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Research Article

Boundedness and Large-Time Behavior Results for a Diffusive Epidemic Model

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We consider a reaction-diffusion system modeling the spread of an epidemic disease within a population divided into the susceptible and infective classes. We first consider the question of the uniform boundedness of the solutions for which we give a positive answer. Then we deal with the asymptotic behavior of the solutions where in particular we are interested in reasonable conditions leading to the extinction of the infection disease as the time goes to infinity.

1. Introduction

In this paper, we consider the following reaction-diffusion system of equations:

$$\begin{aligned} \frac{\partial S}{\partial t} - d_1 \Delta S &= \Lambda - \lambda(t) f(S,I) - \mu S \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial I}{\partial t} - d_2 \Delta I &= \lambda(t) f(S,I) - \sigma I \quad \text{in } \mathbb{R}^+ \times \Omega, \end{aligned} \tag{1.1}$$

with homogeneous Neuman boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.2}$$

and the nonnegative and bounded initial data

$$S(0,x) = S_0(x), \quad I(0,x) = I_0(x) \quad \text{in } \Omega, \tag{1.3}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and outer normal $\nu(x)$. The constants d_1, d_2, Λ, μ are such that

$$d_1 > 0, \qquad d_2 > 0, \qquad \mu > 0, \qquad \sigma > 0, \qquad \Lambda \geq 0. \tag{1.4}$$

We assume that $t \mapsto \lambda(t)$ is a nonnegative and bounded function in $C(\mathbb{R}^+)$ with $0 \leq \lambda(t) \leq \widehat{\lambda}$ and the nonlinearity $f(\xi,\eta)$ is a nonnegative differentiable function in $\mathbb{R}^+ \times \mathbb{R}^+$ such that there exist two increasing nonnegative functions φ and ψ in $C^1(\mathbb{R}^+)$ with

$$\xi \geq 0, \quad \eta \geq 0 \Longrightarrow 0 \leq f(\xi,\eta) \leq \psi(\xi)\varphi(\eta), \tag{1.5}$$

$$\psi(0) = 0, \qquad \varphi(0) = 0, \qquad \lim_{\eta\to+\infty} \frac{\ln(1+\varphi(\eta))}{\eta} = 0. \tag{1.6}$$

The reaction-diffusion system (1.1)–(1.3) may be viewed as a diffusive epidemic model where S and I represent the nondimensional population densities of susceptibles and infectives, respectively. In other words, system (1.1)–(1.3) is a model describing the spread of an infection disease (such as AIDS, e.g.) within a population assumed to be divided into the susceptible and infective classes as precised (for further motivation, see, e.g., [1–3], and the references therein).

A basic question arising in this context is the existence of global solutions in $C(\overline{\Omega})$ as well as their uniform boundedness to system (1.1)–(1.3). When $\Lambda = 0$ (which corresponds to the situation where there is no new supply in the susceptible class), a quite similar question was studied by many authors (see [4–6]) and a positive answer was first given by Haraux and Youkana [7] using the Lyapunov function techniques (see also [8]) and later on by Kanel (see, e.g., [9]) using useful properties inherent to the underlying Green function.

However when $\Lambda > 0$, these studies, while directly leading to conclude a global existence of the solutions, do not seem of a direct application concerning the uniform boundedness. To establish the uniform boundedness of the solutions in this case (i.e., when $\Lambda > 0$), it is worthwhile to mention the method developed by Morgan [10] which can be successfully applied to our case provided that $|\varphi(\eta)| \leq c\eta^{\beta}$, $\beta > 0$. Clearly the class considered in this work of φ satisfying the limit

$$\lim_{\eta\to+\infty} \frac{\ln(1+\varphi(\eta))}{\eta} = 0 \tag{1.7}$$

as it handles nonlinearities of a weakly exponential growth is larger than that required in [10] of nonlinearities of a polynomial growth. Indeed, it is easily observed for instance that $\varphi(\eta) = e^{\eta^{\alpha}} - 1$, $0 < \alpha < 1$, satisfies this limit. Unfortunately for the nonlinearities φ not of a polynomial growth and satisfying this limit, the method in [10] cannot be applied.

In this paper, we first consider this problem of uniform boundedness of the solutions to system (1.1)–(1.3) by proving that the Lyapunov function argument proposed in [7] (or in [8]) can be adapted to our situation. Interestingly, we show that the same Lyapunov function is not necessarily nonincreasing as established in [7, 8] but rather it satisfies a

differential inequality from which the uniform boundedness of the solutions is readily deduced.

