Existence and uniqueness of solution are proved for elastodynamics of Reissner-Mindlin plate model. Higher regularity is proved under the assumptions of smoother data and certain compatibility conditions. A mass scaling is introduced. When the thickness approaches zero, the solution of the clamped Reissner-Mindlin plate is shown to approach the solution of a Kirchhoff-Love plate.

1. Introduction

The Reissner-Mindlin (R-M) theory [13, 14, 15] has been popularly applied to thin-walled structures with moderate thickness. Transient response plays an important role in many aspects of structural analysis. The governing equation of the elastodynamics problem of R-M plate is of an evolutionary type with second-order time derivatives. In this paper, we apply a priori estimate to investigate the elastodynamics problem of R-M plate. This method has been successfully used in developing the theory of various partial differential equations, for example, [7, 10, 11]. Following the line of [7, 10, 11], we prove the existence and uniqueness of the $H^1$ solution. We then apply the approaches of [8] to prove the $H^2$ regularity and higher regularity when the data is smoother and certain compatibility conditions are satisfied.

For static problem under the assumption of load scaling, it is proved in [3] that the solution of the clamped R-M plate approaches the solution of the Kirchhoff-Love (K-L) plate when the thickness approaches zero. This fact has been employed to investigate the finite element method of R-M plate, such as locking-free and uniform convergence, cf. [2, 3, 5, 12, 16, 18]. For dynamics problem, with the introduction of mass scaling [4], we prove that when thickness approaches zero, the $H^2$ strong solution of the clamped R-M plate approaches the $H^2$ weak solution of K-L plate (whose classical solution requires $H^4$ smoothness).

In what follows, we describe the system of equations in Section 2 and prove the existence, uniqueness, and regularity in Section 3. Then we discuss the relation between R-M plate and K-L plate in Section 4. This is followed by a summary in Section 5.
2. Governing equations of Reissner-Mindlin plate for elastodynamics

For elastodynamic bending shear problem modeled by R-M plate theory [13, 14, 15], the displacement components of a generic point at a distance \( z \) to the midsurface are expressed by the deflection \( w \) at the midsurface and the rotations \( (\beta_1, \beta_2) \) of the normal to the midsurface,

\[
U_1 = -z\beta_1, \quad U_2 = -z\beta_2, \quad U_3 = w, \tag{2.1}
\]

\( |z| \leq \zeta/2 \). \( \zeta \) is the thickness of the plate. For dynamics problems, the velocity and acceleration, traditionally denoted by \( \dot{U}_i \) and \( \ddot{U}_i \), respectively, have the same format of (2.1) after differentiation with respect to time. The motion equation of R-M plate can be derived from the general three-dimensional elastodynamics by integration through thickness, or from the energy method using Hamilton’s principle, for example, [9]

\[
I\ddot{\beta}_1 + EA_1(\beta) - \lambda\zeta^{-2}(w_{,1} - \beta_1) = m_1, \\
I\ddot{\beta}_2 + EA_2(\beta) - \lambda\zeta^{-2}(w_{,2} - \beta_2) = m_2, \\
\rho\zeta^{-2}\ddot{w} - \lambda\zeta^{-2}\nabla \cdot (\nabla w - \beta) = g = f_3\zeta^{-2}, \tag{2.2}
\]

where we define

\[
A_1(\beta) = \frac{-(1 + \nu)(\beta_{\alpha\alpha})_{,1} + (1 - \nu)\nabla^2\beta_1}{24(1 - \nu^2)}, \\
A_2(\beta) = \frac{-(1 + \nu)(\beta_{\alpha\alpha})_{,2} + (1 - \nu)\nabla^2\beta_2}{24(1 - \nu^2)}. \tag{2.3}
\]

Here, \( E \) is the Young’s modulus, \( \rho \) is the density, and \( \nu \) is the Poisson ratio. We denote \( I = \rho/12 \) and \( \lambda = G\kappa \), with the shear modulus \( G \) and a shear correction factor \( \kappa \), which is introduced to balance the zero shear stress at the top and bottom surfaces. As analyzed for static problem [2, 3, 5], the lateral loading force \( f_3 \) (per unit volume) is scaled to \( \zeta^2 g \). The convention of summation on repeated indices is also applied, with the Greek index running over the range from 1 to 2. The bold-faced variables are used to denote a two-dimensional vector, for example, \( \beta = (\beta_1, \beta_2) \), and \( \beta_\alpha \) is used to indicate all of the two components involved when the indication is clear. Here, \( w_{,1} \) in (2.2) indicates the partial derivative \( \partial w/\partial x_1 \). The same applies for all the similar cases.

