ON REPRESENTATIONS OF LIE ALGEBRAS OF A GENERALIZED TAVIS-CUMMINGS MODEL

L. A. M. HANNA

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Consider the Lie algebras $L_{r,t}^s: [K_1, K_2] = sK_3$, $[K_3, K_1] = rK_1$, $[K_3, K_2] = -rK_2$, $[K_3, K_4] = 0$, $[K_4, K_1] = -tK_1$, and $[K_4, K_2] = tK_2$, subject to the physical conditions, $K_3$ and $K_4$ are real diagonal operators representing energy, $K_2 = K_1^\dagger$, and the Hamiltonian $H = \omega_1 K_3 + (\omega_1 + \omega_2) K_4 + \lambda(t)(K_1 e^{-i\phi} + K_2 e^{i\phi})$ is a Hermitian operator. Matrix representations are discussed and faithful representations of least degree for $L_{r,t}^s$ satisfying the physical requirements are given for appropriate values of $r, s, t \in \mathbb{R}$.

1. Introduction

Introducing an algebraic method to solve certain types of linear partial differential equations, Steinberg [6] exploited the Lie-algebraic decomposition formulas of Baker, Campbell, Hausdorff, and Zassenhaus (cf. [7]) and their matrix realization. A faithful matrix representation of low degree is required. In [2, 3, 4], the faithful matrix representations of least degree were discussed for the Lie algebra $L_{r}^s$ generated by $K_+^\dagger$, and $K_0$ satisfying the commutation relations: $[K_0, K_\pm] = \pm r K_\pm$ and $[K_+, K_-] = sK_0$ subject to the physical properties $K_- = K_0^\dagger$ († for Hermitian conjugation), $K_0$ is a real diagonal operator, and $(K_+ + K_-)$ is real. The Lie algebra $L_{r}^s$ was introduced as a generalization of the coupled quantized harmonic oscillators [5] namely, the model of light amplifier $L_{1}^{-2}$, and the model of two-level optical atom $L_{1}^{2}$, whose Hamiltonian model $H = K_0 + \lambda (K_+ + K_-)$, $\lambda$ is the coupling parameter. Note that, $L_{1}^{2}$ is exactly the Lie algebra $\mathfrak{sl}(2)$.

In this paper, $L_{r,t}^s$ is considered to be the Lie algebra generated by $K_1$, $K_2$, $K_3$, and $K_4$, satisfying the commutation relations: $[K_1, K_2] = sK_3$, $[K_3, K_1] = rK_1$, $[K_3, K_2] = -rK_2$, $[K_3, K_4] = 0$, $[K_4, K_1] = -tK_1$, and $[K_4, K_2] = tK_2$, subject to the physical conditions, $K_3$ and $K_4$ are real diagonal operators representing energy, $K_2 = K_1^\dagger$, and the Hamiltonian $H = \omega_1 K_3 + (\omega_1 + \omega_2) K_4 + \lambda(t)(K_1 e^{-i\phi} + K_2 e^{i\phi})$ is a Hermitian operator. Matrix representations are discussed and faithful representations of least degree for $L_{r,t}^s$ satisfying the physical requirements are given for appropriate values of $r, s, t \in \mathbb{R}$.
[Lemma 1.1] The Lie algebra $L_{r,t}^s$ can be defined by

\[ [K_1, K_2] = sK_3, \quad [K_3, K_1] = rK_1, \quad [K_4, K_1] = -tK_1, \]  

(1.1)

where $K_3$ and $K_4$ are real diagonal operators and $K_2 = K_1^\dagger$.

**Proof.** Indeed $-rK_2 = -(rK_1)^\dagger = -[K_3, K_1]^\dagger = [K_3, K_2]$ and similarly, for the relation $[K_4, K_2] = tK_2$. Since $K_3$ and $K_4$ are diagonal, they commute. The Hermiticity of the Hamiltonian follows since $\omega_1, \omega_2, \lambda(t) \in \mathbb{R}$. \( \square \)

As a necessity of Lemma 1.1 we have the following lemma.

**Lemma 2.2.** The matrices $A, B, C,$ and $D$ satisfy the following:

(i) $[A, B^T]$ is a symmetric matrix,

(ii) $[A, A^T] + [B, B^T] = sC$,

(iii) $[C, A] = rA, \ [C, B] = rB$,

(iv) $[D, A] = -tA, \ [D, B] = -tB$.

