

GENERALIZED RESOLVENTS AND SPECTRUM FOR A CERTAIN CLASS OF PERTURBED SYMMETRIC OPERATORS

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Received 3 February 2004 and in revised form 20 May 2004

The generalized resolvents for a certain class of perturbed symmetric operators with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula, we give a classification of the spectrum.

1. Introduction

The present paper is concerned with the study of spectral properties for a certain class of linear symmetric operator T , defined in the Hilbert space H of the form $T = A + B$, where A is a closed linear symmetric operator, with nondensely defined domain in general, $D(A) \subset H$, and B is a finite-rank operator of the form

$$Bf = \sum_{k=1}^n a_k(f, y_k) y_k, \quad (1.1)$$

where y_1, y_2, \dots, y_n is a linearly independent system in H , $a_1, a_2, \dots, a_n \in \mathbb{R}$. We remark that the operator T can be considered as a perturbation of the operator A by the finite-rank operator B .

The case when A is a first-order or second-order differential operator in the spaces $L^2(0, 2\pi)$, $L^2(0, \infty)$ or in the Hilbert space of vector-valued functions, and B is a one-dimensional perturbation ($n = 1$), has been studied by many authors (see, e.g., [9, 20, 24]).

In particular, certain integrodifferential equations of the above type occur in quantum mechanical scattering theory [8].

In this paper, the generalized resolvents of perturbed symmetric operator T with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula (see, e.g., [18]), we give a classification of the spectrum. Finally, the obtained results are applied to the study of two classes of first-order and second-order differential operators.

We note that the spectral theory of perturbed symmetric and selfadjoint operators have been investigated using various methods by many authors [3, 4, 5, 6, 11, 12, 13, 14, 15, 16, 17, 21, 22].

2. Preliminaries

Let A be a closed symmetric operator with nondensely defined domain in a separable Hilbert space H with equal deficiency indices (m, m) , and $m < \infty$. We denote by $\rho(A)$ the resolvent set of the operator A , the resolvent operator $R_\lambda(A)$ of A is defined as $R_\lambda(A) = (A - \lambda I)^{-1}$. The complement of $\rho(A)$ in the complex plane is called the spectrum of A and denoted by $\sigma(A)$. There is a decomposition of the spectrum $\sigma(A)$ into three disjoint subsets, at least one of which is not empty [1, 2, 10]:

$$\sigma(A) = P\sigma(A) \cup C\sigma(A) \cup PC\sigma(A), \quad (2.1)$$

$P\sigma(A)$ is called the point spectrum, $C\sigma(A)$ the continuous spectrum, and $PC\sigma(A)$ the point-continuous spectrum. We denote the essential spectrum of the operator A by $\sigma_e(A) = C\sigma(A) \cup PC\sigma(A)$.

For arbitrary $\lambda \in \mathbb{C}$, we denote $P_\lambda = N_\lambda \cap (D(A) \oplus N_{\bar{\lambda}})$, where $N_\lambda = H \ominus (A - \lambda I)D(A)$ is the deficiency subspace of the operator A [1, 2].

It is known [23] that $P_\lambda = \{0\}$ if and only if $\overline{D(A)} = H$, and if $\overline{D(A)} \neq H$, then the subset

$$G_\lambda = \{[\varphi, \psi] \in N_\lambda \times N_{\bar{\lambda}} : \varphi - \psi \in D(A)\} \quad (2.2)$$

is a graph of the isometric operator X_λ with domain P_λ and values in $P_{\bar{\lambda}}$.

We denote by \mathfrak{F} the set of linear operators F defined from N_i to N_{-i} , such that $\|F\| \leq 1$. For each analytic operator-valued function $F(\lambda)$ in \mathbb{C}^+ , with $\mathbb{C}^+ = \{\lambda : \text{Im}\lambda > 0\}$, and values in \mathfrak{F} , we introduce the set $\Omega_F(\infty)$ consisting of elements $h \in N_i$ such that

$$\lim_{\lambda \rightarrow \infty, \lambda \in \mathbb{C}_\varepsilon^+} |\lambda| [\|h\| - \|F(\lambda)h\|] < \infty, \quad (2.3)$$

where $\mathbb{C}_\varepsilon^+ = \{\lambda \in \mathbb{C}^+ : \varepsilon < \arg \lambda < \pi - \varepsilon\}$, $0 < \varepsilon < \pi/2$.