Then we deal with the long-time behavior of the solutions as the time goes to $+\infty$ where in particular we are concerned with reasonable conditions allowing to assure that (S,I) goes to the infection-free state $(\Lambda/\mu,0)$ of system (1.1)–(1.3) as $t \to +\infty$ in the sense

$$\lim_{t\to+\infty} \|I(t,\cdot)\|_\infty = \lim_{t\to+\infty} \left\| S(t,\cdot) - \frac{\Lambda}{\mu} \right\|_\infty = 0, \tag{1.8}$$

which of course can fail for arbitrary λ in $L^\infty(\mathbb{R}^+) \cap C(\mathbb{R}^+)$. More precisely, we will show that this property is valid if $\lambda(t)$ satisfies either assumption (H.1) or assumption (H.2) formulated in what follows.

(H.1) There exists a real number $p \geq 1$ such that

$$\int_0^{+\infty} (\lambda(s))^p ds < +\infty. \tag{1.9}$$

(H.2) The function $\eta \mapsto \varphi(\eta)/\eta$ is increasing on $]0,+\infty[$ and $\lambda(t) \equiv \widehat{\lambda} > 0$ is a positive constant independent of t such that

$$\frac{\widehat{\lambda}}{\sigma} \frac{\varphi(N)}{N} \psi\left(\frac{\Lambda}{\mu}\right) < 1, \tag{1.10}$$

where $N > 0$ is a positive constant independent of t of which the expression will be explicitly given in Lemma 2.3 in the next section.

2. Boundedness of the solutions

The basic existence theory for abstract semilinear differential equations directly leads to conclude a local existence result to system (1.1)–(1.3) (see, e.g., Henry [11] or Pazy [12]). Thus for nonnegative S_0, I_0 in the class $L^\infty(\Omega)$, there exists a unique local nonnegative solution (S,I) of class $C(\overline{\Omega})$ of system (1.1)–(1.3) on $]0,T^*[$, where T^* is the eventual blowing-up time in $L^\infty(\Omega)$.

On the other hand, using the comparison principle, one may also show that

$$0 \leq S(t,x) \leq \max\left(\|S_0\|_\infty, \frac{\Lambda}{\mu}\right) =: K \quad \forall (t,x) \in]0,T^*[\times \Omega, \tag{2.1}$$

from which it follows that the solutions S and I of system (1.1)–(1.3) are global and uniformly bounded as soon as we can show that I is uniformly bounded in $]0,T^*[$.

Following Haraux and Youkana [7], let us consider the function

$$L(t) = \int_\Omega (1+\delta(S+S^2))e^{\varepsilon I} dx \tag{2.2}$$

defined on $]0,T^*[$, where δ and ε are positive constants satisfying

$$0<\delta\le\min\left(\frac{\sigma}{2\Lambda(1+2K)},\frac{8d_1d_2}{(1+2K)^2(d_1+d_2)^2}\right),\qquad 0<\varepsilon\le\frac{\delta}{1+\delta(K+K^2)}. \tag{2.3}$$

The main result of the paper can be stated as follows.

Theorem 2.1. *For the solution (S,I) of system (1.1)–(1.3) in $]0,T^*[$, let $L(t)$ be the function defined by (2.2) with δ and ε satisfying (2.3). Then there exists a nonnegative constant a such that*

$$\frac{d}{dt}L(t)\le-\frac{\sigma}{2}L(t)+a. \tag{2.4}$$

Proof. Let (S,I) be the solution of system (1.1)–(1.3) in $]0,T^*[$. Differentiating $L(t)$ defined by (2.2) with respect to t and using Green's formula, one obtains

$$\frac{d}{dt}L(t)=G+H, \tag{2.5}$$

where

$$\begin{aligned}
G&=-2d_1\delta\int_\Omega e^{\varepsilon I}(\nabla S)^2dx\\
&\quad-(d_1+d_2)\varepsilon\delta\int_\Omega(1+2S)e^{\varepsilon I}\nabla I\nabla S\,dx\\
&\quad-d_2\varepsilon^2\int_\Omega\left(1+\delta(S+S^2)\right)e^{\varepsilon I}(\nabla I)^2dx,\\
H&=\int_\Omega\left(\Lambda\frac{\delta(1+2S)}{1+\delta(S+S^2)}-\mu S\frac{\delta(1+2S)}{1+\delta(S+S^2)}\right)\left(1+\delta(S+S^2)\right)e^{\varepsilon I}dx\\
&\quad+\int_\Omega\lambda(t)\left(\varepsilon-\frac{\delta(1+2S)}{1+\delta(S+S^2)}\right)f(S,I)\left(1+\delta(S+S^2)\right)e^{\varepsilon I}dx\\
&\quad-\int_\Omega\varepsilon\sigma I\left(1+\delta(S+S^2)\right)e^{\varepsilon I}dx.
\end{aligned} \tag{2.6}$$