For simplicity, we consider the equations defined on a smooth bounded domain \( \Omega \) in \( \mathbb{R}^2 \), with homogeneous Dirichlet boundary conditions and general initial conditions

\[
\beta_\alpha(t, x) = 0; \quad w(t, x) = 0 \quad \text{on } \partial \Omega, \\
\beta_\alpha(0, x) = \beta_\alpha^0(x); \quad w(0, x) = W^0(x), \tag{2.4} \\
\dot{\beta}_\alpha(0, x) = \dot{\beta}_\alpha^0(x); \quad \dot{w}(0, x) = W^1(x).
\]
We adopt the usual notations of Sobolev spaces. The Galerkin method yields the following variational equation. For any $t \in [0, T]$, find $\beta_\alpha, w \in V = H^1_0(\Omega)$ such that

$$I(\beta_\alpha, \eta_\alpha) + \rho \zeta^{-2}(\ddot{w}, \nu) + Ea(\beta, \eta) + \lambda \zeta^{-2}(w_{x=\alpha} \beta_\alpha, \nu_{x=\alpha} \eta_\alpha) = (m_\alpha, \eta_\alpha) + (g, \nu), \quad \forall \eta_\alpha, \nu \in V. \quad (2.5)$$

Here, $(\cdot, \cdot)$ denotes the usual $L^2$ inner product, and $\langle \cdot, \cdot \rangle$ denotes the duality on $V' \otimes V$. $a(\cdot, \cdot)$ is a bilinear form on $V \otimes V$ defined as

$$a(\beta, \eta) = \frac{1}{24(1 - \nu^2)}((1 + \nu)(\beta_{x\alpha, x\alpha} + \nu(\nabla_{x\alpha} \nabla \eta_\alpha))). \quad (2.6)$$

It is associated with the operators $A_1$ and $A_2$ such that

$$a(\beta, \eta) = \langle A_1(\beta), \eta_1 \rangle + \langle A_2(\beta), \eta_2 \rangle = \langle A(\beta), \eta \rangle, \quad \forall \beta_\alpha, \eta_\alpha \in V. \quad (2.7)$$

In fact, $a(\cdot, \cdot)$ is symmetric, the same as the two-dimensional elasticity operator with a scalar factor. With Dirichlet boundary conditions, $a(\cdot, \cdot)$ is equivalent to the $H^1$-norm on $V$ [6] and there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|\eta\|^2_1 \leq a(\eta, \eta), \quad \forall \eta_\alpha \in V,$$

$$a(\beta, \eta) \leq \alpha_2 \|\beta\|_1 \|\eta\|_1, \quad \forall \beta_\alpha, \eta_\alpha \in V. \quad (2.8)$$

We denote $\|\eta\|^2_1 = \|\eta_1\|^2_2 + \|\eta_2\|^2_2$ for $\eta_\alpha$ of a functional space $X$. Note that for time-dependent problems, the norm $\|v\|_X$ of a function $v : [0, T] \to X$ is a function of time. We use the following notations for the functional spaces and the measure in time:

$$L^2(X) = L^2(0, T; X)$$

$$= \left\{ v : [0, T] \to X \mid v(t, x) \in X, \|v\|_{L^2(0, T; X)} = \left( \int_0^T \|v\|_X^2 \, dt \right)^{1/2} < \infty \right\}, \quad (2.9)$$

$$L^\infty(X) = L^\infty(0, T; X)$$

$$= \left\{ v : [0, T] \to X \mid v(t, x) \in X, \|v\|_{L^\infty(0, T; X)} = \text{ess sup}_{0 \leq t \leq T} (\|v\|_X) < \infty \right\}. \quad (2.9)$$

3. Existence, uniqueness, and regularity

For linear hyperbolic equations of the second order in time with one function, the existence and uniqueness are proven (see, e.g., [7, 10, 11]) using the method of a priori estimate. The method is also employed in [17] for Navier-Stokes problem whose steady-state case has close relation to the static problems of R-M plate, (cf. [5]). We extend the scheme to the dynamic problems of R-M plate. For the time being, we keep the material parameters explicitly expressed for later use in Section 4.

**Theorem 3.1.** If $m_\alpha, g \in L^2(L^2); B_\alpha^0, W_0^0 \in H^1_0; \text{and } B_\alpha^1, W_1^1 \in L^2$, then there exists a solution $(\beta_\alpha, w)$ of (2.5) (a weak solution of (2.2)) with initial conditions (2.4), $\beta_\alpha, w \in L^\infty(H^1_0)$, $\beta_\alpha, \dot{w} \in L^\infty(L^2)$, and $\dot{\beta}_\alpha, \dot{w} \in L^\infty(H^{-1})$. Moreover, there exists a constant $C > 0$, independent
of the material parameters, such that

\[ I\|\tilde{\beta}_n\|_0^2 + \rho \zeta^{-2} \|\tilde{w}_n\|_0^2 + E \|\tilde{\beta}_n\|_0^2 + \lambda \zeta^{-2} \|\nabla w - \beta_n\|_0^2 \]

\[ \leq C \left( I\|\tilde{B}_1\|_0^2 + \rho \zeta^{-2} \|W_1\|_0^2 + E \|\tilde{B}_0\|_1^2 \right. \]

\[ \left. + \lambda \zeta^{-2} \|\nabla W_0 - \tilde{B}_0\|_0^2 + I^{-1} \|\tilde{m}_n\|_{L^2(\Omega)}^2 + \rho^{-1} \zeta^2 \|\bar{g}\|_{L^2(\Omega)}^2 \right) \]  