It is known [27] that $\Omega_F(\infty)$ is a vector space and for each $h \in \Omega_F(\infty)$,

$$\lim_{\lambda \rightarrow \infty, \lambda \in \mathbb{C}_\varepsilon^+} F(\lambda)h = F_0(\infty)h \quad (2.4)$$

exists in the sense of the strong topology, and $F_0(\infty)$ is an isometric operator.

According to the theory of Štraus [28], the generalized resolvents of A are given by the formula

$$R_\lambda(A) = R_\lambda = (A_{F(\lambda)} - \lambda I)^{-1}, \quad R_{\bar{\lambda}} = R_\lambda^*, \quad \lambda \in \mathbb{C}^+, \quad (2.5)$$

where $A_{F(\lambda)}$ is an extension of A which is determined by the function $F(\lambda)$, whose values are operators from the deficiency subspace N_i to the deficiency subspace N_{-i} such that $\|F(\lambda)\| \leq 1$ and $F(\lambda)$ satisfy the condition

$$F_0(\infty)\psi = X_i\psi, \quad \text{for } \psi = 0 \text{ only}, \quad (2.6)$$

then $A_{F(\lambda)}$ is a restriction on H of a selfadjoint operator defined in a certain extended Hilbert space and is called quasiselfadjoint extension of the operator A [28] defined on $D(A_{F(\lambda)}) = D(A) + (F(\lambda) - I)N_i$ by

$$A_{F(\lambda)}(f + F(\lambda)\varphi - \varphi) = Af + iF(\lambda)\varphi + i\varphi, \quad f \in D(A), \varphi \in N_i. \tag{2.7}$$

For selfadjoint extensions with exit in the space in which acts the considered operators, see, for example, [12, 21] and the references therein.

We denote by \mathfrak{N} the set of analytic operator functions $F(\lambda)$ in \mathbb{C}^+ with values in \mathfrak{F} satisfying the condition (2.6).

Remark 2.1. To each selfadjoint extension of the operator A corresponds a certain constant operator function $F(\lambda) = V$, where V is an isometric operator defined from N_i over N_{-i} satisfying the condition $V\psi = X_i\psi$ for $\psi = 0$ only, and reciprocally.

We denote by \mathring{A} a selfadjoint extension of A and we introduce the operator

$$\mathring{U}_{\lambda\lambda_0} = (\mathring{A} - \lambda_0 I)(\mathring{A} - \lambda I)^{-1}, \quad \text{Im } \lambda > 0. \tag{2.8}$$

We note that (see [19])

$$\mathring{U}_{\lambda\lambda_0} N_{\lambda_0} = N_{\bar{\lambda}}, \quad (\text{Im } \lambda)(\text{Im } \lambda_0) \neq 0. \tag{2.9}$$

We denote by

$$\varphi_i^{(1)}, \varphi_i^{(2)}, \dots, \varphi_i^{(m)} \tag{2.10}$$

a basis of N_{-i} . From (2.9), $\varphi_{\lambda}^{(k)} = \mathring{U}_{\lambda i} \varphi_i^{(k)}$, $k = 1, 2, \dots, m$ form a basis for $N_{\bar{\lambda}}$. In particular, the vectors

$$\varphi_{-i}^{(k)} = \mathring{U} \varphi_i^{(k)}, \quad k = 1, 2, \dots, m, \tag{2.11}$$

where $\mathring{U} = \mathring{U}_{-ii}$ is the Cayley transform [1, 2] of \mathring{A} , form an orthogonal basis of N_i .

To get a convenient formula of the generalized resolvents of A , we will need the following notation:

$$\Phi_{\lambda\mu} = (\lambda - \bar{\mu}) [(\varphi_{\lambda}^{(k)}, \varphi_{\mu}^{(s)})]_{k,s=1}^m, \quad C(\lambda) = \Phi_{\lambda i}^{-1} \Phi_{\lambda(-i)}, \tag{2.12}$$

where E is the identity matrix of order m , $\Omega(\lambda)$ is an analytic matrix function in \mathbb{C}^+ corresponding, in the bases (2.10) and (2.11), to the operator function $F(\lambda) \in \mathfrak{N}$ and $\varphi_{\lambda} = (\varphi_{\lambda}^{(1)}, \dots, \varphi_{\lambda}^{(m)})^t$, $(f, \varphi_{\bar{\lambda}})^t = ((f, \varphi_{\bar{\lambda}}^{(1)}), \dots, (f, \varphi_{\bar{\lambda}}^{(m)}))$, t denotes the transpose, and (φ_{λ}, g) is defined analogously.