We observe that G involves a quadratic form with respect to ∇S and ∇I,

$$Q=2d_1\delta e^{\varepsilon I}(\nabla S)^2+(d_1+d_2)\varepsilon\delta(1+2S)e^{\varepsilon I}\nabla I\nabla S+d_2\varepsilon^2\left(1+\delta(S+S^2)\right)e^{\varepsilon I}(\nabla I)^2, \tag{2.7}$$

which is nonnegative since the constants δ and ε satisfying (2.3) are chosen in such a way that the discriminant

$$\left[(d_1+d_2)\varepsilon\delta(1+2S)e^{\varepsilon I}\right]^2-4\left[2d_1\delta e^{\varepsilon I}\right]\left[d_2\varepsilon^2\left(1+\delta(S+S^2)\right)e^{\varepsilon I}\right] \tag{2.8}$$

is ≤ 0 so that one concludes that $G\le 0$ a.e. on $]0,T^*[$ (see [7]). On the other hand, H may be written as follows:

$$H=H_1+H_2+H_3, \tag{2.9}$$

such that

$$\begin{aligned}
H_1 &= \int_\Omega \left(\Lambda \frac{\delta(1+2S)}{1+\delta(S+S^2)} - \mu S \frac{\delta(1+2S)}{1+\delta(S+S^2)} - \sigma\right)(1+\delta(S+S^2))e^{\varepsilon I}dx,\\
H_2 &= \int_\Omega \lambda(t)\left(\varepsilon - \frac{\delta(1+2S)}{1+\delta(S+S^2)}\right) f(S,I)(1+\delta S+\delta S^2)e^{\varepsilon I}dx,\\
H_3 &= \int_\Omega \sigma(1-\varepsilon I)e^{\varepsilon I}(1+\delta S+\delta S^2)dx.
\end{aligned} \tag{2.10}$$

Again from (2.3) where now

$$0<\delta \le \frac{\sigma}{2\Lambda(1+2K)}, \qquad 0<\varepsilon \le \frac{\delta}{1+\delta(K+K^2)}, \tag{2.11}$$

one checks that

$$\begin{gathered}
\Lambda \frac{\delta(1+2S)}{1+\delta(S+S^2)} - \mu S \frac{\delta(1+2S)}{1+\delta(S+S^2)} - \sigma \le \Lambda\delta(1+2K)-\sigma \le -\frac{\sigma}{2},\\
\varepsilon - \frac{\delta(1+2S)}{1+\delta(S+S^2)} \le \varepsilon - \frac{\delta}{1+\delta(K+K^2)} \le 0,
\end{gathered} \tag{2.12}$$

from which it is obviously deduced that $H_2 \le 0$ and

$$H_1 \le -\frac{\sigma}{2}L(t). \tag{2.13}$$

Concerning H_3, one observes that the function

$$\pi : \eta \longmapsto (1-\varepsilon\eta)e^{\varepsilon\eta} \tag{2.14}$$

is bounded on $\mathbb{R}^+$. Indeed, one has

$$\frac{d\pi}{d\eta}(\eta) = -\varepsilon^2\eta e^{\varepsilon\eta} \le 0, \tag{2.15}$$

so that π is nonincreasing in $[0,+\infty[$ and

$$\max_{\eta\ge 0}(1-\varepsilon\eta)e^{\varepsilon\eta} = 1. \tag{2.16}$$

Let now

$$a := \sigma(1+\delta(K+K^2))|\Omega| \tag{2.17}$$

be chosen on purpose in such a way that $H_3 \le a$. To sum up, one has

$$\frac{d}{dt}L(t) = G+H = G+H_1+H_2+H_3 \le -\frac{\sigma}{2}L(t)+a \tag{2.18}$$

exactly as the theorem claimed. □

We are now ready to establish the global existence and uniform boundedness of the solutions of (1.1)–(1.3).

Theorem 2.2. *If f satisfies conditions (1.5) and (1.6), the solutions S and I of system (1.1)–(1.3) with nonnegative and bounded initial data S_0, I_0 are global and uniformly bounded on* $[0,+\infty[$.

Proof. Let (S,I) be the solution of system (1.1)–(1.3) in $]0,T^*[$. Multiplying inequality (2.4) by $e^{(\sigma/2)t}$ and then integrating over $[0,t]$, we deduce that there exists a positive constant $C>0$ independent of t such that

$$L(t) \le 2\left(1+\delta\left(K+K^2\right)\right)|\Omega| + Ce^{-(\sigma/2)t} \quad \text{on }]0,T^*[. \tag{2.19}$$

In this proof, we will make use of the result established in [13] from which the uniform boundedness of I is derived once,

$$\left|\left|\lambda(t)f(S,I)-\sigma I\right|\right|_p \le C_1(p), \tag{2.20}$$

(where $C_1(p)$ is a positive constant independent of t) for some $p>n/2$. In this direction, we observe that

$$\left|\left|\lambda(t)f(S,I)-\sigma I\right|\right|_p \le \left|\left|\lambda(t)f(S,I)\right|\right|_p + \sigma\|I\|_p \le \hat{\lambda}\psi(K)\left|\left|\varphi(I)\right|\right|_p + \sigma\|I\|_p, \tag{2.21}$$

and both $\varphi(\eta)$ and η satisfy

$$\lim_{\eta\to+\infty}\frac{\ln\left(1+\varphi(\eta)\right)}{\eta} = \lim_{\eta\to+\infty}\frac{\ln(1+\eta)}{\eta} = 0, \tag{2.22}$$

so that it is quite sufficient to establish that

$$\left|\left|\varphi(I)\right|\right|_p \le C_2(p), \tag{2.23}$$

(where $C_2(p)$ is a positive constant independent of t) for some $p>n/2$.

