BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS FOR A LINEAR THIRD-ORDER EQUATION

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We prove the existence and uniqueness of a strong solution for a linear third-order equation with integral boundary conditions. The proof uses energy inequalities and the density of the range of the generated operator.

1. Introduction

In the rectangle $\Omega = [0,1] \times [0,T]$, we consider the equation

$$\mathcal{L}u = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right) = f(x,t),$$  \hspace{1cm} (1.1a)

with the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1),$$  \hspace{1cm} (1.1b)

the final condition

$$\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1),$$  \hspace{1cm} (1.1c)

the Dirichlet condition

$$u(0,t) = 0, \quad \forall t \in (0,T),$$  \hspace{1cm} (1.1d)
and the integral condition

$$\int_0^1 u(x,t) \, dx = 0, \quad \forall t \in (0,T). \quad (1.1e)$$

In addition, we assume that the function $a(x,t)$ is bounded with

$$0 < a_0 \leq a(x,t) \leq a_1,$$

and has bounded partial derivatives such that

$$c'_k \leq \frac{\partial^k a}{\partial t^k}(x,t) \leq c_k, \quad \forall x \in (0,1), \ t \in (0,T), \ k = 1,3, \text{ with } c'_1 \geq 0,$$

$$\left| \frac{\partial a}{\partial x}(x,t) \right| \leq b_1, \quad \text{for } (x,t) \in \Omega. \quad (1.3)$$

Various problems arising in heat conduction [4, 6, 14, 15], chemical engineering [9], underground water flow [13], thermoelasticity [21], and plasmaphysics [19] can be reduced to the nonlocal problems with integral boundary conditions. This type of boundary value problems has been investigated in [1, 2, 3, 5, 6, 7, 9, 14, 15, 16, 20, 23] for parabolic equations, in [18, 22] for hyperbolic equations, and in [10, 11, 12] for mixed-type equations. The basic tool in [4, 10, 11, 12, 16, 23] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation. This type of problems is encountered in the study of thermal conductivity [17] and microscale heat transfer [8].

2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of problem (1.1). For this, we consider the solution of problem (1.1) as a solution of the operator equation $Lu = \mathcal{F}$, where $L$ is the operator with domain of definition $D(L)$ consisting of functions $u \in E$ such that

$$\sqrt{1-x} (\partial^{k+1} u / \partial t \partial x)(x,t) \in L^2(\Omega), \ k = 0,3 \text{ and } u \text{ satisfies conditions (1.1d) and (1.1e).}$$

The operator $L$ is considered from $E$ to $F$, where $E$ is the Banach space of the functions $u, \ u \in L^2(\Omega)$, with the finite norm

$$\|u\|^2_L = \int_\Omega \left( \frac{(1-x)^2}{2} \left( \left\| \frac{\partial^3 u}{\partial t^3} \right\|^2 + \left\| \frac{\partial^2 u}{\partial x^2} \right\|^2 \right) + \int_\Omega \left( \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) \, dx \, dt, \quad (2.1)$$
\[ \mathcal{F} = (f, 0, 0, 0), \ f \in L^2(\Omega), \ \text{with the finite norm} \]
\[ \| \mathcal{F} \|^2_{\mathcal{F}} = \int_{\Omega} (1-x)^2 |f|^2 \, dx \, dt. \quad (2.2) \]

Then we establish an energy inequality
\[ \| u \|_E \leq k \| Lu \|_{\mathcal{F}}, \ \forall u \in D(L), \quad (2.3) \]
and we show that the operator \( L \) has the closure \( \overline{L} \).

**Definition 2.1.** A solution of the operator equation \( \overline{L}u = \mathcal{F} \) is called a strong solution of problem (1.1).

Inequality (2.3) can be extended to \( u \in D(\overline{L}) \), that is,
\[ \| u \|_E \leq k \| \overline{L}u \|_{\mathcal{F}}, \ \forall u \in D(\overline{L}). \quad (2.4) \]

From this inequality, we obtain the uniqueness of a strong solution, if it exists, and the equality of the sets \( R(\overline{L}) \) and \( R(L) \). Thus, to prove the existence of a strong solution of problem (1.1) for any \( \mathcal{F} \in F \), it remains to prove that the set \( R(L) \) is dense in \( F \).