In what follows, we denote by Φ the set of matrices $\Omega(\lambda)$, $\lambda \in \mathbb{C}^+$, associated in the bases (2.10) and (2.11) to the operator functions $F(\lambda) \in \mathfrak{K}$.

According to the notation used in [7], the generalized resolvents of A are given by

$$\begin{aligned} R_\lambda(A)f &= R_\lambda f = \overset{\circ}{R}_\lambda f + (f, \varphi_{\bar{\lambda}})^t [E - \Omega(\lambda)][C(\lambda)\Omega(\lambda) - E]^{-1} \Phi_{\bar{\lambda}i}^{-1} \varphi_\lambda, \\ R_{\bar{\lambda}} &= R_\lambda^*, \quad \lambda \in \mathbb{C}^+, \end{aligned} \quad (2.13)$$

where $\overset{\circ}{R}_\lambda$ is the resolvent of $\overset{\circ}{A}$ and $\Omega(\lambda) \in \Phi$.

Remark 2.2. The formula (2.13) defines a resolvent of a selfadjoint extension of A if and only if $\Omega(\lambda)$ is a unitary constant matrix.

3. Resolvent and spectrum of a symmetric perturbed operator

Let $T = A + B$ be defined on $D(T) = D(A)$, where A is a linear closed symmetric operator in H and B is a finite-rank operator.

LEMMA 3.1. For $\lambda \in \rho(A) \cap \rho(T)$, the resolvent $R_\lambda(T)$ of the operator T is given by

$$R_\lambda(T) = R_\lambda(A) - R_\lambda(A)[I + BR_\lambda(A)]^{-1}BR_\lambda(A). \quad (3.1)$$

Proof. For $\lambda \in \rho(A) \cap \rho(T)$, the operator

$$R_\lambda(A)[I + BR_\lambda(A)]^{-1} = R_\lambda(T) \quad (3.2)$$

exists and is bounded. Then, we get

$$\begin{aligned} (T - \lambda I)[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\ = (A - \lambda I + B)[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\ = I + BR_\lambda(A) - (I + BR_\lambda(A))(I + BR_\lambda(A))^{-1}BR_\lambda(A) = I \end{aligned} \quad (3.3)$$

as required. \square

Remark 3.2. If $\|BR_\lambda(A)\| < 1$, then from (3.1), we obtain

$$R_\lambda(T) = R_\lambda(A)(I + BR_\lambda(A))^{-1} = R_\lambda(A) \sum_{k=0}^{\infty} (-1)^k [BR_\lambda(A)]^k. \quad (3.4)$$

Now, the aim is to give a convenient expression of $(I + BR_\lambda(A))^{-1}$ in a more specific case.

So, we study in detail the case when B is a finite-rank operator. Then,

$$Bf = \sum_{k=1}^n a_k(f, y_k) y_k, \quad f \in H, \quad (3.5)$$

where $a_1, a_2, \dots, a_n \in \mathbb{R}$; $\{y_1, y_2, \dots, y_n\}$ is a linearly independent system in H . If we put

$$(I + BR_\lambda(A))^{-1}BR_\lambda(A)f = y, \quad (3.6)$$

we have

$$y = BR_\lambda(A)f - BR_\lambda(A)y, \tag{3.7}$$

then, $y \in \text{Im}B$, so that

$$y = \sum_{k=1}^n c_k y_k. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\sum_{k=1}^n c_k y_k = BR_\lambda(A)f - \sum_{k=1}^n c_k BR_\lambda(A)y_k, \tag{3.9}$$

with

$$c_k + a_k \sum_{j=1}^n c_j (R_\lambda(A)y_j, y_k) = a_k (R_\lambda(A)f, y_k). \tag{3.10}$$