(3.1)

**Proof.** We apply the scheme for hyperbolic equations developed in [7, 10, 11]. The space \( V \) is separable. We construct an approximation of order \( n \), with a countable basis \( \{\psi_i(x), i = 1, 2, \ldots\} \) of \( V \):

\[ \beta_n(t, X) = \sum_{j=1}^{\infty} \beta_n^j(t) \psi_j(X), \quad w_n(t, X) = \sum_{j=1}^{\infty} w_n^j(t) \psi_j(X), \]

(3.2)

\[ I(\ddot{\beta}_n, \eta_a) + \rho \zeta^{-2} (\ddot{\bar{w}}_n, \nu) + Ea(\beta_n, \eta) + \lambda \zeta^{-2} (w_{n,a} - \beta_n, \nu_{a} - \eta_a) \]

\[ = (g, \nu) + (m_a, \eta_a), \quad \forall \eta_a, \nu \in V_n = \text{span} \{\psi_1, \ldots, \psi_n\}. \]

The approximation problem (3.2) leads to a linear system of second-order ordinary differential equations. With the approximations of (2.4) for the initial conditions

\[ \sum_{j=1}^{n} B_{an}^{0j} \psi_j(X) = B_n^0 \rightarrow B_0^n, \quad \sum_{j=1}^{n} W_n^{0j} \psi_j(X) = W_n^0 \rightarrow W_0^n, \]

\[ \sum_{j=1}^{n} B_{an}^{1j} \psi_j(X) = B_n^1 \rightarrow B_1^n, \quad \sum_{j=1}^{n} W_n^{1j} \psi_j(X) = W_n^1 \rightarrow W_1^n, \]

(3.3)

we have unique solution

\[ \{\beta_n^j(t), \beta_n^0(t), w_n^j(t), j = 1, \ldots, n\} \in H^2([0, T]). \]

(3.4)

Now using \( \eta_a = \dot{\beta}_n(t, X) \) and \( \nu = \dot{w}_n(t, X) \) in (3.2), then integrating from \( t = 0 \) to \( T \), we have

\[ I\|\dot{\beta}_n\|_0^2 + \rho \zeta^{-2} \|\dot{w}_n\|_0^2 + E\|\dot{\beta}_n\|_0^2 + \lambda \zeta^{-2} \|\nabla w_n - \beta_n\|_0^2 \]

\[ = I\|\dot{\beta}_n(0)\|_0^2 + \rho \zeta^{-2} \|\dot{w}_n(0)\|_0^2 + E\|\dot{\beta}_n(0)\|_0^2 \]

\[ + \lambda \zeta^{-2} \|\nabla w_n(0) - \beta_n(0)\|_0^2 + 2 \int_0^T \left((m_a, \dot{\beta}_n) + (g, \dot{w}_n)\right) dt \]

\[ \leq I\|B_n^0\|_0^2 + \rho \zeta^{-2} \|W_n^0\|_0^2 + E\|B_n^0\|_1^2 + \lambda \zeta^{-2} \|\nabla W_n^0 - B_n^0\|_0^2 \]

\[ + \int_0^T \left(I^{-1} \|\tilde{m}_n\|_0^2 + \rho^{-1} \zeta^2 \|\bar{g}\|_0^2 \right) dt + \int_0^T \left(I\|\dot{\beta}_n\|_0^2 + \rho \zeta^{-2} \|\dot{w}_n\|_0^2 \right) dt. \]

(3.5)
Applying Gronwall inequality, we obtain
\[
I ||\dot{\beta}_n||_0^2 + \rho \zeta^{-2} ||w_n||_0^2 + \alpha_1 E ||\dot{\beta}_n||_1^2 + \lambda \zeta^{-2} ||\nabla w_n - \beta_n||_0^2 \\
\leq C(I ||B_n||_0^2 + \rho \zeta^{-2} ||W_n||_0^2 + \alpha_2 E ||B_n||_1^2 \\
+ \lambda \zeta^{-2} ||\nabla W_n - B_0||_0^2 + I^{-1} ||m||_{L^2(L^2)}^2 + \rho^{-1} \zeta^2 ||g||_{L^2(L^2)}^2).
\]