**Lemma 1.3.** Let $L, M,$ and $K$ be $n \times n$ matrices such that $[L, M] = aK, \ a \neq 0$, then $\text{trace}(K) = 0$.

**Lemma 1.4.** Let $p, q \in \mathbb{N},$ and $\sigma = (pq)$ be a transposition. The representation obtained by applying $\sigma$ to the rows as well as to the columns of $X, Y, C,$ and $D$ is a conjugate representation for $L_{r,t}^s$ and satisfies the physical requirements.
Proof. Let $P$ be the elementary matrix obtained by applying $\sigma$ to the rows of $I_n$. Since $P = P^{-1} = P^T = P^\dagger$, then the proof of the lemma follows. □

Since $[C, X] = rX$, then for all $i, j \in \mathbb{N}$ we have,

$$a_{ij} (c_{ii} - c_{jj} - r) = 0, \quad b_{ij} (c_{ii} - c_{jj} - r) = 0. \quad (1.2)$$

Similarly, from Lemma 1.2(iv),

$$a_{ij} (d_{ii} - d_{jj} + t) = 0, \quad b_{ij} (d_{ii} - d_{jj} + t) = 0. \quad (1.3)$$

If $x_{ij} \neq 0$, then from (1.2) and (1.3)

$$c_{ii} - c_{jj} = r, \quad d_{jj} - d_{ii} = t. \quad (1.4)$$

Since $[X, Y] = sC$, then for each $i \in \mathbb{N}$ we have,

$$sc_{ii} = \sum_{l=1}^{n} \left( |x_{il}|^2 - |x_{li}|^2 \right) = \sum_{l=1}^{n} \left( a_{il}^2 - a_{li}^2 + b_{il}^2 - b_{li}^2 \right). \quad (1.5)$$

**Lemma 1.5.** If $t^2 + r^2 \neq 0$, then

1. $x_{ii} = 0$, for all $i \in \mathbb{N}$,
2. if $x_{ij} \neq 0$ then $x_{ji} = 0$, for all $i, j \in \mathbb{N}$.

**Proof.** If $r \neq 0$, then from (1.2) we have, for each $i \in \mathbb{N}$, that $x_{ii} = 0$. Also, if $x_{ij} \neq 0$, then $c_{jj} - c_{ii} - r = -2r$, thus $x_{ji} = 0$. Similarly, when $t \neq 0$. □

**Lemma 1.6.** If $s \neq 0$, then

1. trace($C$) = 0,
2. if $x_{ij} \neq 0$ then, for $i, j \in \mathbb{N}$

$$r = \frac{1}{s} \left( \sum_{l=1}^{n} \left( |x_{il}|^2 - |x_{li}|^2 - |x_{jl}|^2 + |x_{lj}|^2 \right) \right). \quad (1.6)$$

**Proof.** Since $[X, Y] = sC$ then from Lemma 1.3, trace($C$) = 0. The proof of (2), follows from (1.4) and (1.5). □

We build the representation matrices starting with $C$.

**Remark 1.7.** Using Lemma 1.4, $C$ can be rearranged into $k$ diagonal blocks, the $i$th diagonal block consists of the $k_i$ scalar matrices, $\{c_i I_{m_{i0}}, (c_i - r) I_{m_{i1}}, \ldots, (c_i - r (k_i - 1)) I_{m_{ik_i - 1}} \}$, where $m_{ij}$ is the repetitions of $(c_i - rj)$
in the diagonal of $C$; for $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, k_i - 1$. Thus,
\[
C = \text{diag} \{ c_1 I_{m_{1,0}}, (c_1 - r) I_{m_{1,1}}, \ldots, [c_1 - r (k_1 - 1)] I_{m_{1,(k_1 - 1)}}, \ldots, \\
c_i I_{m_{i,0}}, (c_i - r) I_{m_{i,1}}, \ldots, [c_i - r (k_i - 1)] I_{m_{i,(k_i - 1)}}, \ldots, \\
c_k I_{m_{k,0}}, (c_k - r) I_{m_{k,1}}, \ldots, [c_k - r (k_k - 1)] I_{m_{k,(k_k - 1)}} \},
\]
where
\[
c_i \neq c_j, \quad \text{whenever } i \neq j, \quad \text{for } i, j = 1, 2, \ldots, k, \\
[c_i - r j] - c_{i+1} \neq r, \quad \text{for } j = 0, \ldots, k_i - 1; \ i = 1, 2, \ldots, k - 1.
\]
The $i$th diagonal block of $C$ is called the $c_i$-block and $k_i$ is its length. Any diagonal entry $c$ of $C$ such that $c = c_i - rl$, for $l \geq 0$ then $0 \leq l \leq k_i - 1$ for some $i = 1, \ldots, k$, that is, $c$ belongs to the $c_i$-block. If $c_i - l r = c_j - l_2 r$, $0 \leq l_1 \leq k_i - 1$, $0 \leq l_2 \leq k_j - 1$, then $c_i$ and $c_j$ are in the same block, violating (1.9).