To that purpose, let $\delta>0$ and $\varepsilon>0$ be two positive numbers satisfying (2.3). It is readily seen from (1.6) that there exists $\eta_0 \ge 0$ such that

$$\eta \ge \eta_0 \Longrightarrow \max\left(\eta,\varphi(\eta)\right) \le e^{(\varepsilon/n)\eta}, \tag{2.24}$$

from which one gets the following estimates:

$$\begin{aligned}
\left|\left|\varphi(I)\right|\right|_n^n &= \int_{I\le\eta_0}\left(\varphi(I)\right)^n dx + \int_{I\ge\eta_0}\left(\varphi(I)\right)^n dx \\
&\le \left(\varphi(\eta_0)\right)^n|\Omega| + \int_\Omega e^{\varepsilon I}dx \le \left(\varphi(\eta_0)\right)^n|\Omega| + L(t) \\
&\le \left(\varphi(\eta_0)\right)^n|\Omega| + 2\left(1+\delta\left(K+K^2\right)\right)|\Omega| + Ce^{-(\sigma/2)t}.
\end{aligned} \tag{2.25}$$

Hence, one merely lets

$$C_2(n) = \sqrt[n]{(\varphi(\eta_0))^n|\Omega| + 2(1+\delta(K+K^2))|\Omega| + C} \tag{2.26}$$

in order to obtain (2.23) and thereby (2.20). As precised, the result established in [13] permits to deduce the uniform boundedness of the solutions of (1.1)–(1.3) and the theorem is completely proved. □

In order to make assumption (H.2) stated in the introduction meaningful, we expose the following result where we establish the expression defining the positive constant N introduced in (H.2) as well as the property it enjoys. It will be soon observed that N depends on positive constants $M(r,n)$ and $C(r,n)$ issued from known embedding theorems. To be more precise, we refer the reader to the appendix where the existence of $M(r,n)$ is shown in (P.2) of Lemma A.1 and that of $C(r,n)$ is claimed in Lemma A.2.

We merely say here that these constants $M(r,n)$ and $C(r,n)$ are supposed to be available in the following lemma.

Lemma 2.3. *Let*

$$N = C\left(\frac{3}{4},n\right)M\left(\frac{3}{4},n\right)(1+6(\hat{\lambda}\psi(K)+\sigma))\{([\varphi(\eta_0)]^n + 2[1+\delta(K+K^2)])|\Omega|\}^{1/n}, \tag{2.27}$$

where K, δ, and η_0 are the constants defined by (2.1), (2.3), and (2.24), respectively. Then for all $(t,x) \in \mathbb{R}^+ \times \Omega$,

$$I(t,x) \le N + C \cdot e^{-(\sigma/2)t}, \tag{2.28}$$

where C is a positive constant.

To keep the flow of the main objectives of this work, we postpone to the appendix the proof of this lemma which is rather technical and somewhat long.

3. Asymptotic behavior of the solutions

In this section, we deal with the large-time behavior of the solutions S and I of system (1.1)–(1.3) as $t \to +\infty$. Before stating the results, let us expose some notations and simple facts concluded from the results of the previous section. First, thanks to Theorem 2.2, let $R > 0$ be a positive constant independent of t such that

$$I(t,x) \le R \quad \text{on } \mathbb{R}^+ \times \Omega, \tag{3.1}$$

and set

$$\theta_q = \sup_{0\le\zeta\le\eta\le R} (\varphi'(\zeta)^q \eta^{q-1}) \tag{3.2}$$

for $q \geq 1$ so that using the mean value theorem, one checks that for all $(t,x) \in \mathbb{R}^+ \times \Omega$ and $q \geq 1$,

$$\varphi(I)^q \leq \theta_q I. \tag{3.3}$$

On the other hand, let us observe that the application of the maximum principle directly implies that

$$0 \leq S(t,x) \leq \frac{\Lambda}{\mu}(1 - e^{-\mu t}) + \|S_0\|_\infty e^{-\mu t} \tag{3.4}$$

so that if we set

$$J = \frac{\Lambda}{\mu}(1 - e^{-\mu t}) + \|S_0\|_\infty e^{-\mu t} - S, \tag{3.5}$$