(3.6)

The right-hand side has limit as \(n \to \infty\) due to (3.3). Therefore, the left-hand side is bounded. Note that \(||w_n||_1 \leq C ||\nabla w_n||_0 \leq C(\|\nabla w_n - \beta_n||_0 + \|\beta_n||_0).\) By compactness, we can find convergent subsequences, still denoted by subscript \(n\), such that
\[
\beta_n \rightharpoonup \beta, \quad w_n \to w \quad \text{weakly star in} \ L^\infty(H_0^1),
\]

(3.7)

\[
\dot{\beta}_n \rightharpoonup \dot{\beta}, \quad \dot{w}_n \to \chi \quad \text{weakly star in} \ L^\infty(L^2).
\]

It is a straightforward task to verify that \(\dot{\beta}_a = \phi_a, \dot{w} = \chi, \dot{\beta}_n \to \dot{\beta}, \dot{w}_n \to \dot{w} \) weakly star in \(L^\infty(H^{-1})\), and \(\{\beta_n, w\}\) satisfy the initial conditions (2.4) and the variational equation (2.5), thus form a weak solution of (2.2). \(\square\)

**Theorem 3.2.** Under the conditions of Theorem 3.1, the solution \(\{\beta, w\}\) is unique, that is, if \(g = 0, m_a = 0, B_0 = w^0 = B_1 = w^1 = 0\), then \(\dot{\beta}_a = w = 0\).

**Proof.** Following the line of [10, 11], we can prove the uniqueness, but omit the details. \(\square\)

**Theorem 3.3.** Under the conditions of Theorem 3.1, if \(m_a, g \in L^2(L^2), B_0, W \in H^2\), and \(B_1, W \in H_0^1\), then the solution \((\beta, w)\) of (2.2) with the initial conditions (2.4) satisfies \(\beta_a, w \in L^\infty(L^2), \beta_a, w \in L^\infty(H_0^1), \beta_a, w \in L^\infty(H^2),\) and

\[
I ||\dot{\beta}||_0^2 + \rho \zeta^{-2} ||\dot{w}||_0^2 + E ||\dot{\beta}||_1^2 + \zeta^{-2} ||\nabla \dot{w} - \dot{\beta}||_0^2 \\
\leq C(I (E \|B_0\|_2^2 + \lambda \zeta^{-4} ||\nabla W^0 - B_0||_0^2 + ||m(0)||_0^2) \\
+ \rho^{-1} \zeta^2 (\lambda^2 \zeta^{-4} ||\nabla W^0 - B_0||_0^2 + ||g(0)||_0^2) \\
+ E \|B_1||_2^2 + \lambda \zeta^{-2} ||\nabla W^1 - B_1||_0^2 + I^{-1} ||m||_{L^2(L^2)}^2 + \rho^{-1} \zeta^2 ||g||_{L^2(L^2)}^2),
\]

(3.8)

\[
E ||\beta||_2 \leq C(I ||\dot{\beta}_0||_0 + \lambda \zeta^{-2} ||\nabla w - \beta||_0 + ||m||_0),
\]

(3.9)

\[
\lambda \zeta^{-2} ||w||_2 \leq C(\rho \zeta^{-2} ||\dot{w}_0||_0 + \lambda \zeta^{-2} ||\beta||_1 + ||g||_0),
\]

(3.10)

where the bounds of \(||\nabla w - \beta||_0\) and \(||\beta||_1\) are established in (3.1).

**Proof.** We apply the method for hyperbolic equations demonstrated in [8]. From Theorem 3.1, we have \(\dot{\beta}_n, \dot{w}_n \in H^1([0, T]).\) Differentiating (3.2) with respect to \(t\), we obtain \(\ddot{\beta}_n, \ddot{w}_n \in L^2([0, T]).\) The a priori estimate like (3.6) holds:

\[
I ||\ddot{\beta}_n||_0^2 + \rho \zeta^{-2} ||\ddot{w}_n||_0^2 + \alpha_1 E ||\dot{\beta}_n||_1^2 + \lambda \zeta^{-2} ||\nabla \ddot{w}_n - \ddot{\beta}_n||_0^2 \\
\leq C(I ||\ddot{\beta}_0||_0^2 + \rho \zeta^{-2} ||\ddot{w}_0||_0^2 + \alpha_2 E ||\dot{\beta}_0||_1^2 \\
+ \lambda \zeta^{-2} ||\nabla \ddot{w}_0(0) - \ddot{\beta}_0(0)||_0^2 + I^{-1} ||m||_{L^2(L^2)}^2 + \rho^{-1} \zeta^2 ||g||_{L^2(L^2)}^2).
\]

(3.11)
From (2.2), we obtain
\[\ddot{\beta}_a(0), \ddot{w}(0) \in L^2,\]
\[I\|\ddot{\beta}_a(0)\|_0 \leq E\|B^0\|_2 + \lambda \zeta^{-2}\|\nabla W^0 - B^0\|_1 + \|m_a(0)\|_0,\]
\[\rho \zeta^{-2}\|\ddot{w}(0)\|_0 \leq \lambda \zeta^{-2}\|\nabla W^0 - B^0\|_1 + \|m_a(0)\|_0.\] (3.12)

The argument of boundedness and compactness leads to the conclusion that \(\ddot{\beta}_a, \ddot{w}\) weakly star in \(L^\infty(L^2)\). Hence, (3.11) implies (3.8).