The determinant $\Delta(\lambda)$ of the system (3.10) is given by

$$\Delta(\lambda) = \det \{ [\delta_{kj} + a_k (R_\lambda(A)y_j, y_k)]_{k,j=1}^n \}, \tag{3.11}$$

where δ_{kj} is the Kronecker symbol. If we suppose that $\Delta(\lambda) \neq 0$, the solution of (3.10) is given by

$$c_k = c_k(\lambda; f) = \frac{(f, \Delta_k(\lambda))}{\Delta(\lambda)}, \quad k = 1, 2, \dots, n, \tag{3.12}$$

where $\Delta_k(\lambda)$ is the determinant obtained from $\overline{\Delta(\lambda)}$ by replacing the k th column by $[a_j R_{\bar{\lambda}}(A)y_j]_{j=1}^n$. So, from (3.1), we have

$$R_\lambda(T)f = R_\lambda(A)f - \sum_{k=1}^n \frac{(f, \Delta_k(\lambda))}{\Delta(\lambda)} R_\lambda(A)y_k. \tag{3.13}$$

This completes the proof of the following theorem.

THEOREM 3.3. *Let $\lambda \in \rho(A)$ such that $\Delta(\lambda) \neq 0$. Then, $\lambda \in \rho(T)$ and the resolvent of the operator T is given by (3.13).*

Remark 3.4. From (3.13), we note that the resolvent $R_\lambda(T)$ is a perturbation of $R_\lambda(A)$ by a finite-rank operator.

Remark 3.5. For the particular case $n = 1$ and $a_1 = 1$, the formula (3.13) was established in [9].

Remark 3.6. If $\lambda \in \rho(A)$ such that $\Delta(\lambda) = 0$, then λ is an eigenvalue of the operator T .

Proof. We can show that there exists an element

$$\psi = \sum_{k=1}^n \alpha_k y_k \quad (3.14)$$

such that $R_\lambda(A)\psi$ is an eigenvector of the operator T , corresponding to the eigenvalue λ . Consequently, we have

$$a_k \sum_{j=1}^n \alpha_j (R_\lambda(A)y_j, y_k) + \alpha_k = 0, \quad k = \overline{1, n}. \quad (3.15)$$

Since the determinant of this system $\Delta(\lambda) = 0$, it admits a nontrivial solution, which gives the desired result. \square

THEOREM 3.7. *Let μ be a fixed complex number. Then, the following holds.*

- (a) *If $\mu \in \rho(A)$ and $\Delta(\mu) \neq 0$, then $\mu \in \rho(T)$.*
- (b) *If $\mu \in \rho(A)$ and $\Delta(\mu) = 0$, then $\mu \in P\sigma(T)$ and the multiplicity of μ as an eigenvalue of T is equal to the order of the zero of $\Delta(\lambda)$ at μ .*
- (c) *If $\mu \in P\sigma(A)$ and μ of multiplicity $k > 0$ and if μ is a pole of $\Delta(\lambda)$ of multiplicity p ($k \geq p$), then*
 - (1) *for $k > p$, it holds that $\mu \in P\sigma(T)$ of multiplicity $(k - p)$,*
 - (2) *for $k = p$, it holds that $\mu \in \rho(T)$.*
- (d) *If $\mu \in P\sigma(A)$ is neither a zero, nor a pole of $\Delta(\lambda)$, then $\mu \in P\sigma(T)$.*
- (e) *If $\mu \in P\sigma(A)$ of multiplicity k and μ is a root of the function $\Delta(\lambda)$ of order p , then $\mu \in P\sigma(T)$ of order $(k + p)$.*
- (f) *The essential spectra $\sigma_e(A)$ and $\sigma_e(T)$, respectively of the operators A and T , coincide.*

Proof. It is sufficient to evaluate the function

$$C(\lambda) = \det \{I + BR_\lambda(A)\}. \quad (3.16)$$

To this end, let $y \in \text{Im}B$. Then,

$$BR_\lambda(A)y = \sum_{k=1}^n a_k (y, R_\lambda^*(A)y_k) y_k, \quad (3.17)$$

it is clear that $C(\lambda) = \Delta(\lambda)$, and the function $\Delta(\lambda)$ is meromorphic in $\rho(A) \cup P\sigma(A)$. From the formula of Weinstein and Aronszajn [18], we have

$$\overline{\vartheta}(\lambda; T) = \overline{\vartheta}(\lambda; A) + \vartheta(\lambda; \Delta), \quad (3.18)$$