### 3. An energy inequality and its applications

**Theorem 3.1.** For any function \( u \in D(L) \), there exists the a priori estimate
\[ \| u \|_E \leq k \| Lu \|_{\mathcal{F}}, \quad (3.1) \]
where
\[ k^2 = \frac{17 \exp(ct)[5 + 4(b_1)^2/(c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2)] + 1}{\min (1, a_0^2c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2)}, \quad (3.2) \]
with the constant \( c \) satisfying
\[ \sup_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} \right) \leq c < \inf_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} + 1 \right), \quad (3.3) \]
\[ c_3 - 3cc_2 + 3c^2c_1' - c^3a_1 - (b_1)^2 > 0, \]
\[ c_2 - 2cc_1' + c^2a_1 - c_1' + ca_1 < 0. \]
Proof. Let
\[ Mu = (1-x)^2 \frac{\partial^3 u}{\partial t^3} + 2(1-x)J_x \frac{\partial^3 u}{\partial t^3}, \]  
where
\[ J_x u = \int_0^x u(\zeta, t) d\zeta. \]

We consider the quadratic form
\[ \Phi(u,u) = \text{Re} \int_\Omega \exp(-ct)\bar{u}Mu \, dx \, dt, \]  
with the constant \( c \) satisfying (3.3), obtained by multiplying (1.1a) by \( \exp(-ct)\bar{u}u \), integrating over \( \Omega \), and taking the real part. Substituting the expression of \( Mu \) in (3.6), we obtain
\[ \text{Re} \int_\Omega \exp(-ct)\bar{u}Mu \, dx \, dt \]
\[ = \text{Re} \int_\Omega \exp(-ct)(1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt \]
\[ + 2 \text{Re} \int_\Omega \exp(-ct)(1-x) \frac{\partial^3 u}{\partial t^3} J_x \frac{\partial^3 u}{\partial t^3} \, dx \, dt \]
\[ + \text{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right) \bar{u}Mu \, dx \, dt. \]

Integrating the last two terms on the right-hand side by parts with respect to \( x \) in (3.7) and using the Dirichlet condition (1.1d), we obtain
\[ 2 \text{Re} \int_0^1 (1-x) \exp(-ct) \frac{\partial^3 u}{\partial t^3} J_x \frac{\partial^3 \bar{u}}{\partial t^3} \, dx = \int_0^1 \exp(-ct) \left| J_x \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx, \]
\[ \text{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right) \bar{u}Mu \, dx \, dt \]
\[ = - \text{Re} \int_\Omega \exp(-ct)(1-x)^2 a \frac{\partial u}{\partial x} \frac{\partial^4 \bar{u}}{\partial x^4} \, dx \, dt \]
\[ - 2 \text{Re} \int_\Omega \exp(-ct) \frac{\partial a}{\partial x} J_x \frac{\partial^3 \bar{u}}{\partial t^3} \, dx \, dt \]
\[ - 2 \text{Re} \int_\Omega \exp(-ct) au \frac{\partial^3 \bar{u}}{\partial t^3} \, dx \, dt. \]
Integrating each term by parts in (3.9) with respect to $t$ and using the initial and final conditions (1.1b) and (1.1c), we get

\[
\text{Re} \int_{\Omega} \exp(-ct) \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) M u \, dx \, dt
\]

\[
= -2 \text{Re} \int_{\Omega} \exp(-ct) \frac{\partial a}{\partial x} u J_x \frac{\partial^3 u}{\partial t^3} \, dx \, dt
\]

\[
+ \int_{\Omega} \exp(-ct) \left( \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^3 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right)
\]

\[
\times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \, dx \, dt
\]

\[
- 3 \int_{\Omega} \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left[ \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] \, dx \, dt
\]

\[
+ \int_0^1 \exp(-ct) a \left[ \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] \bigg|_{T=t} \, dx
\]

\[
- \int_0^1 \exp(-ct) \left( \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a \right) \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \bigg|_{T=t} \, dx
\]

\[
+ \text{Re} \int_0^1 \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left\{ (1-x)^2 \frac{\partial^2 u}{\partial t \partial x} \frac{\partial u}{\partial x} + 2u \frac{\partial u}{\partial t} \right\} \bigg|_{T=t} \, dx.
\]

\[(3.10)\]