We use the notations given in Remark 1.7.

2. Faithful representations for $L^s_{r,t}$ where $rs \neq 0$

Lemma 2.1. The matrices $A$ and $B$ can be partitioned into submatrices of the same size corresponding to those of $C$. The nonzero submatrices of $A$ and $B$ are all off-diagonal submatrices.

Proof. From (1.2), the diagonal submatrices of $A$ and $B$ are square zero submatrices of orders $m_{1,0}, \ldots, m_{k,(k_k - 1)}$, in respective to those of $C$. Let $c_{ii}$, $c_{jj}$, and $c_{il}$; $i, j, l \in \mathbb{N}$, be from different diagonal submatrices of $C$, and suppose that $a_{ii} \neq 0$ and $a_{il} \neq 0$, then from (1.2), $c_{ii} = c_{jj}$ contradicting (1.8). Similarly, if $a_{ji}$ and $a_{li}$ are from different submatrices in $A$ they cannot be both nonzero. In view of (1.2), only the off-diagonal submatrices of $A$ may be nonzero. Thus we have, $A = [A_{ij}]$ where $A_{ij} = 0$, for $j \neq i + 1$. And similarly for $B$. \hfill \square

Lemma 2.2. For $k > 1$, if $k_i = 1$, for some $i = 1, 2, \ldots, k$, then $L^s_{r,t}$ has a representation of degree $n - m_{1,0}$. Moreover, if the entries in the $i$th row and the $i$th column of $X$ are all zeros, then $L^s_{r,t}$ has a representation of degree $n - 1$.

Proof. We use Lemma 1.4 so that the $c_i$-block becomes the first block of the main diagonal of $C$. Since for all $j \in \mathbb{N}$, $1 \leq i \leq m_{1,0}$, $|c_{ii} - c_{jj}| \neq r$, otherwise $k_i > 1$, then from (1.2) the representation is fully reducible since, $A = \begin{bmatrix} 0 & 0 \\ 0 & A' \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$, $C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$, and $D = \begin{bmatrix} D'_1 & 0 \\ 0 & D'_2 \end{bmatrix}$. The matrices
\[ X' = A' + iB', \quad Y' = X'^{\dagger}, \quad C'_2, \quad \text{and} \quad D'_2 \text{ are all of degree } n - m_{i,0} \text{ and satisfy the lemma. Similar argument holds when the entries in the } i \text{th row and the } \quad i \text{th column of } X \text{ are all zeros.} \]

So, it can be assumed that if \( k > 1 \) then \( k_i > 1; \ i = 1, \ldots, k \). And for \( X \not\in O \), if the entries of the \( i \)th row of \( X \) are all zeros, then those of the \( i \)th column are not all zeros, and vice versa, in such cases, we get from (1.5) that \( sc_{i,i} \not= 0 \).

**Theorem 2.3.** If \( rs < 0 \), then \( X = Y = C = O \).

**Proof.** If \( k = 1 \) and \( k_1 = 1 \), then from (1.2) \( X = Y = O \). If \( X = O \), then from (1.5) \( C = O \). Suppose that \( X \not\in O \), there are only two cases to consider namely, the case where \( k = 1 \) and \( k_1 > 1 \), and the case where \( k > 1 \). In both cases \( k_1 > 1 \), from Lemma 2.1 the first \( m_{1,0} \) columns of \( X \) are zero columns, and from Lemma 2.2 there must be an \( x_{i_1,j} \neq 0 \) for some \( m_{1,0} < j \leq (m_{1,0} + m_{1,1}) \). Thus from (1.5),

\[
sc_{11} = sc_1 = \sum_{l=1}^{n} \left( |x_{1j}|^2 - 0 \right) > 0. \tag{2.1}
\]