where

$$\begin{aligned} \bar{\vartheta}(\lambda; A) &= \begin{cases} 0 & \text{if } \lambda \in \rho(A), \\ k & \text{if } \lambda \in P\sigma(A) \text{ and of multiplicity } k, \\ +\infty & \text{otherwise,} \end{cases} \\ \vartheta(\lambda; \Delta) &= \begin{cases} k & \text{if } \lambda \text{ is a zero of } \Delta(\lambda) \text{ of order } k, \\ -k & \text{if } \lambda \text{ is a pole of } \Delta(\lambda) \text{ of order } k, \\ 0 & \text{for other } \lambda \in \Omega, \end{cases} \end{aligned} \tag{3.19}$$

which gives the desired result. \square

4. Generalized resolvents

Now, we suppose that A is a symmetric operator with deficiency indices (m, m) , $m < \infty$.

LEMMA 4.1. *Let $\lambda \in \mathbb{C}$ such that $\text{Im } \lambda > 0$ and $\varphi_\lambda(A) \in N_{\bar{\lambda}}(A)$. Then, the element $\varphi_\lambda(T)$, defined by the formula*

$$\varphi_\lambda(T) = D(\lambda)\varphi_\lambda(A) = \varphi_\lambda(A) - \sum_{k=1}^n \frac{(\varphi_\lambda(A), \mathring{g}_k(\lambda))}{\mathring{\Delta}(\lambda)} R_\lambda(\mathring{A})y_k, \tag{4.1}$$

is an element of the deficiency subspace $N_{\bar{\lambda}}(T)$, where

$$D(\lambda) = I - R_\lambda(\mathring{A})[I + BR_\lambda(\mathring{A})]^{-1}B = I - R_\lambda(\mathring{T})B, \quad \mathring{g}_k(\lambda) = (\mathring{A} - \bar{\lambda}I)\mathring{\Delta}_k(\lambda), \tag{4.2}$$

$\mathring{\Delta}(\lambda)$ and $\mathring{\Delta}_k(\lambda)$ are defined similarly as $\Delta(\lambda)$ and $\Delta_k(\lambda)$ in the formula (3.13) by putting the operator \mathring{A} instead of the operator A .

Proof. Since the operators \mathring{A} and $\mathring{T} = \mathring{A} + B$ are selfadjoint and λ is nonreal, then $\lambda \in \rho(\mathring{A}) \cap \rho(\mathring{T})$. In addition, from Theorem 3.3 we have $\mathring{\Delta}(\lambda) \neq 0$. Furthermore, for each $f \in D(A) = D(T)$, we have

$$\begin{aligned} ([\mathring{T} - \bar{\lambda}I]f, D(\lambda)\varphi_\lambda(A)) &= (D^*(\lambda)[\mathring{T} - \bar{\lambda}I]f, \varphi_\lambda(A)) \\ &= ([I - BR_{\bar{\lambda}}(\mathring{T})](\mathring{T} - \bar{\lambda}I)f, \varphi_\lambda(A)) \\ &= ((\mathring{A} - \bar{\lambda}I)f, \varphi_\lambda(A)) \\ &= 0, \end{aligned} \tag{4.3}$$

and the equality

$$\varphi_\lambda(T) = \varphi_\lambda(A) - \sum_{k=1}^n \frac{(\varphi_\lambda(A), \mathring{g}_k(\lambda))}{\mathring{\Delta}(\lambda)} R_\lambda(\mathring{A})y_k \tag{4.4}$$

results from (3.13). \square

Remark 4.2. We note that if $\varphi_\lambda(A) \neq 0$, then $\varphi_\lambda(T) \neq 0$.

Proof. If we suppose the contrary, we obtain $R_\lambda(\mathring{T})B\varphi_\lambda(A) = \varphi_\lambda(A)$, which gives $\mathring{A}\varphi_\lambda(A) = \lambda\varphi_\lambda(A)$. This leads to a contradiction, since a selfadjoint operator can not have nonreal eigenvalues. \square

Remark 4.3. If $D(A)$ is dense in H , then $\varphi_\lambda(A)$ and $\varphi_\lambda(T)$ are, respectively, eigenfunctions of the operators A^* and T^* , corresponding to the eigenvalues $\bar{\lambda}$.