Substituting (3.8) and (3.10) in (3.7) and using conditions (1.2), (1.3), and (3.3), we obtain

\[
\int_{\Omega} \exp(-ct) (1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt
\]

\[
+ \int_{\Omega} \exp(-ct) \{ c_3 - 3cc_2 + 3c^2 c_1' - c^3 a_1 - b_1^2 \}
\]

\[
\times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \, dx \, dt
\]

\[
\leq \text{Re} \int_{\Omega} \exp(-ct) \varepsilon u M u \, dx \, dt.
\]

\[(3.11)\]

Again, substituting the expression of $Mu$ in (3.11) and using elementary inequality, we get
Boundary value problem with integral conditions

\[ \int_{\Omega} \exp(-ct) \left( \frac{(1-x)^2}{2} \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right) dx dt + \int_{\Omega} \exp(-ct) \left( c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2 \right) \times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] dx dt \leq 17 \int_{\Omega} \exp(-ct)(1-x)^2|f|^2 dx dt. \] (3.12)

By virtue of (1.1a), we have

\[ \int_{\Omega} a_0 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \frac{(1-x)^2}{2} dx dt \leq \int_{\Omega} (1-x)^2|f|^2 dx dt + \int_{\Omega} 2(1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx dt + 4 \int_{\Omega} b_1^2 \left\{ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right\} dx dt. \] (3.13)

This last inequality combined with (3.12) yields

\[ \int_{\Omega} \frac{(1-x)^2}{2} \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx dt + \int_{\Omega} \left( c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2 \right) \left\{ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right\} dx dt + \int_{\Omega} a_0 \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial x^2} \right|^2 dx dt \leq \left\{ 17 \exp(cT) \left[ 5 + \frac{4b_1^2}{c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2} \right] + 1 \right\} \times \int_{\Omega} (1-x)^2|f|^2 dx dt. \] (3.14)

Thus, this inequality implies

\[ \int_{\Omega} \frac{(1-x)^2}{2} \left\{ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right\} dx dt + \int_{\Omega} \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 dx dt \leq k^2 \int_{\Omega} (1-x)^2|f|^2 dx dt, \] (3.15)
where
\[ k^2 = \frac{17 \exp(cT) \left[ 5 + 4b_1^2 \right] / \left( c_3 - 3c_2 + 3c_1 c_2' - c_3 a_1 - b_1^2 \right) }{ \min \left( 1, a_0^2, c_3 - 3c_2 + 3c_1 c_2' - c_3 a_1 - b_1^2 \right) } + 1. \] (3.16)

Then,
\[ \|u\|_E \leq k\|Lu\|_F, \quad \forall u \in D(L). \] (3.17)

Thus, we obtain the desired inequality. □

**Lemma 3.2.** The operator $L$ from $E$ to $F$ admits a closure.

**Proof.** Suppose that $(u_n) \in D(L)$ is a sequence such that
\[ u_n \to 0 \quad \text{in} \ E, \quad Lu_n \to \emptyset \quad \text{in} \ F. \] (3.18)

We need to show that $\emptyset = 0$. We introduce the operator
\[ \mathcal{E}_0 v = -(1-x)^2 \frac{\partial^3 v}{\partial t^3} + \frac{\partial}{\partial x} \left\{ a(x,t) \frac{\partial}{\partial x} \left[ (1-x)^2 v \right] \right\}, \] (3.19)

with domain $D(\mathcal{E}_0)$ consisting of functions $v \in W^{2,3}_2(\Omega)$ satisfying
\[ v|_{t=0} = 0, \quad \frac{\partial v}{\partial t} \bigg|_{t=0} = 0, \quad \frac{\partial^2 v}{\partial t^2} \bigg|_{t=0} = 0, \quad v|_{x=0} = 0, \quad \frac{\partial v}{\partial x} \bigg|_{x=0} = 0. \] (3.20)

