ in the disk D (resp., to an L^p complex-valued

function $u = u_1 + iu_2$ on the circle C or to a 2π -periodic complex-valued function $f = f_1 + if_2 \in L^p[-\pi, \pi]$) is a coordinate approximant given by the formula

$$\left(\frac{m}{m+1}\right)_u(z) = \operatorname{Re}\left(\frac{m_1}{m_1+1}\right)_{u_1}(z) + i\operatorname{Re}\left(\frac{m_2}{m_2+1}\right)_{u_2}(z) \quad (z \in D), \quad (3.1)$$

respectively, by the formula

$$\left(\frac{m}{m+1}\right)_u(z) = \operatorname{Re}\left(\frac{m_1}{m_1+1}\right)_{u_1}(z) + i\operatorname{Re}\left(\frac{m_2}{m_2+1}\right)_{u_2}(z) \quad (z \in C) \quad (3.2)$$

or

$$\left(\frac{m}{m+1}\right)_f(t) = \operatorname{Re}\left(\frac{m_1}{m_1+1}\right)_{f_1}(z) + i\operatorname{Re}\left(\frac{m_2}{m_2+1}\right)_{f_2}(t), \quad (3.3)$$

where $-\pi \leq t \leq \pi$ (see [7]).

Set

$$\begin{aligned} L^p_{\mathbb{C}}(C) &:= \{u \in L^p(C) : u \text{ is complex-valued}\}, \\ L^p_{\mathbb{C},(2\pi\text{-per})}[-\pi, \pi] &:= \{f \in L^p[-\pi, \pi] : f \text{ is complex-valued and} \\ &\quad 2\pi\text{-periodic } (f(-\pi) = f(\pi))\}, \\ H_{\mathbb{C}}(D) &:= \{u : D \longrightarrow \mathbb{C} : u \text{ is harmonic and complex-valued}\}. \end{aligned} \quad (3.4)$$

From Theorems 2.3 and 2.9, the following theorem follows immediately.

THEOREM 3.1. *For $j = 1, 2$, let $M^{(j)} = (\pi_{m_j, k}^{(j)})_{m_j \geq 0, 0 \leq k \leq m_j}$ be an infinite triangular interpolation matrix with complex entries $\pi_{m_j, k}^{(j)} \in D$, and, for any $m_j \geq 0$, let $G_{m_j}^{(j)}(x, z)$ be the unique polynomial of degree at most m_j which interpolates the function $(1 - xz)^{-1}$ at $x = \pi_{m_j, 0}^{(j)}, \pi_{m_j, 1}^{(j)}, \dots, \pi_{m_j, m_j}^{(j)}$ (where z is regarded as a parameter).*

If $G_{m_j}^{(j)}(x, z) = \sum_{v=0}^{m_j} g_v^{(j, m_j)}(z) \cdot x^v$, denote by $\overline{G_{m_j}^{(j)}}(x, z)$ the polynomial

$$\sum_{v=0}^{m_j} \overline{g_v^{(j, m_j)}}(z) \cdot x^v. \quad (3.5)$$

Put

$$B_{m_j}^{(j)}(x, z) = 4\pi \cdot \overline{G_{m_j}^{(j)}}(x, z) - 1. \quad (3.6)$$

(a) For any $u = u_1 + i \cdot u_2 \in L^2_{\mathbb{C}}(C)$, the corresponding composed Padé-type approximant $(m/(m+1))_u(z)$ to $u(z)$ has the following integral representation

$$\left(\frac{m}{m+1}\right)_u(z) = \frac{1}{2\pi} \cdot \int_C \left\{ u_1(\zeta) \cdot \operatorname{Re} \left[\frac{B_{m_1}^{(1)}(\zeta, z)}{\zeta i} \right] + i \cdot u_2(z) \cdot \operatorname{Re} \left[\frac{B_{m_2}^{(2)}(\zeta, z)}{i\zeta} \right] \right\} d\zeta \quad (|z| = 1), \quad (3.7)$$

or equivalently

$$\left(\frac{m}{m+1}\right)_u(e^{it}) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \left\{ u_1(e^{i\theta}) \cdot \operatorname{Re} B_{m_1}^{(1)}(e^{i\theta}, e^{it}) + i \cdot u_2(e^{i\theta}) \cdot \operatorname{Re} B_{m_2}^{(2)}(e^{i\theta}, e^{it}) \right\} d\theta \quad (-\pi \leq t \leq \pi). \quad (3.8)$$

(b) For any $f = f_1 + i \cdot f_2 \in L^2_{\mathbb{C}, (2\pi\text{-per})}[-\pi, \pi]$, the corresponding composed Padé-type approximant $(m/(m+1))_f(t)$ to $f(t)$ has the following integral representation:

$$\left(\frac{m}{m+1}\right)_f(t) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \left\{ f_1(\theta) \cdot \operatorname{Re} B_{m_1}^{(1)}(e^{i\theta}, e^{it}) + i \cdot f_2(\theta) \cdot \operatorname{Re} B_{m_2}^{(2)}(e^{i\theta}, e^{it}) \right\} d\theta \quad (-\pi \leq t \leq \pi). \quad (3.9)$$