On the other hand, we rewrite (2.2):
\[EA_1(\beta) = m_1 - I\ddot{\beta}_1 + \lambda \zeta^{-2}(w_1 - \beta_1),\]
\[EA_2(\beta) = m_2 - I\ddot{\beta}_2 + \lambda \zeta^{-2}(w_2 - \beta_2),\]
\[-\lambda \zeta^{-2}\nabla^2 w = g - \rho \zeta^{-2}\ddot{w} + \lambda \zeta^{-2}\beta_{a,a}.\] (3.13)

For any fixed time \(t\), the right-hand sides of these equations are in \(L^2\). We have the elasticity operator and the Laplace operator in the left-hand side. According to the theory of elliptic equations, with a smooth domain \(\Omega\), we have \(\beta_a, w \in H^2\), and the bounds (3.9) and (3.10).

We are ready to extend the method for higher regularity of hyperbolic equation [8] to the transient dynamics of R-M plate. For simplicity, the dependence on the material parameters is not explicitly expressed and will have more discussion in Section 4.

**Theorem 3.4.** Assume for any integer \(P \geq 0\),
\[B^0_a, W^0 \in H^{P+1} \cap H^1_0, \quad B^1_a, W^1 \in H^P \cap H^1_0,\]
\[\frac{\partial^k m_a}{\partial t^k}, \quad \frac{\partial^k g}{\partial t^k} \in L^2(H^{P-k}), \quad k = 0,1,\ldots,P,\] (3.14)
and that the following compatibility conditions hold for \(P \geq 2\):
\[B^{k+2}_a = I^{-1}\left(\frac{\partial^k m_a(0)}{\partial t^k} - EA_a(B^k) + \lambda \zeta^{-2}(\nabla W^k - B^k)\right) \in H^1_0, \quad k = 0,1,\ldots,P-2;\]
\[W^{k+2} = (\rho \zeta^{-2})^{-1}\left(\frac{\partial^k g(0)}{\partial t^k} + \lambda \zeta^{-2}\nabla \cdot (\nabla W^k - B^k)\right) \in H^1_0, \quad k = 0,1,\ldots,P-2.\] (3.15)
Then the solution of (2.2) with (2.4) satisfy, for \(k = 0,1,\ldots,P+1,\)
\[\left\|\frac{\partial^k \beta_a}{\partial t^k}\right\|_{P+1-k} + \left\|\frac{\partial^k w}{\partial t^k}\right\|_{P+1-k} \leq C\left(\sum_{j=0}^{P} \left(\left\|\frac{\partial^j m_a}{\partial t^j}\right\|_{L^2(H^{P-j})} + \left\|\frac{\partial^j g}{\partial t^j}\right\|_{L^2(H^{P-j})}\right)\right) + \|B^0\|_{P+1} + \|W^0\|_{P+1} + \|B^1\|_P + \|W^1\|_P.\] (3.17)
Then we can use (3.19) to estimate

\[ \tilde{B}_a = \hat{B}_a, \quad \tilde{w} = \hat{w}, \]
\[ \tilde{m}_a = \hat{m}_a, \quad \tilde{g} = \hat{g}, \]
\[ \tilde{g}^k = B_a^{k+1}, \quad \tilde{W}^k = W^{k+1}, \quad k = 0, 1, \ldots, Q. \tag{3.18} \]

Proof. The cases of \( P = 0 \) and \( P = 1 \) are proved in Theorems 3.1 and 3.3, respectively. Using the method of induction, we assume that the theorem is true for \( P \leq Q \) and assume that the conditions (3.14) and (3.15) are valid for \( P = Q + 1 \). Denote

\[ \frac{\partial^k \tilde{B}_a}{\partial t^k}, \quad \frac{\partial^k \tilde{w}}{\partial t^k} \in L^\infty(H^{Q+1-k}), \quad (3.20) \]
\[ \left\| \frac{\partial^k \tilde{B}_a}{\partial t^k} \right\|_{Q+1-k} + \left\| \frac{\partial^k \tilde{w}}{\partial t^k} \right\|_{Q+1-k} \]
\[ \leq C \left( \sum_{j=0}^{Q} \left( \left\| \frac{\partial^j \tilde{m}_a}{\partial t^j} \right\|_{L^2(H^{Q-j})} + \left\| \frac{\partial^j \tilde{g}}{\partial t^j} \right\|_{L^2(H^{Q-j})} \right) \right) \]
\[ + \left\| \tilde{B}^0 \right\|_{Q+1} + \left\| \tilde{W}^0 \right\|_{Q+1} + \left\| \tilde{B}^1 \right\|_Q + \left\| \tilde{W}^1 \right\|_Q \bigg). \tag{3.21} \]