We note that $D(\mathcal{E}_0)$ is dense in the Hilbert space obtained by completing $L^2(\Omega)$ with respect to the norm
\[ \int_\Omega (1-x)^2 |v|^2 \, dx \, dt = \|v\|^2. \] (3.21)

Since
\[ \int_\Omega (1-x)^2 f \bar{v} \, dx \, dt = \lim_{n \to +\infty} \int_\Omega (1-x)^2 \mathcal{E} u_n \bar{v} \, dx \, dt \]
\[ = \lim_{n \to +\infty} \int_\Omega u_n \mathcal{E}_0 \bar{v} \, dx \, dt = 0, \] (3.22)

for any function $v \in D(\mathcal{E}_0)$, it follows that $f = 0$. □
Theorem 3.1 is valid for a strong solution, then we have the inequality

$$\|u\|_E \leq k\|Lu\|_F, \quad \forall u \in D(L).$$

(3.23)

Hence we obtain the following corollary.

**Corollary 3.3.** A strong solution of problem (1.1) is unique if it exists, and depends continuously on $\varphi$.

**Corollary 3.4.** The range $R(L)$ of the operator $L$ is closed in $F$, and $R(L) = R(L)$.

4. Solvability of problem (1.1)

To prove the solvability of problem (1.1), it is sufficient to show that $R(L)$ is dense in $F$. The proof is based on the following lemma.

**Lemma 4.1.** Suppose that $a(x,t)$ and its derivatives $\partial^4 a/\partial t^3 \partial x$ and $\partial^3 a/\partial t \partial x$ are bounded. Let $D_0(L) = \{u \in D(L) : u(x,0) = 0, (\partial u/\partial t)(x,0) = 0, (\partial^2 u/\partial t^2)(x,T) = 0\}$. If, for $u \in D_0(L)$ and for some functions $w \in L^2(\Omega)$,

$$\int_\Omega (1-x)Lu \overline{w} \, dx \, dt = 0,$$

(4.1)

then $w = 0$.

**Proof.** Equality (4.1) can be written as follows:

$$\int_\Omega (1-x)\overline{w} \frac{\partial^3 u}{\partial t^3} \, dx \, dt = -\int_\Omega \frac{\partial}{\partial x} \left( a(1-x) \frac{\partial u}{\partial x} \right) \left\{ \overline{w} - \int_0^x \frac{\overline{w}(\xi)}{1-\xi} \, d\xi \right\} \, dx \, dt.$$

(4.2)

For a given $w(x,t)$, we introduce the function $v(x,t)$ such that

$$v(x,t) = w(x,t) - \int_0^x \frac{w(\xi,t)}{1-\xi} \, d\xi.$$

(4.3)

From (4.3), we conclude that $\int_0^1 v(x,t) \, dx = 0$, and thus, we have

$$\int_\Omega \frac{\partial^3 u}{\partial t^3} \frac{\partial v}{\partial t} \, dx \, dt = -\int_\Omega A(t)u \overline{v} \, dx \, dt,$$

(4.4)

where $A(t)u = (\partial/\partial x)(a(1-x)(\partial u/\partial x))$ and $Nv = (1-x)v + Jv$. 

