We study the asymptotic behavior of $L_p(\sigma)$ extremal polynomials with respect to a measure of the form $\sigma = \alpha + \gamma$, where $\alpha$ is a measure concentrated on a rectifiable Jordan curve in the complex plane and $\gamma$ is a discrete measure concentrated on an infinite number of mass points.

1. Introduction

Let $F$ be a compact subset of the complex plane $\mathbb{C}$ and let $B$ be a metric space of functions defined on $F$. We suppose that $B$ contains the set of monic polynomials. Then the extremal or general Chebyshev polynomial $T_n$ of degree $n$ is a monic polynomial that minimizes the distance between zero and the set of all monic polynomials of degree $n$, that is,

$$\text{dist} (T_n, 0) = \min \{ \text{dist} (Q_n, 0) : Q_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \} = m_n(B). \quad (1.1)$$

Recently, a series of results concerning the asymptotic of the extremal polynomials was established for the case of $B = L_p(F, \sigma)$, $1 \leq p \leq \infty$, where $\sigma$ is a Borel measure on $F$; see, for example, [3, 7, 8, 12]. When $p = 2$, we have the special case of orthogonal polynomials with respect to the measure $\sigma$. A lot of research work has been done on this subject; see, for example, [1, 4, 5, 9, 11, 13]. The case of the spaces $L_p(F, \sigma)$, where $0 < p < \infty$ and $F$ is a closed rectifiable Jordan curve with some smoothness conditions, was studied by Geronimus [2]. An extension of Geronimus’s result has been given by Kaliaguine [3] who found asymptotics when $0 < p < \infty$ and the measure $\sigma$ has a decomposition of the form

$$\sigma = \alpha + \gamma, \quad (1.2)$$

where $\alpha$ is a measure supported on a closed rectifiable Jordan curve $E$ as defined in [2] and $\gamma$ is a discrete measure with a finite number of mass points.

In this paper, we generalize Kaliaguine’s work [3] in the case where $1 \leq p < \infty$ and the support of the measure $\sigma$ is a rectifiable Jordan curve $E$ plus an infinite discrete set of
mass points which accumulate on \( E \). More precisely, \( \sigma = \alpha + \gamma \), where the measure \( \alpha \) and its support \( E \) are defined as in [3], that is,

\[
d\alpha(\xi) = \rho(\xi)|d\xi|, \quad \rho \geq 0, \rho \in L^1(E, |d\xi|); \tag{1.3}\]

\( \gamma \) is a discrete measure concentrated on \( \{z_k\}_{k=1}^{\infty} \subseteq \text{Ext}(E) \) (\( \text{Ext}(E) \) is the exterior of \( E \)), that is,

\[
y = \sum_{k=1}^{+\infty} A_k \delta(z - z_k), \quad A_k > 0, \sum_{k=1}^{+\infty} A_k < \infty. \tag{1.4}\]

Note that the result of the special case \( p = 2 \) is also a generalization of [4]. More precisely, in the proof of Theorem 4.3, we show that condition [4, page 265, (17)] imposed on the points \( \{z_k\}_{k=1}^{\infty} \) is a redundant.

2. The \( H^p(\Omega, \rho) \) spaces \((1 \leq p < \infty)\)

Let \( E \) be a rectifiable Jordan curve in the complex plane, \( \Omega = \text{Ext}(E), G = \{z \in \mathbb{C}, |z| > 1\} \) (\( \infty \) belongs to \( \Omega \) and \( G \)).

We denote by \( \Phi \) the conformal mapping of \( \Omega \) into \( G \) with \( \Phi(\infty) = \infty \) and \( 1/C(E) = \lim_{z \to \infty} (\Phi(z)/z) > 0 \), where \( C(E) \) is the logarithmic capacity of \( E \). We denote \( \Psi = \Phi^{-1} \).

Let \( \rho \) be an integrable nonnegative weight function on \( E \) satisfying the Szegö condition

\[
\int_E (\log \rho(\xi)) |\Phi'(\xi)| |d\xi| > -\infty. \tag{2.1}\]

Condition (2.1) allows us to construct the so-called Szegö function \( D \) associated with the curve \( E \) and the weight function \( \rho \):

\[
D(z) = \exp \left\{ -\frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{w + e^{it}}{w - e^{it}} \log \left( \frac{\rho(\xi)}{|\Phi'(\xi)|} \right) dt \right\} \quad (w = \Phi(z), \xi = \Psi(e^{it})) \tag{2.2}\]

such that

(i) \( D \) is analytic in \( \Omega, D(z) \neq 0 \) in \( \Omega \), and \( D(\infty) > 0 \);

(ii) \( |D(\xi)|^{-p} |\Phi'(\xi)| = \rho(\xi) \) a.e. on \( E \), where \( D(\xi) = \lim_{z \to \xi} D(z) \).