one obtains

$$\begin{aligned} \frac{\partial J}{\partial t} - d_1 \Delta J &= \lambda(t) f(S,I) - \mu J \quad \text{in } \mathbb{R}^+ \times \Omega, \\ J(0,x) = J_0(x) &= \|S_0\|_\infty - S_0(x) \quad \text{in } \Omega, \\ \frac{\partial J}{\partial \nu} &= 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \end{aligned} \tag{3.6}$$

and $0 \leq J(t,x) \leq (\Lambda/\mu)(1 - e^{-\mu t}) + \|S_0\|_\infty e^{-\mu t}$. We observe that both I and J satisfy a parabolic equation of the same kind, namely

$$\begin{aligned} \frac{\partial V}{\partial t} - d \Delta V &= \lambda(t) f(S,I) - \rho V \quad \text{in } \mathbb{R}^+ \times \Omega, \\ V(0,x) &= V_0(x) \quad \text{in } \Omega, \\ \frac{\partial V}{\partial \nu} &= 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \end{aligned} \tag{3.7}$$

with

$$V = \begin{cases} I & \text{if } d = d_2,\ \rho = \sigma,\ V_0 = I_0, \\ J & \text{if } d = d_1,\ \rho = \mu,\ V_0 = J_0. \end{cases} \tag{3.8}$$

The results of this section are based on the preliminary lemma below.

Lemma 3.1. *Suppose that $\int_0^{+\infty} \int_\Omega I \, dx \, ds < +\infty$, where (S,I) is the global and bounded solution to system (1.1)–(1.3). Then as $t \to +\infty$,*

$$\left\| S(t,\cdot) - \frac{\Lambda}{\mu} \right\|_\infty \longrightarrow 0, \tag{3.9}$$

$$\|I(t,\cdot)\|_\infty \longrightarrow 0. \tag{3.10}$$

Proof. Let us multiply by V the parabolic equation (3.7)-(3.8) satisfied by V, integrate over Ω, and use Green's formula so that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega V^2dx+d\int_\Omega(\nabla V)^2dx\\ =\lambda(t)\int_\Omega Vf(S,I)dx-\rho\int_\Omega V^2dx\le\hat{\lambda}\left(\int_\Omega Vf(S,I)dx\right)-\rho\int_\Omega V^2dx.\end{aligned}\tag{3.11}$$

As a consequence, letting $R^*=\max(K,R)$,

$$E(V)=\frac{1}{2}\int_\Omega V(t,x)^2dx+d\int_0^t\int_\Omega(\nabla V)^2dx\,ds+\rho\int_0^t\int_\Omega V^2dx\,ds,\tag{3.12}$$

using (2.1), (3.1), (3.3), (3.6), and integrating over $(0,t)$,

$$\begin{aligned}E(V)&\le\hat{\lambda}\int_0^t\int_\Omega Vf(S,I)dx\,ds+\frac{1}{2}\int_\Omega V(0,x)^2dx\\ &\le\hat{\lambda}R^*\psi(K)\theta_1\int_0^t\int_\Omega I\,dx\,ds+\frac{1}{2}\int_\Omega V(0,x)^2dx,\end{aligned}\tag{3.13}$$

from which one obviously deduces that

$$V(t,\cdot)\in L^2(\Omega),\quad\int_0^{+\infty}\int_\Omega(\nabla V)^2dx\,dt<+\infty,\quad\int_0^{+\infty}\int_\Omega V^2dx\,dt<+\infty,\tag{3.14}$$

so that Barbalate's lemma (see [14, Lemma 1.2.2]) permits to conclude that

$$\lim_{t\to+\infty}\|V(t,\cdot)\|_2=0.\tag{3.15}$$

On the other hand, since the orbit $\{V(t,\cdot)/t\ge0\}$ of the equation verified by V is (on account of the uniform boundedness of S and I) relatively compact (see, e.g., [13]), it readily follows that

$$\lim_{t\to+\infty}\|V(t,\cdot)\|_\infty=0.\tag{3.16}$$

Hence limit (3.9) is verified. Since

$$\begin{aligned}\left\|S(t,\cdot)-\frac{\Lambda}{\mu}\right\|_\infty&=\left\|\frac{\Lambda}{\mu}(1-e^{-\mu t})+\|S_0\|_\infty e^{-\mu t}-S+e^{-\mu t}\left(\frac{\Lambda}{\mu}-\|S_0\|_\infty\right)\right\|_\infty\\ &\le\|J(t,\cdot)\|_\infty+e^{-\mu t}\left|\frac{\Lambda}{\mu}-\|S_0\|_\infty\right|,\end{aligned}\tag{3.17}$$

limit (3.10) is also valid and the lemma is proved. □

Our first result of this section regarding the asymptotic behavior can be stated as follows.