Let \( \alpha = m_{1,0} + m_{1,1} + \cdots + m_{1,(k_1-2)} \). If \( k > 1 \), we get from (1.9), \( [c_1 - r(k_1 - 1)] - c_2 \not= r \), thus from (1.2), the rows \( \alpha + 1, \alpha + 2, \ldots, \alpha + m_{1,(k_1-1)} \) are zero rows of \( X \). If \( k = 1 \) and \( k_1 > 1 \), we get from Lemma 2.1 that the mentioned rows are zero rows of \( X \), being the last rows of \( X \). In both cases, from Lemma 2.2 there must be an \( x_{i,\alpha+1} \neq 0 \) for some \( [\alpha - m_{1,(k_1-2)}] < i \leq \alpha \). From (1.5),

\[
sc_{\alpha+1,\alpha+1} = s[c_1 - r(k_1 - 1)] = \sum_{l=1}^{n} \left( 0 - |x_{l,\alpha+1}|^2 \right) < 0. \tag{2.2}
\]

If \( s > 0 \), then \( c_1 > 0 \) by (2.1), since \( r < 0 \), then \( [c_1 - r(k_1 - 1)] > 0 \), violating (2.2). Similarly, if \( s < 0 \), we get from (2.1), \( [c_1 - r(k_1 - 1)] < 0 \), violating (2.2). \( \square \)

We conclude this section by introducing the \( 2 \times 2 \) representation matrices \( X, Y, C, \) and \( D \) of \( K_1, K_2, K_3, \) and \( K_4, \) respectively, for \( rs > 0, t \in \mathbb{R} \)

\[
X = \begin{bmatrix} 0 & a \pm i\sqrt{rs/2 - a^2} \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ a \mp i\sqrt{rs/2 - a^2} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} r/2 & 0 \\ 0 & -r/2 \end{bmatrix}, \quad D = \begin{bmatrix} b & 0 \\ 0 & b + t \end{bmatrix}. \tag{2.3}
\]
for any $a, b \in \mathbb{R}$ such that $|a| \leq \sqrt{rs}/2$ and for the linear independency of $C$ and $D$, take $b \neq -t/2$. These representations are faithful. The $2 \times 2$ representation matrices $X, Y, C, \text{ and } D$ generalize those given in [1].

Clearly, the vector space spanned by $X, Y, \text{ and } C$ is $\text{sl}(2, \mathbb{C})$, a vector space. The representation matrices of $L_{rs, t}$, in [2], are for the special cases, $a^2 = rs/2$.

3. Faithful representations for $L_{rs, t}$ where $rst = 0$

The case where $rs \neq 0$ and $t = 0$ was considered in the previous section. So, if $s \neq 0$ we only need to consider the case where $r = 0$ and $t$ is any real number.

3.1. For $s \neq 0, r = 0, \text{ and } t \in \mathbb{R}$

Since $r = 0$ then any $c_i$-block of the matrix $C$ has length $k_i = 1$. So, we have $C = \text{diag}(c_1I_{m_1}, \ldots, c_kI_{m_k})$ where $c_i \neq c_j$ whenever $i \neq j; i, j = 1, \ldots, k$.

**Remark 3.1.** If $X$ commutes with $Y = X^\dagger$, then $X$ is a normal matrix, and there exists a unitary matrix $U$ such that $X = U^\dagger Z U$ for some complex diagonal matrix $Z$. If $U$ commutes with $C$ and $D$, then the diagonal matrices $Z, \tilde{Z}, C, \text{ and } D$ are representation matrices for $K_1, K_2, K_3, \text{ and } K_4$, respectively, and satisfy the physical requirements. We take $U = I_n$ when $X$ is diagonal.

**Lemma 3.2.** If $C = \text{diag}(c_1I_{m_1}, \ldots, c_kI_{m_k})$ for different $c_i$'s, then the representation is fully reducible into representations of degrees $m_1, \ldots, m_k$.

**Proof.** The matrix $D$ is diagonal and from (1.2), $x_{ij} = x_{ji} = y_{ij} = y_{ji} = 0$, whenever $c_{ii} \neq c_{jj}; i, j \in \mathbb{N}$. □

**Lemma 3.3.** Let $K = [K_{ij}]$ be a partitioned matrix which is normal whose diagonal blocks are $k$ square matrices. If $K_{ij} = O$ whenever $j \neq i + 1$ (or $j \neq i - 1$); $i, j = 1, \ldots, k$. Then $K = O$.