Let $\varphi_i^{(k)}(T) = D(i)\varphi_\lambda^{(k)}(A)$, $k = 1, 2, \dots, m$, defined by the formula (4.1). If $\varphi_i^{(1)}(A)$, $\varphi_i^{(2)}(A), \dots, \varphi_i^{(m)}(A)$ is a basis of the deficiency subspace $N_i(A)$ of the operator A , then $\varphi_i^{(1)}(T), \varphi_i^{(2)}(T), \dots, \varphi_i^{(m)}(T)$ is a basis of the deficiency subspace $N_i(T)$ of the operator T . Putting

$$\begin{aligned} \mathring{U}_{\lambda\lambda_0}(\mathring{T}) &= (\mathring{T} - \lambda_0 I)R_\lambda(\mathring{T}), & \varphi_\lambda^{(k)}(T) &= \mathring{U}_{\lambda i}(\mathring{T})\varphi_i^{(k)}(T), & k &= 1, 2, \dots, m, \\ \varphi_\lambda(T) &= (\varphi_\lambda^{(1)}(T), \dots, \varphi_\lambda^{(m)}(T))^t, & \Phi_{\lambda\mu}(T) &= (\lambda - \bar{\mu})[(\varphi_\lambda^{(k)}(T), \varphi_\mu^{(j)}(T))]_{k,j=1}^m, \end{aligned} \tag{4.5}$$

$C(\lambda) = \Phi_{\lambda i}^{-1}(T)\Phi_{\lambda(-i)}(T)$ denotes the characteristic matrix of the operator T , and $\omega(\lambda)$ the corresponding matrix of order $m \times m$, in the bases $\varphi_i^{(1)}(T), \varphi_i^{(2)}(T), \dots, \varphi_i^{(m)}(T)$ and $\varphi_{-i}^{(1)}(T), \varphi_{-i}^{(2)}(T), \dots, \varphi_{-i}^{(m)}(T)$.

THEOREM 4.4. *The set of all generalized resolvents of the operator T is given by*

$$R_\lambda(T)f = R_\lambda(\mathring{T})f + (f, \varphi_{\bar{\lambda}}(T))^t [E - \omega(\lambda)][C(\lambda)\omega(\lambda) - E]^{-1}\Phi_{\lambda i}^{-1}(T)\varphi_\lambda(T), \quad \forall f \in H, \tag{4.6}$$

where

$$R_\lambda(\mathring{T})f = R_\lambda(\mathring{A})f - \sum_{k=1}^n \frac{(f, \mathring{\Delta}_k(\lambda))}{\mathring{\Delta}(\lambda)} R_\lambda(\mathring{A})y_k. \tag{4.7}$$

Proof. The proof results from Lemma 4.1 and formula (2.13). \square

We denote, respectively, by A_ω and T_ω the quasiselfadjoint extensions of operators A and T corresponding to the operator function $F(\lambda) \in \mathfrak{F}$, defined by the matrix $\omega(\lambda)$.

Remark 4.5. To selfadjoint extensions of these operators correspond the constant unitary matrices $\omega = [\omega_{ij}]$.

THEOREM 4.6. *Suppose that $y_1, y_2, \dots, y_n \in \text{Im } A$, μ is an eigenvalue of the quasiselfadjoint extension A_ω of the operator A , $\mu \in P\sigma(A_\omega)$. If $\mu \in \rho(\mathring{A})$ and $\mathring{\Delta}(\mu) \neq 0$, then μ is an eigenvalue of the operator $T_\omega = A_\omega + B$ and the corresponding eigenfunction $\varphi_\mu(T_\omega)$ is given by*

$$\varphi_\mu(T_\omega) = D(\mu)\varphi_\mu(A_\omega) = \varphi_\mu(A_\omega) - \sum_{k=1}^n \frac{(\varphi_\mu(A_\omega), \mathring{g}_k(\mu))}{\mathring{\Delta}(\mu)} R_\mu(\mathring{A})y_k, \tag{4.8}$$

where $\varphi_\mu(A_\omega)$ is the eigenfunction of the operator A_ω , corresponding to the eigenvalue μ .