It implies that, for \( k = 1, \ldots, Q+2 \),

\[ \frac{\partial^k \tilde{B}_a}{\partial t^k}, \quad \frac{\partial^k \tilde{w}}{\partial t^k} \in L^\infty(H^{Q+2-k}), \quad (3.22) \]
\[ \left\| \frac{\partial^k \tilde{B}_a}{\partial t^k} \right\|_{Q+2-k} + \left\| \frac{\partial^k \tilde{w}}{\partial t^k} \right\|_{Q+2-k} \]
\[ \leq C \left( \sum_{j=0}^{Q+1} \left( \left\| \frac{\partial^j \tilde{m}_a}{\partial t^j} \right\|_{L^2(H^{Q+1-j})} + \left\| \frac{\partial^j \tilde{g}}{\partial t^j} \right\|_{L^2(H^{Q+1-j})} \right) \right) \]
\[ + \left\| \tilde{B}^1 \right\|_{Q+1} + \left\| \tilde{W}^1 \right\|_{Q+1} + \left\| \tilde{B}^2 \right\|_Q + \left\| \tilde{W}^2 \right\|_Q \bigg). \tag{3.23} \]

We can use (3.19) to estimate \( B^2 \) and \( W^2 \) in (3.23) with

\[ \left\| m(0) \right\|_Q \leq C \left\| m \right\|_{C^0(H^0)} \leq C \left( \left\| m \right\|_{L^2(H^0)} + \left\| \tilde{m} \right\|_{L^2(H^0)} \right), \]
\[ \left\| g(0) \right\|_Q \leq C \left\| g \right\|_{C^0(H^0)} \leq C \left( \left\| g \right\|_{L^2(H^0)} + \left\| \tilde{g} \right\|_{L^2(H^0)} \right). \tag{3.24} \]
Therefore, (3.17) is true for \( P = Q + 1 \) and \( k = 1, \ldots, Q + 2 \). Now the right-hand sides of (3.13) are bounded in \( H^Q \). We have
\[
\| \beta \|_{Q+2}^2 \leq C \| I \tilde{\beta} - \lambda \zeta^{-2}(\nabla w - \beta) - m \|_{Q}^2 \leq C \| \tilde{\beta} \|_{Q}^2 + \| w \|_{Q+1}^2 + \| \beta \|_{Q}^2 + \| m \|_{Q}^2) \leq \infty,
\]
\[
\| w \|_{Q+2}^2 \leq C \| \rho \zeta^{-2} \tilde{w} + \lambda \zeta^{-2} \beta_{a,a} - g \|_{Q}^2 \leq C \| \tilde{w} \|_{Q}^2 + \| \beta \|_{Q+1}^2 + \| g \|_{Q}^2) \leq \infty.
\]
(3.25)

Therefore,
\[
\| \beta \|_{Q+2} + \| w \|_{Q+2} \leq C \left( \sum_{j=0}^{Q+1} \left( \| \frac{\partial^j m}{\partial t^j} \|_{L^2(H^{0^{(q+1-j)}})} + \| \frac{\partial^j g}{\partial t^j} \|_{L^2(H^{0^{(q+1-j)}})} \right) + \| B^0 \|_{Q+2} + \| W^0 \|_{Q+2} + \| B^1 \|_{Q+1} + \| W^1 \|_{Q+1} \right).
\]
(3.26)

Thus, (3.17) also holds for \( P = Q + 1 \) and \( k = 0 \). The case of \( P = Q + 1 \) of the induction is true. □

4. Relation to Kirchhoff-Love plate

For static problem, it is understood that when the thickness \( \zeta \to 0 \), the solution of the clamped R-M plate approaches the solution of a K-L plate (see, e.g., [3] for a proof). The convergence is for the systems with load scaling, in the sense that \( \beta_{a} \to \tilde{\beta}_{a} \), \( w \to \tilde{w} \), and
\[
\tilde{\beta} = \nabla \tilde{w},
\]
(4.1)
\[
D_0 \nabla^4 \tilde{w} = g,
\]
(4.2)

where \( D_0 = E/12(1-\nu^2) = D \zeta^{-3} \). \( D \) is the usual bending stiffness. Due to the load scaling, the K-L equation (4.2) is independent of thickness. Physically, when the thickness approaches zero, the bending stiffness approaches zero faster with a factor of \( \zeta^3 \). The unscaled loading, which contributes to the external work, is proportional to the thickness and will not give a meaningful solution. This fact is used for investigating the thickness-independent convergence of finite element method, for example, [2, 3, 5, 12, 16, 18] (see [12, 16, 18] for numerical examples).