(c) For any $u = u_1 + i \cdot u_2 \in H_{\mathbb{C}}(D)$, the corresponding composed Padé-type approximant $(m/(m+1))_u(z)$ to $u(z)$ has the following integral representation:

$$\left(\frac{m}{m+1}\right)_u(z) = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} \left\{ u_1(z) \cdot \operatorname{Re} \left[\frac{B_{m_1}^{(1)}(\zeta, z)}{\zeta i} \right] + i \cdot u_2(\zeta) \cdot \operatorname{Re} \left[\frac{B_{m_2}^{(2)}(\zeta, z)}{\zeta i} \right] \right\} d\zeta \quad (|z| < 1). \quad (3.10)$$

In particular, since any Padé-type approximant in the ordinary sense is a composed Padé-type approximant, we can give integral representation for the classical Padé-type approximants to analytic functions.

COROLLARY 3.2. Let $M = (\pi_{m,k})_{m \geq 0, 0 \leq k \leq m}$ be an infinite triangular interpolation matrix with complex entries $\pi_{m,k} \in D$, and, for any $m \geq 0$, let $G_m(x, z)$

be the unique polynomial of degree at most m which interpolates the function $(1 - xz)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \dots, \pi_{m,m}$ (z is regarded as a parameter).

If $G_m(x, z) = \sum_{v=0}^m g_v^{(m)}(z) \cdot x^v$, denote by $\bar{G}_m(x, z)$ the polynomial $\sum_{v=0}^m g_v^{(m)}(z) \cdot x^v$, and put

$$B_m(x, z) = 4\pi \cdot \bar{G}_m(x, z) - 1. \quad (3.11)$$

For any $f \in O(D)$, the corresponding Padé-type approximant $(m/(m+1))_f(z)$ to $f(z)$ (in the Brezinski's sense of [1]) has the following integral representation:

$$\left(\frac{m}{m+1}\right)_f(z) = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} f(\zeta) \cdot \operatorname{Re} \left[\frac{B_m(\zeta, z)}{\zeta i} \right] d\zeta \quad (|z| < 1). \quad (3.12)$$

Under the assumptions of [Theorem 3.1](#), the integral operators

$$\begin{aligned} & \left(\frac{m}{m+1}\right) : L_{\mathbb{C}}^2(C) \longrightarrow L_{\mathbb{C}}^2(C); \\ & u = u_1 + iu_2 \longmapsto \left(\frac{m}{m+1}\right)_u(z) \\ & = \frac{1}{2i} \cdot \int_C \left\{ u_1(\zeta) \cdot \operatorname{Re} \left[\frac{B_{m_1}^{(1)}(\zeta, z)}{\zeta i} \right] + i \cdot u_2(\zeta) \cdot \operatorname{Re} \left[\frac{B_{m_2}^{(2)}(\zeta, z)}{\zeta i} \right] \right\} d\zeta, \\ & \left(\frac{m}{m+1}\right) : L_{\mathbb{C}, (2\pi\text{-per})}^2[-\pi, \pi] \longrightarrow L_{\mathbb{C}, (2\pi\text{-per})}^2[-\pi, \pi]; \\ & f = f_1 + i \cdot f_2 \longmapsto \left(\frac{m}{m+1}\right)_f(t) \\ & = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \left\{ f_1(\theta) \cdot \operatorname{Re} B_{m_1}^{(1)}(e^{i\theta}, e^{it}) + i \cdot f_2(\theta) \cdot \operatorname{Re} B_{m_2}^{(2)}(e^{i\theta}, e^{it}) \right\} d\theta, \\ & \left(\frac{m}{m+1}\right) : H_{\mathbb{C}}(D) \longrightarrow H_{\mathbb{C}}(D); \\ & u = u_1 + iu_2 \longmapsto \left(\frac{m}{m+1}\right)_u(z) \\ & = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} \left\{ u_1(\zeta) \cdot \operatorname{Re} \left[\frac{B_{m_1}^{(1)}(\zeta, z)}{\zeta i} \right] \right. \\ & \quad \left. + i \cdot u_2(\zeta) \cdot \operatorname{Re} \left[\frac{B_{m_2}^{(2)}(\zeta, z)}{\zeta i} \right] \right\} d\zeta \end{aligned} \quad (3.13)$$

are called *composed Padé-type operators* for $L_{\mathbb{C}}^2$, $L_{\mathbb{C},(2\pi\text{-per})}^2[-\pi,\pi]$, and $H_{\mathbb{C}}(D)$, respectively. Under the assumptions of [Corollary 3.2](#), the integral operator

$$\begin{aligned} & \left(\frac{m}{m+1} \right) : O(D) \longrightarrow O(D); \\ f \mapsto \left(\frac{m}{m+1} \right)_f(z) &= \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} f(\zeta) \cdot \operatorname{Re} \left[\frac{B_m(\zeta, z)}{\zeta i} \right] d\zeta \end{aligned} \quad (3.14)$$

is called a *Padé-type operator* for $O(D)$.