Proof. Since $y_1, y_2, \dots, y_n \in \text{Im } A$, then $B\varphi_\mu(A) \in \text{Im } A$. We also have

$$\varphi_\mu(T_\omega) = D(\mu)\varphi_\mu(A_\omega) = \varphi_\mu(A_\omega) - R_\mu(\overset{\circ}{T})B\varphi_\mu(A) = \varphi_\mu(A_\omega) - \psi_\mu, \tag{4.9}$$

where

$$\psi_\mu = R_\mu(\overset{\circ}{T})B\varphi_\mu(A) \in D(A). \tag{4.10}$$

Then,

$$\begin{aligned} T_\omega\varphi_\mu(T_\omega) &= T_\omega(\varphi_\mu(A_\omega) - \psi_\mu) \\ &= (A_\omega + B)\varphi_\mu(A_\omega) - T_\omega R_\mu(\overset{\circ}{T})B\varphi_\mu(A) \\ &= \mu\varphi_\mu(A_\omega) + B\varphi_\mu(A_\omega) - B\varphi_\mu(A_\omega) + \mu R_\mu(\overset{\circ}{T})B\varphi_\mu(A) \\ &= \mu\varphi_\mu(T_\omega). \end{aligned} \tag{4.11}$$

□

5. Applications

5.1. Perturbed first-order differential operator. Consider in $L^2(0, 2\pi)$ the operator $T = A + B$, where A is defined by $Ay = iy'$ with domain $D(A) = H_0^1(0, 2\pi)$ and B is given by

$$(By)(x) = \sum_{k=1}^n a_k(y, y_k)y_k(x), \tag{5.1}$$

where $y_1, y_2, \dots, y_n \in L^2(0, 2\pi)$ and $a_k \in \mathbb{R}$, for all $k = \overline{1, n}$. From [1, 2], the operator A is regular symmetric of deficiency indices $(1, 1)$ and each selfadjoint extension of A has a discrete spectrum.

THEOREM 5.1. *The generalized resolvent $R_\lambda(T_\theta)$ of T , corresponding to the function $\omega(\lambda) = \theta(\lambda)$, is an integral operator with kernel*

$$K(x, t) = \left[1_{[x, 2\pi]}(x) + \frac{1}{\theta(\lambda)e^{2\pi\lambda i} + 1} \right] e^{i\lambda(t-x)} + \sum_{k=1}^n \theta_k(\lambda, x)\phi_k(\lambda, t), \tag{5.2}$$

where $1_{[x, 2\pi]}(x)$ is the characteristic function of the interval $[x, 2\pi]$,

$$\phi_k(\lambda, t) = (\Delta_k^\theta(\lambda))(t), \quad \theta_k(\lambda, x) = \frac{(R_\lambda(A_\theta)y_k)(x)}{\Delta^\theta(\lambda)}, \tag{5.3}$$

where $R_\lambda(A_\theta)$, associated to the function $\theta(\lambda)$, is given by

$$(R_\lambda(A_\theta)y)(x) = \int_0^x y(t)e^{i\lambda(t-x)} dt - \frac{1}{\theta(\lambda)e^{2\pi\lambda i} + 1} \int_0^{2\pi} y(t)e^{\lambda i(t-x)} dt \tag{5.4}$$

with

$$\Delta^\theta(\lambda) = \{\delta_{k_j} + a_k(R_\lambda(A_\theta)y_j, y_k)\}, \tag{5.5}$$

and Δ_k^θ is the determinant obtained from $\overline{\Delta^\theta(\lambda)}$ replacing the k th column by $[a_k R_\lambda(A_\theta)y_k]^n$.

Proof. The proof results from [26] and Theorem 3.3. □

COROLLARY 5.2. *Let T_θ be a selfadjoint extension of T corresponding to the function θ , $|\theta| = 1$.*

(1) *The spectrum of T_θ is simple if and only if the roots of $\Delta^\theta(\lambda)$ are simple and for $k = 0, \pm 1, \pm 2, \dots, \Delta^\theta(1/2 + k - \varphi_0/2\pi) \neq 0$, where $\{1/2 + k - \varphi_0/2\pi\}$ is the spectrum of A_θ , and $\varphi_0 = \arg \theta$.*

(2) *$\sigma(T_\theta) = P\sigma(T_\theta) = E_1 \cup E_2$, where E_1 is the set of points of $\sigma(A_\theta) = \{1/2 + k - \varphi_0/2\pi, k = 0, \pm 1, \pm 2, \dots\}$ in which $\Delta^\theta(\lambda)$ is analytic, E_2 is the set of roots of $\Delta^\theta(\lambda)$.*