For dynamic problem, due to the appearance of the inertia term, which contributes to the kinetic energy, the equation of K-L plate is no longer thickness independent. To keep K-L plate as a reference model, a possible approach is then to scale the mass density [4] along with the load. Assume
\[
\rho = \zeta^2 \rho_0,
\]
\[
I = \zeta^2 I_0, \quad I_0 = \frac{\rho_0}{12}.
\]
(4.3)
We consider the scaled R-M equation (2.2) with \( m_\alpha = 0 \), which does not appear in K-L plate:

\[
I_0 \zeta^2 \ddot{\beta}_1 + EA_1(\beta) - \lambda \zeta^{-2}(w_1 - \beta_1) = 0,
\]
\[
I_0 \zeta^2 \ddot{\beta}_2 + EA_2(\beta) - \lambda \zeta^{-2}(w_2 - \beta_2) = 0,
\]  \( \rho_0 \ddot{w} - \lambda \zeta^{-2} \nabla \cdot (\nabla w - \beta) = g, \)

or the variational equation (2.5):

\[
I_0 \zeta^2 \langle \ddot{\beta}_\alpha, \eta_\alpha \rangle + \rho_0 \langle \ddot{w}, \nu \rangle + Ea(\beta, \eta) + \lambda \zeta^{-2}(w_\alpha - \beta_\alpha, \nu_\alpha - \eta_\alpha) = (g, \nu), \quad \forall \eta_\alpha, \nu \in V.
\]  \( \text{(4.5)} \)

As a parallel study to the static problem, we consider a special case of elastodynamics with zero initial conditions:

\[
B^0_\alpha = B^1_\alpha = W_0 = W_1 = 0.
\]  \( \text{(4.6)} \)

**Theorem 4.1.** Assume \( g \in H^1(L^2) \), \( \dot{g} \in L^2(L^2) \), and \( (\beta_\alpha, w) \in L^\infty(H_0^1) \) is the solution of (4.4) (or (4.5)) with initial conditions (4.6). Then as \( \zeta \to 0 \), there exists a sequence of \( (\tilde{\beta}_\alpha, \tilde{w}) \) with the same notation for simplicity such that

\[
w \rightharpoonup \tilde{w} \quad \text{weakly star in } L^\infty(H^2),
\]
\[
\beta_\alpha \rightharpoonup \tilde{\beta}, \quad \dot{\tilde{w}} \rightharpoonup \dot{\tilde{w}} \quad \text{weakly star in } L^\infty(H^1),
\]
\[
\ddot{w} \rightharpoonup \ddot{\tilde{w}} \quad \text{weakly star in } L^\infty(L^2).
\]  \( \text{(4.7)} \)

Moreover,

\[
\tilde{\beta} = \nabla \tilde{w}
\]  \( \text{(4.8)} \)

and \( \tilde{w} \) is the solution of a K-L plate problem of elastodynamics with clamped boundary conditions

\[
\rho_0 \ddot{\tilde{w}} + D_0 \nabla^4 \tilde{w} = g \quad \text{or}
\]
\[
\rho_0(\ddot{w}, \nu) + D_0(\nabla^2 \ddot{w}, \nabla^2 \nu) = (g, \nu), \quad \forall \nu \in H_0^2,
\]
\[
\tilde{w} \big|_{\partial \Omega} = \frac{\partial \tilde{w}}{\partial n} \big|_{\partial \Omega} = 0,
\]
\[
\tilde{w}(0, x) = \tilde{w}(0, x) = 0.
\]  \( \text{(4.9)} \)

**Proof.** By Theorems 3.1 to 3.3, we have a unique solution \( (\beta_\alpha, w) \in L^\infty(H_0^1) \cap L^\infty(H^2) \) for (4.4) (or (4.5)), where the generic constant \( C > 0 \) used in the a priori estimates is independent of material parameters. With (4.6), the a priori estimate (3.1) is reduced to

\[
\sqrt{I_0 \zeta \| \dot{\beta} \|_0 + \sqrt{\rho_0} \| \ddot{w} \|_0 + \sqrt{E} \| \dot{\beta} \|_1 + \sqrt{\lambda \zeta^{-2}} \| \nabla w - \beta \|_0} \leq C \| g \|_{L^2(L^2)}. \]  \( \text{(4.10)} \)