Theorem 3.2. *Let assumption (H.1) hold and let (S,I) be the solution of (1.1)–(1.3) in $[0,+\infty[$. Then as $t \to +\infty$,*

$$\begin{gathered} \left\| S(t,\cdot) - \frac{\Lambda}{\mu} \right\|_\infty \longrightarrow 0, \\ \|I(t,\cdot)\|_\infty \longrightarrow 0. \end{gathered} \tag{3.18}$$

Proof. According to assumption (H.1), there exists $p \geq 1$ such that

$$\int_0^{+\infty} (\lambda(t))^p dt = \alpha < +\infty. \tag{3.19}$$

Let $q > 1$ be the dual number of p, that is

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{3.20}$$

We assume for simplicity that $p > 1$ and $q < +\infty$ since the cases $p = 1$ and $q = +\infty$ can be treated similarly. Integrating the parabolic equation satisfied by I over Ω and using Holder's inequality and (3.3), we get

$$\begin{aligned} \frac{\partial}{\partial t} \int_\Omega I\,dx &= \lambda(t) \int_\Omega f(S,I)dx - \sigma \int_\Omega I\,dx \\ &\leq \lambda(t)\psi(K) \int_\Omega \varphi(I)dx - \sigma \int_\Omega I\,dx \\ &\leq \lambda(t)\psi(K)|\Omega|^{1/p} \left(\int_\Omega (\varphi(I))^q dx \right)^{1/q} - \sigma \int_\Omega I\,dx \\ &\leq \lambda(t)\theta_q^{1/q}\psi(K)|\Omega|^{1/p} \left(\int_\Omega I dx \right)^{1/q} - \sigma \int_\Omega I\,dx. \end{aligned} \tag{3.21}$$

Therefore integrating over $[0,t]$ and using again Holder's inequality, we obtain

$$\begin{aligned} \int_\Omega I\,dx + \sigma \int_0^t \int_\Omega I\,dx\,ds &\leq \theta_q^{1/q}\psi(K)|\Omega|^{1/p} \int_0^t \lambda(s) \left(\int_\Omega I\,dx \right)^{1/q} ds + \int_\Omega I_0 dx \\ &\leq \theta_q^{1/q}\psi(K)|\Omega|^{1/p} \left(\int_0^t (\lambda(s))^p ds \right)^{1/p} \left(\int_0^t \int_\Omega I\,dx\,ds \right)^{1/q} + \|I_0\|_\infty |\Omega|. \end{aligned} \tag{3.22}$$

Let $B(t) = (\int_0^t \int_\Omega I\,dx\,ds)^{1/q}$ so that

$$\sigma B(t)^q - \theta_q^{1/q}\psi(K)(\alpha|\Omega|)^{1/p} B(t) - \|I_0\|_\infty |\Omega| \leq 0, \tag{3.23}$$

and consequently

$$\left(\int_0^t \int_\Omega I\,dx\,ds \right)^{1/q} \leq \omega, \tag{3.24}$$

where ω is the unique positive root of

$$\sigma X^q - A_q \psi(K)(\alpha|\Omega|)^{1/p} X - ||I_0||_\infty |\Omega| \tag{3.25}$$

in $\mathbb{R}^+$. We directly deduce that

$$\int_0^{+\infty} \int_\Omega I\,dx\,ds < +\infty. \tag{3.26}$$

By virtue of Lemma 3.1, limits (3.9) and (3.10) are satisfied and Theorem 3.2 is completely proved. □

The second result of this section concerns also the large-time behavior of the solutions and can be stated as follows.

Theorem 3.3. *Let assumption (H.2) hold with N defined by expression (2.27) introduced in Lemma 2.3. Then as $t \to +\infty$,*

$$\begin{gathered} \left|\left|S(t,\cdot) - \frac{\Lambda}{\mu}\right|\right|_\infty \longrightarrow 0, \\ ||I(t,\cdot)||_\infty \longrightarrow 0. \end{gathered} \tag{3.27}$$

Proof. Let us consider the parabolic equation below satisfied by I:

$$\begin{aligned} \frac{\partial I}{\partial t} - d_2 \Delta I &= \lambda(t) f(S,I) - \sigma I \quad \text{in } \mathbb{R}^+ \times \Omega, \\ I(0,x) &= I_0(x) \quad \text{in } \Omega, \\ \frac{\partial I}{\partial \nu} &= 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega. \end{aligned} \tag{3.28}$$