Following [23], we introduce the smoothing operators

\[ J^{-1}_\epsilon = \left( I - \epsilon \left( \frac{\partial^3}{\partial t^3} \right) \right)^{-1}, \quad (J^{-1}_\epsilon)^* = \left( I + \epsilon \left( \frac{\partial^3}{\partial t^3} \right) \right)^{-1}, \]  

(4.5)

with respect to \( t \), which provide the solutions of the respective problems

\[ g - \epsilon \frac{\partial^3 g}{\partial t^3} = g, \quad g(0) = 0, \quad \frac{\partial g}{\partial t} (0) = 0, \quad \frac{\partial^2 g}{\partial t^2} (T) = 0, \]

\[ g^* + \epsilon \frac{\partial^3 g^*}{\partial t^3} = g, \quad g^*(0) = 0, \quad \frac{\partial g^*}{\partial t} (T) = 0, \quad \frac{\partial^2 g^*}{\partial t^2} (T) = 0. \]  

(4.6)

We also have the following properties: for any \( g \in L^2(0,T) \), the functions \( J^{-1}_\epsilon (g) \), \( (J^{-1}_\epsilon)^* g \in W^2(0,T) \). If \( g \in D(L) \), then \( J^{-1}_\epsilon (g) \in D(L) \) and we have

\[ \lim \| (J^{-1}_\epsilon)^* g - g \|_{L^2[0,T]} = 0 \quad \text{for} \quad \epsilon \to 0, \]

\[ \lim \| (J^{-1}_\epsilon) g - g \|_{L^2[0,T]} = 0 \quad \text{for} \quad \epsilon \to 0. \]  

(4.7)

Substituting the function \( u \) in (4.4) by the smoothing function \( u_\epsilon \) and using the relation

\[ A(t)u_\epsilon = J^{-1}_\epsilon Au - \epsilon J^{-1}_\epsilon \beta_\epsilon(t)u_\epsilon, \]  

(4.8)

where

\[ \beta_\epsilon(t)u_\epsilon = 3 \frac{\partial^2 A(t)}{\partial t^2} \frac{\partial u_\epsilon}{\partial t} + 3 \frac{\partial A(t)}{\partial t} \frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial^3 A(t)}{\partial t^3} u_\epsilon, \]  

(4.9)

we obtain

\[ - \int_\Omega uN \frac{\partial^3 \overline{v_\epsilon}}{\partial \overline{v_\epsilon}} d\Gamma dt = \int_\Omega A(t)u \overline{v_\epsilon} dx dt - \epsilon \int_\Omega \beta_\epsilon(t)u_\epsilon \overline{v_\epsilon} dx dt. \]  

(4.10)

Passing to the limit, the equality in the relation (4.10) remains true for all functions \( u \in L^2(\Omega) \) such that \((1 - x)(\partial u/\partial x), (\partial/\partial x)((1 - x)(\partial u/\partial x)) \in L^2(\Omega)\), and satisfying condition (1.1d).
The operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$A^{-1}(t)g = -\int_0^x \frac{1}{1-\zeta} \frac{1}{a(\zeta,t)} \int_0^\zeta g(\eta,t) d\eta d\zeta$$

$$+ C(t) \int_0^x \frac{1}{1-\zeta} \frac{1}{a(\zeta,t)} d\zeta,$$

where

$$C(t) = \frac{\int_0^1 (d\zeta / a(\zeta,t)) \int_0^\zeta g(\eta,t) d\eta}{\int_0^1 (d\zeta / a(\zeta,t))}.$$  \hfill (4.11)

Then, we have $\int_0^1 A^{-1}(t)g dx = 0$, hence the function $u_\epsilon = (J_\epsilon)^{-1}u$ can be represented in the form

$$u_\epsilon = (J_\epsilon)^{-1} A^{-1}(t) A(t)u.$$  \hfill (4.13)

Then

$$B_\epsilon(t)g = \frac{\partial^4 a}{\partial t^3 \partial x} J_\epsilon^{-1} \left[ \frac{1}{a(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right) \right]$$

$$+ \frac{\partial^3 a}{\partial t^3} J_\epsilon^{-1} \left[ \frac{g}{a} - \frac{a_x}{a^2(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right) \right]$$

$$+ 3 \frac{\partial}{\partial t} \frac{\partial^2 a}{\partial t^2 \partial x} \frac{\partial}{\partial t} J_\epsilon^{-1} \frac{1}{a(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right)$$

$$+ \frac{\partial a}{\partial t} \frac{\partial}{\partial t} J_\epsilon^{-1} \frac{g}{a} - \frac{a_x}{a^2(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right).$$  \hfill (4.14)