**Proof.** Let $K = [k_{ij}]$ be an $n \times n$ matrix, then for each $i \in \mathbb{N}$,

$$
\sum_{l=1}^{n}|k_{il}|^2 = \sum_{l=1}^{n}|k_{lj}|^2. \quad (3.1)
$$

Let the diagonal blocks of $K$ be of degrees $i_1, \ldots, i_k$, respectively. If $K_{ij} = O$ whenever $j \neq i + 1; i, j = 1, \ldots, k$, then the first $i_1$ rows of $K$ are zeros, thus from (3.1) the first $i_1$ columns of $K$ are zeros. Continuing like that in less
than $k$ steps, it can be shown that $K = O$. Hence the proof of the lemma follows. 

**Theorem 3.4.** The matrix $C = O$, in any representation of $L_{0,t}^s$. If $st \neq 0$, then $X = Y = O$.

**Proof.** Suppose $C \neq O$, we use Lemma 1.4 so that $c_1 \neq 0$, from (1.5) and Lemma 3.2, $m_1 s c_1 = \sum_{i=1}^{m_1} s c_{ii} = \sum_{i=1}^{m_1} \sum_{i=1}^{m_1} (|x_{i,i}|^2 - |x_{i,j}|^2) = 0$, but $m_1 s c_1 \neq 0$. Then $C = O$. Thus from Lemma 1.1, $X$ is a normal matrix. If $t \neq 0$, we use Lemma 1.4, so that

$$D = \text{diag} \left\{ d_1 I_{m_1,i}, (d_1 + t) I_{m_1,i}, \ldots, [d_1 + t(k_1' - 1)] I_{m_1,k_1' - 1}, \ldots, d_i I_{m_i,i}, \ldots, (d_i + t) I_{m_i,i}, \ldots, [d_i + t(k_i' - 1)] I_{m_i,k_i' - 1}, \ldots, d_k I_{m_k,i} \right\},$$

where $m_{i,j}'$ is the repetitions of $(d_i + tj)$ in the diagonal of $D$; for $i = 1, \ldots, k'$ and $j = 0, \ldots, k' - 1$ such that

$$d_i \neq d_j, \quad \text{whenever } i \neq j, \quad \text{for } i, j = 1, 2, \ldots, k',
\quad d_{i+1} - [d_i + tj] \neq t, \quad \text{for } j = 0, \ldots, k_i' - 1; \quad i = 1, 2, \ldots, k' - 1. \quad (3.3)$$

From (1.3), $X$ can be partitioned into submatrices of the same sizes corresponding to those of $D$, whose nonzero submatrices are off-diagonal submatrices. Then by Lemma 3.3 $X = Y = O$. 

If $t = 0$ then from Lemma 1.1, the generators commute and such a case can be considered as a special case of $L_{0,0}^0$ of Section 3.3, with $C = O$.

3.2. For $s = 0$ and $r^2 + t^2 \neq 0$

From (1.5) as $s = 0$, then (3.1) holds. If the $i$th row (or column) of $X$ consists entirely of zeros, the $i$th column (or row) also, consists entirely of zeros and both can be omitted by the following lemma whose proof is analogous to that of Lemma 2.2. So, if $X \neq O$, it can be considered that $X$ has no zero row or zero column.

**Lemma 3.5.** If $X$ has $m$ zero rows (or columns), where $0 \leq m < n$, then $L_{s,t}^s$ has a representation of degree $n - m$. 

Theorem 3.6. If \( s = 0 \) and \( r^2 + t^2 \neq 0 \), \( L_{r,t}^s \) has no faithful representations. In any representation, \( X = Y = O \).

Proof. If \( r \neq 0 \), arrange \( C \) as in Remark 1.7 otherwise, let \( D \) as in the proof of Theorem 3.4. In view of Lemma 1.5, \( X \) can be partitioned into submatrices of the same sizes corresponding to those of \( C \) when \( r \neq 0 \) or to those of \( D \) otherwise. The nonzero submatrices of \( X \) are all off diagonal submatrices. As \( s = 0 \) then \( X \) is normal and from Lemma 3.3, we get \( X = Y = O \). \[\square\]

3.3. For \( s = r = t = 0 \)

Although physically is not applicable, but for the sake of completeness, we consider the case when \( K_1, K_2, K_3, \) and \( K_4 \) are commutant operators.

Theorem 3.7. The representations of \( L_{0,0}^0 \) are conjugate to representations where \( K_1, K_2, K_3, \) and \( K_4 \) are represented by diagonal matrices.