We say that \( f \in H^p(\Omega, \rho) \) if and only if \( f \) is analytic in \( \Omega \) and \( f_0 \Psi/D_0 \Psi \in H^p(G) \).

For \( 1 \leq p < \infty \), \( H^p(\Omega, \rho) \) is a Banach space. Each function \( f \in H^p(\Omega, \rho) \) has limit values a.e. on \( E \) and

\[
\|f\|_{H^p(\Omega, \rho)} = \int_E |f(\xi)|^p \rho(\xi) |d\xi| = \lim_{R \to 1^+} \frac{1}{R} \int_{E_R} |f(z)|^p |\Phi'(z)dz|, \tag{2.3}\]

where \( E_R = \{z \in \Omega : |\Phi(z)| = R\} \).

**Lemma 2.1** [3]. If \( f \in H^p(\Omega, \rho) \), then for every compact set \( K \subset \Omega \), there is a constant \( C_K \) such that

\[
\sup \{|f(z)| : z \in K\} \leq C_K \|f\|_{H^p(\Omega, \rho)}, \tag{2.4}\]
3. The extremal problems

Let $1 \leq p < \infty$; we denote $\sigma_l = \alpha + \sum_{k=1}^{l} A_k \delta(z - z_k)$ and by $\mu(p), \mu(l), \mu(\infty)(p), m_{n,p}(\rho), m_{n,p}(l), \text{ and } m_{n,p}(\sigma)$ the extremal values of the following problems, respectively:

\[
\mu(p) = \inf \{ \| \varphi \|_{H^p(\Omega, \rho)}^p : \varphi \in H^p(\Omega, \rho), \varphi(\infty) = 1 \}, \quad (3.1)
\]
\[
\mu(l) = \inf \{ \| \varphi \|_{H^p(\Omega, \rho)}^p : \varphi \in H^p(\Omega, \rho), \varphi(\infty) = 1, \varphi(z_k) = 0, k = 1, 2, \ldots, l \}, \quad (3.2)
\]
\[
\mu(\infty)(p) = \inf \{ \| \varphi \|_{H^p(\Omega, \rho)}^p : \varphi \in H^p(\Omega, \rho), \varphi(\infty) = 1, \varphi(z_k) = 0, k = 1, 2, \ldots \}, \quad (3.3)
\]
\[
m_{n,p}(\rho) = \min \{ \| Q_n \|_{L_p(\rho)} : Q_n(z) = z^n + \cdots \}, \quad (3.4)
\]
\[
m_{n,p}(l) = \min \{ \| Q_n \|_{L_p(\sigma_l)} : Q_n(z) = z^n + \cdots \}, \quad (3.5)
\]
\[
m_{n,p}(\sigma) = \min \{ \| Q_n \|_{L_p(\sigma)} : Q_n(z) = z^n + \cdots \}. \quad (3.6)
\]

As usual,
\[
\| f \|_{L_p(\sigma)} := \left( \int_E | f(\xi) |^p d\sigma(\xi) \right)^{1/p}. \quad (3.7)
\]

We denote by $\varphi^*$ and $\psi^\infty$ the extremal functions of problems (3.1) and (3.3), respectively.

Let $T_{n,p}^l(z)$ and $T_{n,p}(z)$ be the extremal polynomials with respect to the measures $\sigma_l$ and $\sigma$, respectively, that is,
\[
\| T_{n,p}^l \|_{L_p(\sigma_l)} = m_{n,p}(l), \quad \| T_{n,p} \|_{L_p(\sigma)} = m_{n,p}(\sigma). \quad (3.8)
\]

Lemma 3.1. Let $\varphi \in H^p(\Omega, \rho)$ such that $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$ for $k = 1, 2, \ldots$, and let
\[
B_\infty(z) = \prod_{k=1}^{+\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z) \Phi(z_k) - 1} \frac{|\Phi(z_k)|^2}{\Phi(z_k)}, \quad (3.9)
\]
be the Blaschke product. Then

(i) $B_\infty \in H^p(\Omega, \rho), B_\infty(\infty) = 1, |B_\infty(\xi)| = \prod_{k=1}^{+\infty} |\Phi(z_k)| (\xi \in E);$  
(ii) $\varphi/B_\infty \in H^p(\Omega, \rho)$ and $(\varphi/B_\infty)(\infty) = 1.$

Proof. This lemma is proved for $p = 2$ in [1]. The proof is based on the fact that if $f \in H^2(U)$, where $U = \{ z \in \mathbb{C}, |z| < 1 \}$, and $B$ is the Blaschke product formed by the zeros of $f$, then $f/B \in H^2(U)$. It remains true in $H^p(U)$ for $1 \leq p < \infty$; see [6, 10].
Lemma 3.2. An extremal function $\psi^\infty$ of problem (3.3) is given by $\psi^\infty = \varphi^* B_\infty$; in addition,

$$\mu^\infty(\rho) = \prod_{k=1}^{+\infty} (|\Phi(z_k)|)^p \mu(\rho).$$

(3.10)