The adjoint of $B_\epsilon(t)$ has the form

$$B_\epsilon^*(t) = \frac{1}{a} (J_\epsilon^{-1})^* \left[ \frac{\partial^3 a}{\partial t^3} \right] + 3 \frac{a}{a} (J_\epsilon^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} \frac{\partial h}{\partial t} \right)$$

$$+ (G_\epsilon h)(x) - \frac{\int_0^x (1/a(\eta,t)) d\eta}{\int_0^1 (1/a(x,t)) dx} (G_\epsilon h)(1),$$  \hfill (4.15)
where

\[
(G_{\epsilon}h)(x) = \int_{0}^{x} \left( -\frac{3}{a(\zeta,t)}(J_{\epsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial^2 h}{\partial t \partial \zeta} \right) \right. \\
+ \frac{3}{\partial \zeta} \frac{1}{a^2(\zeta,t)}(J_{\epsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a \partial h}{\partial t} \right) \left. \right) d\zeta.
\]

(4.16)

Consequently, equality (4.10) becomes

\[
-\int_{\Omega} u N \frac{\partial^3 v_{\epsilon}^*}{\partial x^3} dx dt = \int_{\Omega} A(t) u h_{\epsilon} dx dt,
\]

(4.17)

where \( h_{\epsilon} = v_{\epsilon}^* - \epsilon B_{\epsilon}^* v_{\epsilon}^* \).

The left-hand side of (4.17) is a continuous linear functional of \( u \). Hence the function \( h_{\epsilon} \) has the derivatives \( (1-x)(\partial h_{\epsilon}/\partial x) \), \( (\partial/\partial x)((1-x)(\partial h_{\epsilon}/\partial x)) \) \( \in L^2(\Omega) \) and the following conditions are satisfied: \( h_{\epsilon}|_{x=0} = 0 \), \( h_{\epsilon}|_{x=1} = 0 \), and \( (1-x)(\partial h_{\epsilon}/\partial x)|_{x=1}=0 \).

From the equality

\[
(1-x)(\partial h_{\epsilon}/\partial x) = \left[ I - \frac{1}{a}(J_{\epsilon}^{-1})^* \frac{\partial^3 a}{\partial x^3} \right] (1-x)(\partial v_{\epsilon}^*/\partial x) \left. \right) d\zeta,
\]

(4.18)

and since the operator \( (J_{\epsilon}^{-1})^* \) is bounded in \( L^2(\Omega) \), for sufficiently small \( \epsilon \), we have \( \|\epsilon(1/a)(J_{\epsilon}^{-1})^* \frac{\partial^3 a}{\partial x^3}\| < 1 \). Hence the operator \( I - \epsilon(1/a)(J_{\epsilon}^{-1})^* \frac{\partial^3 a}{\partial x^3} \) has a bounded inverse in \( L^2(\Omega) \). We conclude that \( (1-x)(\partial v_{\epsilon}^*/\partial x) \) \( \in L^2(\Omega) \).

Similarly, we conclude that \( (\partial/\partial x)((1-x)(\partial v_{\epsilon}^*/\partial x)) \) exists and belongs to \( L^2(\Omega) \), and the following conditions are satisfied:

\[
\left. v_{\epsilon}^*/\partial x \right|_{x=0} = 0, \quad \left. v_{\epsilon}^*/\partial x \right|_{x=1} = 0, \quad (1-x)(\partial v_{\epsilon}^*/\partial x)|_{x=1} = 0.
\]

(4.19)

Substituting \( u = \int_{0}^{t} \int_{\Omega} \int_{\zeta} \exp(ct)\nu_{\epsilon}^*(\tau) d\tau d\zeta d\eta \) in (4.4), where the constant \( c \) satisfies (3.3), we obtain

\[
\int_{\Omega} \exp(ct)\nu_{\epsilon}^* N\nu d\tau = -\int_{\Omega} A(t) u \nu d\tau dt.
\]