Inequality (3.8) yields

\[
\sqrt{I_0 \zeta \| \dot{\beta} \|_0 + \sqrt{\rho_0} \| \ddot{w} \|_0 + \sqrt{E} \| \dot{\beta} \|_1 + \sqrt{\lambda \zeta^{-2}} \| \nabla \dot{w} - \dot{\beta} \|_0} \leq C(\| g(0) \|_0 + \| g \|_{L^2(L^2)}). \]  \( \text{(4.11)} \)
Inequality (3.10) results in

$$\|w\|_2 \leq C(\|g(0)\|_0 + \|g\|_{L^2(L^2)} + \|g\|_{L^2(L^1)}).$$  \hfill (4.12)$$

Furthermore, \(\|\dot{w}\|_1 \leq C(\|\dot{w}\|_0 + \|\nabla \dot{w}\|_0) \leq C(\|\dot{w}\|_0 + \|\nabla \dot{w} - \dot{\beta}\|_0 + \|\dot{\beta}\|_0).\) Therefore, the boundedness of \(w\) in \(L^\infty(H^2),\ \beta_a, \dot{\beta}_a, \dot{\beta}\) in \(L^\infty(H^1),\) and \(\zeta, \dot{\beta}, \dot{\beta}\) in \(L^\infty(L^2)\) are all uniform with respect to the thickness. We can extract the convergent sequences

$$w \rightarrow \tilde{w} \quad \text{weakly star in } L^\infty(H^2),$$

$$\beta_a \rightarrow \tilde{\beta}_a, \quad \dot{\beta}_a \rightarrow \tilde{\dot{\beta}_a}, \quad \dot{w} \rightarrow \tilde{\dot{w}} \quad \text{weakly star in } L^\infty(H^1),$$

$$\tilde{w} \rightarrow \hat{w} \quad \text{weakly star in } L^\infty(L^2),$$

where, for simplicity, no \(\zeta\)-dependence notation is used for the sequences. The relation with time differentiation is trivial.

The initial conditions \(\hat{w}(0, x) = \dot{w}(0, x) = 0\) are a direct result of (4.6). Inequality (4.10) implies \(\sqrt{\lambda} \|\nabla w - \beta\|_0 \leq C\zeta \|g\|_{L^2(L^2)} \rightarrow 0 \Rightarrow \nabla \hat{w} - \hat{\beta} = 0.\) Meanwhile, the boundary conditions on \((\beta_a, w)\) lead to \(\hat{\beta}_a |_{\partial \Omega} = \hat{w}|_{\partial \Omega} = \nabla \hat{w}|_{\partial \Omega} = 0.\) The last equation implies, for smooth domain, \(\partial \hat{w}/\partial n|_{\partial \Omega} = 0.\)

On the other hand, by (4.11), \(\zeta^2 \|\hat{\beta}\|_0 \rightarrow 0.\) Thus, the first two equations of (4.4) yield \(EA_a(\beta) - \lambda \zeta^{-2}(w_a - \beta_a) \rightarrow 0.\) That means \(\lambda \zeta^{-2}(w_a - \beta_a) \rightarrow EA_a(\beta) = EA_a(\nabla \hat{w}).\) Then the third equation of (4.4) gives \(\rho_0 \ddot{w} - E \nabla \cdot (A(\nabla \hat{w})) = \rho_0 \ddot{w} + D_0 \nabla^4 \hat{w} = g.\) The last equality can be easily verified with the definition of the operator \(A\) and considered in the weak sense. Similar statement for the variational equation is straightforward.

**Remark 4.2.** The generic constant \(C\) involved in the inequalities derived in Theorem 3.4 is thickness dependent. So the boundedness of \(\beta\) and \(w\) in higher spaces may not be uniform with respect to the thickness. It is worth noting that the boundary layer is found for static problems of R-M plate [1]. With clamped boundary conditions, as in our case, \(\partial^3 \beta/\partial n^3 = O(\zeta^{-1})\) near the boundary, that is, \(\|\beta\|_3 = O(\zeta^{-1/2}).\) The boundary layer is expected for the dynamics problem too, which warrants further investigation. Since \(\dot{\beta} = \nabla \hat{w},\) it is not optimistic that we can have the convergence of \(w \rightarrow \hat{w}\) in the sense of \(H^4,\) although the corresponding K-L plate can have a strong solution \(\hat{w}\) in \(H^4.\)

## 5. Summary

Existence and uniqueness of \(H^1\) solution of R-M plate for elastodynamics with homogeneous Dirichlet boundary conditions and general initial conditions were proved. The solution with smoother data was further investigated and proved to be in \(H^2.\) Furthermore, with higher smoothness of data and certain compatibility requirements satisfied, higher regularity of the solution was proved. With the introduction of mass scaling, along with the load scaling, the \(H^2\) solution of R-M plate was proved to approach the \(H^2\) weak solution of K-L plate when the thickness approaches zero.
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References


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