Therefore, thanks to Lemma 2.3, we obtain

$$\frac{\partial I}{\partial t} - d_2 \Delta I \le \hat{\lambda} \varphi(I) \psi(S) - \sigma I \le \left(\hat{\lambda} \frac{\varphi(N + Ce^{-(\sigma/2)t})}{N + Ce^{-(\sigma/2)t}} \psi(J+S) - \sigma\right) I \tag{3.29}$$

since, owing to assumption (H.2), $\varphi(I) \le (\varphi(N + Ce^{-(\sigma/2)t})/(N + Ce^{-(\sigma/2)t})I)$. On the other hand, one has

$$\begin{gathered} \lim_{t \to +\infty} \frac{\varphi(N + Ce^{-(\sigma/2)t})}{N + Ce^{-(\sigma/2)t}} = \frac{\varphi(N)}{N}, \\ \lim_{t \to +\infty} \psi(J(t,x) + S(t,x)) = \psi\left(\frac{\Lambda}{\mu}\right). \end{gathered} \tag{3.30}$$

As a consequence by applying assumption (H.2) once more where $\varphi(N)/N < \sigma/\hat{\lambda}\psi(\Lambda/\mu)$, it follows that there exist $T \ge 1$ and $\kappa > 0$ such that

$$t \ge T \Longrightarrow \hat{\lambda} \frac{\varphi(N + Ce^{-(\sigma/2)t})}{N + Ce^{-(\sigma/2)t}} \psi(J+S) - \sigma \le -\kappa < 0. \tag{3.31}$$

The application of the maximum principle directly yields

$$t \geq T, \quad x \in \Omega \Longrightarrow 0 \leq I(t,x) \leq e^{-\kappa(t-T)} \|I(T,\cdot)\|_\infty, \tag{3.32}$$

from which it follows that

$$\|I(t,\cdot)\|_\infty \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty. \tag{3.33}$$

More importantly, the integral over $\mathbb{R}^+ \times \Omega$,

$$\begin{aligned} \int_0^{+\infty} \int_\Omega I\,dx\,ds &= \int_0^T \int_\Omega I\,dx\,ds + \int_T^{+\infty} \int_\Omega I\,dx\,ds \\ &\leq \int_0^T \int_\Omega I\,dx\,ds + \frac{1}{\kappa} M|\Omega| < +\infty, \end{aligned} \tag{3.34}$$

is finite so that by virtue of Lemma 3.1, limits (3.9) and (3.10) are valid and Theorem 3.3 is completely proved. □

Remark 3.4. In the light of the proof of Theorem 3.3, it is clear that the constant N defined by (2.27) and required in assumption (H.2) might be replaced by any other positive constant, say N', such that for all $(t,x) \in \mathbb{R}^+ \times \Omega$,

$$I(t,x) \leq N' + \varepsilon(t), \tag{3.35}$$

where $\varepsilon(t)$ is a nonnegative function with $\lim_{t\to 0} \varepsilon(t) = 0$.

Appendix

A. Proof of Lemma 2.3

The positive constant N, defined by (2.27) and satisfying the estimate $I(t,x) \leq N + C \cdot e^{-(\sigma/2)t}$ with C a positive constant, is constructed by applying variation of constants and by introducing fractional powers. In this respect, this appendix is logically divided into two subsections A.1 and A.2. While in the second subsection we proceed to the effective proof of Lemma 2.3, the first one is devoted to brief statements of some known aspects on the semigroup formulation and the fractional powers.

Preliminary estimates. Let us recall some classical facts about the semigroup formulation and the fractional powers by following [6]. For $p > 1$, let us define the operator A on $L^p(\Omega)$ by

$$\begin{gathered} A_p u = d_2 \Delta u \quad \text{for } u \in D(A), \\ D(A_p) = \left\{ u \in W^{2,p}(\Omega) \Big/ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega \right\}, \end{gathered} \tag{A.1}$$

where $W^{2,p}(\Omega)$ is the usual Sobolev space. It is well known that A generates a compact analytic semigroup

$$\mathcal{S}_p = \{ e^{tA_p} / t \geq 0 \} \tag{A.2}$$

of bounded linear operators on $L^p(\Omega)$ and that

$$\left|\left|e^{tA_p}u\right|\right|_p \leq \|u\|_p \quad \text{for } t \geq 0,\ u \in L^p(\Omega), \tag{A.3}$$

where

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}. \tag{A.4}$$

It is also a well-known fact that for $r > 0$, the fractional powers $(I - A_p)^{-r}$ exist and are injective bounded linear operators on $L^p(\Omega)$ (see, e.g., [12]).

For $r > 0$, let $B_p^r = ((I - A_p)^{-r})^{-1}$ and recall that $D(B_p^r)$ is a Banach space with the graph norm $|||u||\|_{r,p} = \|B_p^r u\|_p$ and that if $r > s \geq 0$ (where conventionally $L^p(\Omega) = D(B_p^0)$), then $D(B_p^r)$ is a dense space of $D(B_p^s)$ with the inclusion $D(B_p^r) \subset D(B_p^s)$ compact (see, e.g., [12]). Here we will make use of the following two lemmas.