(4.20)
Using the properties of smoothing operators, we have

\[
\int_{\Omega} \exp(\epsilon t)v_{\epsilon}^* \bar{N} \overline{v}_n dx \ dt = - \int_{\Omega} A(t)u \overline{v}_n^* dx \ dt - \epsilon \int_{\Omega} A(t)u \frac{\partial^3 \overline{v}_n^*}{\partial t^3} dx \ dt,
\]

(4.21)

and from

\[
\epsilon \text{Re} \int_{\Omega} A(t)u \frac{\partial^3 \overline{v}_n^*}{\partial t^3} dx \ dt = \epsilon \int_{\Omega} (1-x) a \frac{\partial u}{\partial x} \frac{\partial}{\partial x} \frac{\partial^3 \overline{v}_n^*}{\partial t^3} dx \ dt
\]

\[
= - \epsilon \text{Re} \int_{\Omega} (1-x) \frac{\partial a}{\partial t} \frac{\partial u}{\partial x} \frac{\partial^2 \overline{v}_n^*}{\partial t^2} \frac{\partial}{\partial x} dx \ dt
\]

\[
+ \epsilon \text{Re} \int_{\Omega} (1-x) \frac{\partial a}{\partial t} \frac{\partial^2 u}{\partial t \partial x} \frac{\partial \overline{v}_n^*}{\partial t} \frac{\partial}{\partial x} dx \ dt
\]

\[
+ \epsilon \int_{\Omega} a \exp(-\epsilon t)(1-x) \left| \frac{\partial \overline{v}_n^*}{\partial x} \right|^2 dx \ dt
\]

\[
+ \epsilon \text{Re} \int_{\Omega} (1-x) \frac{\partial a}{\partial t} \frac{\partial^2 u}{\partial t \partial x} \frac{\partial \overline{v}_n^*}{\partial t} dx \ dt,
\]

(4.22)

we have

\[
\epsilon \text{Re} \int_{\Omega} A(t)u \frac{\partial^3 \overline{v}_n^*}{\partial t^3} dx \ dt
\]

\[
\geq \epsilon \int_{\Omega} a \exp(+\epsilon t)(1-x) \left| \frac{\partial \overline{v}_n^*}{\partial x} \right|^2 dx \ dt
\]

\[
- \epsilon \int_{\Omega} (1-x) \frac{1}{4a} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-\epsilon t) \left| \frac{\partial^3 u}{\partial t^3 \partial x} \right|^2 dx \ dt
\]

\[
- \epsilon \int_{\Omega} a \exp(+\epsilon t)(1-x) \left| \frac{\partial \overline{v}_n^*}{\partial x} \right|^2 dx \ dt
\]

\[
- \epsilon \int_{\Omega} \frac{1-x}{2} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-\epsilon t) \left| \frac{\partial u}{\partial x} \right|^2 dx \ dt
\]

\[
- \epsilon \int_{\Omega} \exp(+\epsilon t) \frac{1-x}{2} \left| \frac{\partial^2 \overline{v}_n^*}{\partial t^2 \partial x} \right|^2 dx \ dt
\]

\[
- \epsilon \int_{\Omega} \exp(+\epsilon t) \frac{1}{2} \left| \frac{\partial^2 \overline{v}_n^*}{\partial t \partial x} \right|^2 dx \ dt
\]

\[
- \epsilon \int_{\Omega} \frac{1-x}{2} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-\epsilon t) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx \ dt.
\]

(4.23)
Integrating the first term on the right-hand side by parts in (4.21), we obtain

\[
\text{Re} \int_{\Omega} A(t)u \bar{v}_t^e dx \, dt \\
\geq -\frac{3}{2} \int_{\Omega} (1-x) \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \frac{\partial^2 \overline{u}}{\partial t \partial x}^2 \, dx \, dt \\
+ \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left( a - \left| \frac{\partial a}{\partial t} - ca \right| \right) \left| \frac{\partial^2 \overline{u}}{\partial t \partial x} \right|^2 \left| \frac{\partial \overline{u}}{\partial x} \right| \left| \frac{\partial a}{\partial t} - ca \right| \bigg|_{t=T} \, dx \\
- \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left\{ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + \left| \frac{\partial a}{\partial t} - ca \right| \right\} \left| \frac{\partial \overline{u}}{\partial x} \right|^2 \left| \frac{\partial a}{\partial t} - ca \right| \bigg|_{t=T} \, dx \\
+ \frac{1}{2} \int_{\Omega} (1-x) \exp(-ct) \left\{ \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right\} \left| \frac{\partial \overline{u}}{\partial x} \right|^2 \, dx \, dt.
\]