Proof. The proof results from (5.4), Theorem 3.7, and Lemma 4.1. □

5.2. Perturbed second-order differential operator. Consider in $L^2(0, \infty)$ the operator $T = A + B$, where A is defined by

$$Ay = -y'' + x^2y \tag{5.6}$$

with domain $D(A)$ consisting of all variables y which satisfy

- (i) $y \in L^2(0, \infty)$,
- (ii) y' is absolutely continuous on all compact subintervals of $[0, \infty[$,
- (iii) $Ay \in L^2(0, \infty)$,
- (IV) $y(0) = y(\infty) = \lim_{x \rightarrow \infty} y(x) = 0, y'(0) = y'(\infty) = 0$,

and B is given by

$$(By)(x) = \sum_{k=1}^n a_k(y, y_k)y_k(x), \tag{5.7}$$

where $y_1, y_2, \dots, y_n \in L^2(0, 2\pi)$ and $a_k \in \mathbb{R}$, for all $k = \overline{1, n}$.

From [1, 2], the operator A is symmetric of deficiency indices $(1, 1)$. Let u_1, u_2 be two solutions of (5.6), satisfying the initial conditions

$$\begin{aligned} u_1(0, \lambda) &= 1, & u_1'(x, \lambda)|_{x=0} &= 0, \\ u_2(0, \lambda) &= 0, & u_2'(x, \lambda)|_{x=0} &= -1. \end{aligned} \tag{5.8}$$

There exists a function $m(\lambda)$ [29] analytic in $\mathbb{C} \setminus \mathbb{R}$ such that

$$\psi(x, \lambda) = u_2(x, \lambda) + m(\lambda)u_1(x, \lambda) \in L^2(0, \infty). \tag{5.9}$$

THEOREM 5.3. *The generalized resolvents $R_\lambda(T_\theta)$ of the operator T are defined by*

$$R_\lambda(T_\theta)y = R_\lambda(A_\theta)y - \sum_{k=1}^n \frac{(y, \Delta_k^\theta(\lambda))}{\Delta^\theta(\lambda)} R_\lambda(A_\theta)y_k, \quad \text{Im } \lambda > 0, \tag{5.10}$$

where

$$R_\lambda(A_\theta)y = \psi(x, \lambda) \int_0^x y(s)u_1(s, \lambda)ds + u_1(x, \lambda) \int_x^\infty y(s)\psi(s, \lambda)ds - \frac{\psi(x, \lambda)}{\theta(\lambda) + m(\lambda)} \int_0^\infty y(s)\psi(s, \lambda)ds, \tag{5.11}$$

$$\Delta^\theta(\lambda) = \det \{ \sigma_{jk} + a_k (R_\lambda(A_\theta)y_j, y_k) \}, \quad \lambda \in \mathbb{C}^+, \tag{5.12}$$

with $\theta(\lambda)$ an arbitrary function analytic in \mathbb{C}^+ and such that $\text{Im}\theta(\lambda) \geq 0$ or $\theta(\lambda)$ is an infinite constant.

Proof. First, we show that for $\lambda \in \mathbb{C}^+$, $\Delta^\theta(\lambda) \neq 0$ (then, $\Delta^\theta \neq 0$). We know (see [1, 2]) that for each quasiselfadjoint extension of a symmetric operator, \mathbb{C}^+ is contained in the set of regular points of this operator. Then, if $\lambda \in \mathbb{C}^+$, we have $\lambda \in \rho(A_\theta)$ and $\lambda \in \rho(T_\theta)$. If we suppose that $\lambda \in \mathbb{C}^+$ and $\Delta^\theta(\lambda) = 0$, from Theorem 3.7, we obtain $\lambda \in P\sigma(T_\theta)$, which is a contradiction. The formula (5.11) results from [25]. Using Theorem 3.3, we end the proof. □

COROLLARY 5.4. *Let T_θ be a selfadjoint extension associated to $\theta \in \overline{IR}$, let $\lambda_1, \lambda_2, \dots$ be the roots of $\Delta^\theta(\lambda)$ in $\rho(A_\theta)$ and let z_1, z_2, \dots be the poles of $\Delta^\theta(\lambda)$. Then,*

$$P\sigma(T_\theta) = (P\sigma(A_\theta) \setminus \{z_i\}_1^\infty) \cup \{\lambda_j\}_1^\infty. \tag{5.13}$$

Proof. The proof results from (b) and (c) of Theorem 3.7. □

Acknowledgment

The author is grateful to the editor and the anonymous referees for their valuable comments and helpful suggestions which have much improved the presentation of the paper.

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