The continuity of these integral operators follows directly from the arguments of [Section 2](#) and leads to the following result.

THEOREM 3.3. *Under the assumptions and notations of [Theorem 3.1](#) and [Corollary 3.2](#),*

- (a) *if the sequence $\{u_n \in L_{\mathbb{C}}^2(C) : n = 0, 1, 2, \dots\}$ converges to $u \in L_{\mathbb{C}}^2(C)$ in the L^2 -norm, then $\lim_{n \rightarrow \infty} (m/(m+1))_{u_n}(z) = (m/(m+1))_u(z)$ in the L^2 -norm;*
- (b) *if the sequence $\{f_n \in L_{\mathbb{C},(2\pi\text{-per})}^2[-\pi,\pi] : n = 0, 1, 2, \dots\}$ converges to $f \in L_{\mathbb{C},(2\pi\text{-per})}^2[-\pi,\pi]$, with respect to the L^2 -norm, then $\lim_{n \rightarrow \infty} (m/(m+1))_{f_n}(t) = (m/(m+1))_f(t)$ in the L^2 -norm;*
- (c) *if the sequence $\{u_n \in H_{\mathbb{C}}(D) : n = 0, 1, 2, \dots\}$ converges to $u \in H_{\mathbb{C}}(D)$ compactly in D , then $\lim_{n \rightarrow \infty} (m/(m+1))_{u_n}(z) = (m/(m+1))_u(z)$ compactly in D ;*
- (d) *if the sequence $\{f_n \in O(D) : n = 0, 1, 2, \dots\}$ converges to $f \in O(D)$ compactly in D , then $\lim_{n \rightarrow \infty} (m/(m+1))_{f_n}(z) = (m/(m+1))_f(z)$ compactly in D .*

Especially, for series of functions, there is an obvious consequence of this theorem.

COROLLARY 3.4. *Under the assumptions of [Theorem 3.1](#) and [Corollary 3.2](#),*

- (a) *if the series of functions $u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z)$ ($a_n \in \mathbb{C}$, $u_n \in L_{\mathbb{C}}^2(C)$) converges in the L^2 -norm, then $(m/(m+1))_u(z) = \sum_{n=0}^{\infty} a_n \cdot (m/(m+1))_{u_n}(z)$ in the L^2 -norm;*
- (b) *if the series of functions $f(t) = \sum_{n=0}^{\infty} a_n \cdot f_n(t)$ (where $a_n \in \mathbb{C}$, $f_n \in L_{\mathbb{C},(2\pi\text{-per})}^2[-\pi,\pi]$) converges in the L^2 -norm, then $(m/(m+1))_f(t) = \sum_{n=0}^{\infty} a_n \cdot (m/(m+1))_{f_n}(t)$ in the L^2 -norm;*
- (c) *if the series of functions $u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z)$ ($a_n \in \mathbb{C}$, $u_n \in H_{\mathbb{C}}(D)$) converges compactly in the disk D , then $(m/(m+1))_u(z) = \sum_{n=0}^{\infty} a_n \cdot (m/(m+1))_{u_n}(z)$ compactly in D ;*

- (d) if the series of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n \cdot f_n(z)$ ($a_n \in \mathbb{C}$, $f_n O(D)$) converges compactly in D , then $(m/(m+1))_f(z) = \sum_{n=0}^{\infty} a_n \cdot (m/(m+1))_{f_n}(z)$ compactly in D .

Remark 3.5. Padé and Padé-type approximants to arbitrary series of functions were first considered by Brezinski in [1, 2].

References

- [1] C. Brezinski, *Rational approximation to formal power series*, J. Approx. Theory **25** (1979), no. 4, 295–317.
- [2] ———, *Padé approximants: old and new*, Yearbook: Surveys of Mathematics 1983, Bibliographisches Institut, Mannheim, 1983, pp. 37–63.
- [3] ———, *Duality in Padé-type approximation*, J. Comput. Appl. Math. **30** (1990), no. 3, 351–357.
- [4] N. J. Daras, *Continuity of distributions and global convergence of Padé-type approximants in Runge domains*, Indian J. Pure Appl. Math. **26** (1995), no. 2, 121–130.
- [5] ———, *Rational approximation to harmonic functions*, Numer. Algorithms **20** (1999), no. 4, 285–301.
- [6] ———, *Padé and Padé-type approximation for 2π -periodic L^p functions*, Acta Appl. Math. **62** (2000), no. 3, 245–343.
- [7] ———, *Composed Padé-type approximation*, J. Comput. Appl. Math. **134** (2001), no. 1-2, 95–112.

Nicholas J. Daras: Department of Mathematics, Hellenic Air Force Academy, Dekeleia Attikis, Greece

Current address: Jean Moreas 19, 152 32 Chalandri, Athens, Greece

E-mail address: njdaras@myflash.gr