Combining (4.23) and (4.24), we get

\[
\text{Re} \int_{\Omega} \exp(ct)v_t^e N \overline{v} \, dx \, dt \\
\leq -\frac{3}{2} \int_{\Omega} (1-x) \exp(-ct) \left( c_1 - ca \right) \left| \frac{\partial^2 \overline{u}}{\partial t \partial x} \right|^2 \, dx \, dt \\
- \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left\{ a_0 - c_1' - ca \right\} \left| \frac{\partial^2 \overline{u}}{\partial t \partial x} \right|^2 \bigg|_{t=T} \, dx \\
+ \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left\{ c_2 - 2c_1' c - c^2 a_1 - c_1' + ca \right\} \left| \frac{\partial \overline{u}}{\partial x} \right|^2 \bigg|_{t=T} \, dx \\
- \frac{1}{2} \int_{\Omega} (1-x) \exp(-ct) \left\{ c_3 - 3c_2 c + 3c^2 c_1' - c^3 a \right\} \left| \frac{\partial \overline{u}}{\partial x} \right|^2 \, dx \, dt \\
+ \varepsilon \left( \int_{\Omega} (1-x) \exp(-ct) \frac{c_1^2}{4a_0} \left| \frac{\partial^3 \overline{u}}{\partial t \partial x} \right|^2 \, dx \, dt \right) \\
+ \int_{\Omega} (1-x) \exp(-ct) \frac{c_1^2}{2} \left| \frac{\partial \overline{u}}{\partial x} \right|^2 \, dx \, dt \\
+ \int_{\Omega} \frac{1-x}{2} \exp(ct) \left| \frac{\partial^3 \overline{v}_t^e}{\partial t^2 \partial x} \right|^2 \, dx \, dt \\
+ \int_{\Omega} (1-x) \exp(-ct) \frac{c_1^2}{2} \left| \frac{\partial^2 \overline{u}}{\partial t \partial x} \right|^2 \, dx \, dt \\
+ \int_{\Omega} \frac{1-x}{2} \exp(ct) \left| \frac{\partial^3 \overline{v}_t^e}{\partial t \partial x} \right|^2 \, dx \, dt \right).
\]

(4.25)
Boundary value problem with integral conditions

Using conditions (3.3) and inequalities (4.23) and (4.24), we obtain

\[ \text{Re} \int_{\Omega} \exp(ct)v N \overline{v} \, dx \, dt \leq 0, \quad \text{as } \varepsilon \rightarrow 0. \]  

(4.26)

Since \( \text{Re} \int_{\Omega} \exp(ct)v J_x \overline{v} \, dx \, dt = 0 \), then \( v = 0 \) a.e.

Finally, from the equality \((1 - x)v + J_x v = (1 - x)w\), we conclude \( w = 0 \).

\[ \square \]

**Theorem 4.2.** The range \( R(\overline{L}) \) of \( \overline{L} \) coincides with \( F \).

**Proof.** Since \( F \) is Hilbert space, then \( R(\overline{L}) = F \) if and only if the relation

\[ \int_{\Omega} (1 - x)^2 \mathcal{E} u f \, dx \, dt = 0, \]  

(4.27)

for arbitrary \( u \in D_0(L) \) and \( \mathfrak{F} \in F \), implies that \( f = 0 \).

Taking \( u \in D_0(L) \) in (4.27) and using Lemma 4.1, we obtain that \( w = (1 - x)f = 0 \), then \( f = 0 \).

\[ \square \]

**References**


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