Proof. Let \( X = U^\dagger ZU \) for a unitary matrix \( U \) and a complex diagonal matrix \( Z \). We claim that \( U \) commutes with \( C \) and \( D \), then the theorem holds by using Remark 3.1. We induce on \( n \), the degree of the representation and prove the cases when \( X \) is not diagonal.

For \( n = 2 \): if \( X \) is not diagonal then from (1.4), both \( C \) and \( D \) are scalar matrices and both commute with \( U \).

For \( n = 3 \): if the diagonal elements of \( C \) (or \( D \)) are all different, then \( X \) must be diagonal. If \( X \) has two nonzero elements \( x_{ij} \) and \( x_{lm} \), from (1.4), both are nondiagonal elements where \( x_{lm} \) is not the \( x_{ji} \), then \( C \) and \( D \) are scalar matrices and both commute with \( U \). Otherwise, we use Lemma 1.4, so that \( X = \begin{bmatrix} X & O \\ O & g \end{bmatrix} \), thus from (1.2) and (1.3) \( C = \begin{bmatrix} cI & O \\ O & a \end{bmatrix} \) and \( D = \begin{bmatrix} dI & O \\ O & b \end{bmatrix} \), for some \( a, b, c, d \in \mathbb{R}; \ g \in \mathbb{C} \), where \( X' \) is not a diagonal matrix. That requires \( X' \) to be a normal matrix. So, there exists a unitary matrix \( U' \) such that \( X' = U'^\dagger MU' \), for some complex diagonal matrix \( M \). Obviously, \( U' \) commutes with \( cI \) and \( dI \). Let \( U = \begin{bmatrix} U' & O \\ O & 1 \end{bmatrix} \), and \( Z = \text{diag}(M, g) \) then \( U \) commutes with \( C \) and \( D \).

Assume that the theorem is true for \( n < m \).

For \( n = m \): if both \( C \) and \( D \) are scalar matrices, then \( U \) commutes with \( C \) and \( D \). If either \( C \) or \( D \) is not a scalar matrix, \( C \) say, then we use Lemma 1.4 to rearrange \( C \) so that \( C = \text{diag}(c_1 I_{m_1}, \ldots, c_k I_{m_k}) \) for different \( c_i \)'s, from (1.2) \( X = \text{diag}(X_1, \ldots, X_k) \) where \( X_i \) is a square matrix of order \( m_i < m \). Also, \( D \) can be considered as \( D = \text{diag}(D_1, \ldots, D_k) \) where \( D_i \) is a diagonal matrix of degree \( m_i \). Hence, the representation is fully reducible into representations of degrees \( m_i, i = 1, \ldots, k \). Since \( X \) is normal then \( X_i \) is normal for \( i = 1, \ldots, k \). Thus there exists a unitary matrix \( U_i \) such that
\[ X_i = U_i^\dagger Z_i U_i \] for some complex diagonal matrix \( Z_i \), \( i = 1, \ldots, k \). From the induction \( U_i \) commutes with \( c_i I_m \) and \( D_i \). Let \( U = \text{diag}(U_1, \ldots, U_k) \) and \( Z = \text{diag}(Z_1, \ldots, Z_k) \), then \( U \) commutes with \( C \) and \( D \). \( \square 

**Theorem 3.8.** The Lie algebra \( L^0_{0,0} \) has faithful representations of degree 4 as the least degree.

**Proof.** Any linearly independent diagonal matrices \( Z, \bar{Z}, C, \) and \( D \), of degree 4, with \( C \) and \( D \) are real, are representation matrices for \( K_1, K_2, K_3, \) and \( K_4 \), respectively, of a faithful representation. \( \square 

We conclude the paper by mentioning the cases where \( L^s_{r,t} \) has faithful matrix representations satisfying the physical requirements.

**Summary 3.9.** It is assumed that all representations of \( L^s_{r,t} \) must satisfy the physical requirements.

1. For \( rs > 0, t \in \mathbb{R} \), \( L^s_{r,t} \) has faithful representations of degree 2 as the least degree.
2. For \( r = s = t = 0 \), \( L^0_{0,0} \) has faithful representation of degree 4 as the least degree where the representation matrices are linearly independent diagonal matrices, with \( C \) and \( D \) are real matrices.

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**References**


L. A. M. Hanna: Department of Mathematics and Computer Science, Faculty of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait

E-mail address: hannalam@mcs.sci.kuniv.edu.kw
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