Proof. If $\varphi \in H^p(\Omega, \rho)$, $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$ for $k = 1, 2, \ldots$. Then by Lemma 2.1, we have $f = \varphi/B_\infty \in H^p(\Omega, \rho)$, $f(\infty) = 1$, and $|B_\infty(\xi)| = \prod_{k=1}^{+\infty} |\Phi(z_k)|$ for $\xi \in E$. These lead to

$$\|f\|^p = \left( \prod_{k=1}^{+\infty} |\Phi(z_k)| \right)^{-p} \|\varphi\|^p.$$  

(3.11)

Thus

$$\mu(\rho) \leq \left( \prod_{k=1}^{+\infty} |\Phi(z_k)| \right)^{-p} \mu^\infty(\rho).$$

(3.12)

On the other hand, since the function $\psi^\infty = \varphi^* B_\infty \in H^p(\Omega, \rho)$, $\psi(\infty) = 1$ and $\psi(z_k) = 0$ for $k = 1, 2, \ldots$, we get

$$\mu^\infty(\rho) \leq \|\psi^\infty\|^p = \left( \prod_{k=1}^{+\infty} |\Phi(z_k)| \right)^p \mu(\rho).$$

(3.13)

Finally, the lemma follows from (3.12) and (3.13). $\square$

4. The main results

Definition 4.1. A measure $\sigma = \alpha + \gamma$ is said to belong to a class $A$ if the absolutely continuous part $\alpha$ and the discrete part $\gamma$ satisfy conditions (1.3), (1.4), and (2.1) and Blaschke’s condition, that is,

$$\sum_{k=1}^{+\infty} (|\Phi(z_k)| - 1) < \infty.$$  

(4.1)

We denote $\lambda_n = \Phi^n - \Phi_n$, where $\Phi_n$ is the polynomial part of the Laurent expansion of $\Phi^n$ in the neighborhood of infinity.

Definition 4.2 [2]. A rectifiable curve $E$ is said to be of class $\Gamma$ if $\lambda_n(\xi) \to 0$ uniformly on $E$.

Theorem 4.3. Let a measure $\sigma = \alpha + \gamma$ satisfy conditions (1.3), (1.4) and Blaschke’s condition (4.1); then

$$\lim_{l \to +\infty} m_{n,p}(l) = m_{n,p}(\sigma).$$

(4.2)
Proof. The extremal property of \( T_{n,p}(z_k) \) gives

\[
(m_{n,p}(\sigma))^p \leq \int_E |T_{n,p}(\xi)|^p \rho(\xi) d\xi + \sum_{k=1}^{l} A_k |T_{n,p}(z_k)|^p + \sum_{k=l+1}^{+\infty} A_k |T_{n,p}(z_k)|^p
\]

\[
= (m_{n,p}(l))^p + \sum_{k=l+1}^{+\infty} A_k |T_{n,p}(z_k)|^p.
\]

(4.3)

On the other hand, from the extremal property of \( T_{n,p}(z_k) \), we can write

\[
m_{n,p}(l) \leq \left( \int_E |T_{n,p}(\xi)|^p \rho(\xi) d\xi + \sum_{k=1}^{l} A_k |T_{n,p}(z_k)|^p \right)^{1/p}
\]

\[
\leq m_{n,p}(\sigma) = C_n < \infty.
\]

(4.4)

Note that \( C_n \) does not depend on \( l \); so for all \( l = 1, 2, 3, \ldots \),

\[
\left( \int_E |T_{n,p}(\xi)|^p \rho(\xi) d\xi \right)^{1/p} < C_n.
\]

(4.5)

This implies that there is a constant \( C'_n \) independent of \( l \) such that for all \( l = 1, 2, 3, \ldots \),

\[
\max \{|T_{n,p}(z)|^p : |z| \leq 2\} < C'_n.
\]

(4.6)

Using (4.6) in (4.3) for large enough \( l \) with (4.4), we get

\[
(m_{n,p}(l))^p \leq (m_{n,p}(\sigma))^p \leq (m_{n,p}(l))^p + C'_n \sum_{k=l+1}^{+\infty} A_k.
\]

(4.7)

Letting \( l \to \infty \), we obtain

\[
\lim_{l \to \infty} m_{n,p}(l) = m_{n,p}(\sigma).
\]

(4.8)

Theorem 4.4. Let \( 1 \leq p < \infty \), \( E \in \Gamma \), and let \( \sigma = \alpha + \gamma \) be a measure which belongs to \( A \). In addition, for all \( n \) and \( l \),

\[
m_{n,p}(l) \leq \left( \prod_{k=1}^{l} |\Phi(z_k)| \right) m_{n,p}(\rho).
\]

(4.9)

Then the monic orthogonal polynomials \( T_{n,p}(z) \) with respect to the measure \( \sigma \) have the following asymptotic behavior:

(i) \( \lim_{n \to \infty} (m_{n,p}(\sigma)/\rho_{n}(\sigma))^p = (\mu_{n}(\rho))^{1/p} \);

(ii) \( \lim_{n \to \infty} \|T_{n,p}/[C(E)\Phi]\|_{H^p(\Omega,\rho)} = 0 \);

(iii) \( T_{n,p}(z) = [C(E)\Phi(z)]^p [\psi^\infty(z) + \epsilon_n(z)] \),

where \( \epsilon_n(z) \to 0 \) uniformly on compact subsets of \( \Omega \) and \( \psi^\infty \) is an extremal function of problem (3.3).
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Remark 4.5. For $p = 2$ and $E$ the unit circle, condition (4.9) is proved (see [5, Theorem 5.2]). In this case, this condition can be written as $\gamma_n / \gamma_n' \leq \prod_{k=1}^{\infty} |z_k|$, where $\gamma_n' = 1/m_{n,2}(l)$ and $\gamma_n = 1/m_{n,2}(\rho)$ are, respectively, the leading coefficients of the orthonormal polynomials associated to the measures $\sigma_l$ and $\alpha$.

Proof of Theorem 4.4. Taking the limit when $l$ tends to infinity in (4.9) and using Theorem 4.3, we get

\[
\frac{m_{n,p}(\sigma)}{(C(E))^n} \leq \left( \prod_{k=1}^{\infty} |\Phi(z_k)| \right)^{m_{n,p}(\rho)} \frac{m_{n,p}(\rho)}{(C(E))^n}. \tag{4.10}
\]

On the other hand, it is proved in [2] that

\[
\lim_{n \to \infty} \frac{m_{n,p}(\rho)}{(C(E))^n} = (\mu(\rho))^{1/p}. \tag{4.11}
\]

Using (4.10), (4.11), and Lemma 3.2, we obtain

\[
\limsup_{n \to \infty} \frac{m_{n,p}(\sigma)}{(C(E))^n} \leq \left( \prod_{k=1}^{\infty} |\Phi(z_k)| \right)(\mu(\rho))^{1/p} = (\mu^{\infty}(\rho))^{1/p}. \tag{4.12}
\]

It is well known that (see [3, page 231])

\[
\forall l > 0, \quad \mu(l) = \mu(\rho) \left( \prod_{k=1}^{l} |\Phi(z_k)| \right)^p. \tag{4.13}
\]

We also have (see [3, Theorem 2.2])

\[
\lim_{n \to \infty} \frac{m_{n,p}(l)}{(C(E))^n} = (\mu(l))^{1/p}. \tag{4.14}
\]

From (4.4), we deduce that

\[
\forall l > 0, \quad \frac{m_{n,p}(\sigma)}{(C(E))^n} \geq \frac{m_{n,p}(l)}{(C(E))^n}. \tag{4.15}
\]

By passing to the limit when $n$ tends to infinity in (4.15) and taking into account (4.13) and (4.14), we get

\[
\forall l > 0, \quad \liminf_{n \to \infty} \frac{m_{n,p}(\sigma)}{(C(E))^n} \geq \left( \prod_{k=1}^{l} |\Phi(z_k)| \right)(\mu(\rho))^{1/p}. \tag{4.16}
\]
Finally, by using Lemma 3.2, we obtain
\[
\liminf_{n \to \infty} \frac{m_{n,p}(\sigma)}{(C(E))^n} \geq \left( \prod_{k=1}^{+\infty} |\Phi(z_k)| \right) (\mu(\rho))^{1/p} = (\mu(\sigma))^{1/p}.
\] (4.17)

Inequalities (4.12) and (4.17) prove Theorem 4.4(i).
We obtain (ii) by proceeding as in [3, pages 234, 235]. To prove (iii), we consider the function
\[
\epsilon_n = \frac{T_{n,p}}{(C(E)\Phi)^n} - \psi^n
\] (4.18)
which belongs to the space $H^p(\Omega, \rho)$. Then by applying Lemma 2.1, we obtain
\[
\sup \left\{ \left| \frac{T_{n,p}(z)}{(C(E)\Phi(z))^n} - \psi(z) \right| : z \in K \right\} = \sup \{ |\epsilon_n(z)| : z \in K \} \leq C_K \|\epsilon_n\|_{H^p(\Omega, \rho)} \to 0
\] (4.19)
for all compact subsets $K$ of $\Omega$. This achieves the proof of the theorem. □

References

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