Lemma A.1. *For the semigroup $\mathcal{S}_p$ and the fractional powers B_p^r just considered, one has*

$$t > 0, \quad u \in L^p(\Omega) \Longrightarrow e^{tA_p}u \in D(B_p^r), \tag{P.1}$$

$$t > 0, \quad u \in L^p(\Omega) \Longrightarrow \left|\left|B_p^r e^{tA_p}u\right|\right|_p \leq M(r,p)t^{-r}\|u\|_p, \tag{P.2}$$

$$t > 0, \quad u \in L^p(\Omega) \Longrightarrow B_p^r e^{tA_p}u = e^{tA_p}B_p^r u, \tag{P.3}$$

where $M(r,p) > 0$ is a positive constant independent of t.

Proof. For the proof of this lemma, we refer the reader to Pazy [12, page 74, Theorem 6.13]. □

Lemma A.2. *Suppose that a fractional power B_p^r (defined above) is such that $r > n/2p$. Then $D(B_p^r) \subset L^\infty(\Omega)$ and*

$$\|u\|_\infty \leq C(r,p)\left|\left|B_p^r u\right|\right|_p, \tag{A.5}$$

where $C(r,p) > 0$ is a positive constant.

Proof. The proof of this lemma can be readily deduced by applying Theorem 1.6.1 exposed in [11, page 39]. □

Construction of the constant N. In the sequel, we assume that $C > 0$ is a generic positive constant changing values from line to line. In the proof of Theorem 2.2, we have in fact shown because of (2.24) that

$$\begin{aligned} \left|\left|I(t)\right|\right|_n, \left|\left|\varphi(I(t))\right|\right|_n &\leq \left\{[\varphi(\eta_0)]^n|\Omega| + 2[1+\delta(K+K^2)]|\Omega| + Ce^{-(\sigma/2)t}\right\}^{1/n} \\ &\leq \left\{[\varphi(\eta_0)]^n|\Omega| + 2[1+\delta(K+K^2)]|\Omega|\right\}^{1/n} + Ce^{-(\sigma/2)t}, \end{aligned} \tag{A.6}$$

where $I(t)(x) = I(t,x)$ and $\varphi(I(t))(x) = \varphi(I(t,x))$. Accordingly, let

$$G(t)(x) = \lambda(t)f(S(t,x),I(t,x)) - \sigma I(t,x). \tag{A.7}$$

Applying variation of constants, one can write for $t_0 \geq 0$ and $r > 0$ that

$$\begin{aligned} I(t) &= e^{(t-t_0)A_n} I(t_0) + \int_{t_0}^{t} e^{(t-\tau)A_n} G(\tau)\,d\tau, \\ B_n^r I(t) &= B_n^r e^{(t-t_0)A_n} I(t_0) + \int_{t_0}^{t} B_n^r e^{(t-\tau)A_n} G(\tau)\,d\tau, \end{aligned} \tag{A.8}$$

and using Lemma A.1,

$$\begin{aligned} \|B_n^r I(t)\|_n &\leq \|B_n^r e^{(t-t_0)A_n} I(t_0)\|_n + \int_{t_0}^{t} \|B_n^r e^{(t-\tau)A_n} G(\tau)\|_n\,d\tau \\ &\leq M(r,n)\left[(t-t_0)^{-r}\|I(t_0)\|_n + \int_{t_0}^{t}(t-\tau)^{-r}\|G(\tau)\|_n\,d\tau\right] \\ &\leq M(r,n)\gamma\left[(t-t_0)^{-r} + (\hat{\lambda}\psi(K)+\sigma)\int_{t_0}^{t}(t-\tau)^{-r}\,d\tau\right] \\ &\quad + C\int_{t_0}^{t}(t-\tau)^{-r}e^{-(\sigma/2)\tau}\,d\tau \end{aligned} \tag{A.9}$$

with the constant $M(r,n) > 0$ given in Lemma A.1 and

$$\gamma = \{([\varphi(\eta_0)]^n + 2[1+\delta(K+K^2)])|\Omega|\}^{1/n}. \tag{A.10}$$

Set $t_0 = \lfloor t \rfloor - 1$, where $\lfloor t \rfloor$ denotes the floor of t (i.e., the largest integer less than or equal t). We have for $t \geq 1$ that

$$\begin{aligned} \|B_n^r I(t)\|_n &\leq M(r,n)\gamma\left(1 + \frac{(\hat{\lambda}\psi(K)+\sigma)}{1-r}(t-t_0)^{1-r}\right) + Ce^{-(\sigma/2)t} \\ &\leq M(r,n)\gamma\left(1 + \frac{(\hat{\lambda}\psi(K)+\sigma)}{1-r}2^{1-r}\right) + Ce^{-(\sigma/2)t}. \end{aligned} \tag{A.11}$$

Now, we set $r = 3/4 > n/2n$ so that by virtue of Lemma A.2 with the positive constant $C(3/4,n) > 0$ introduced therein, one claims that

$$I(t,x) \leq N + Ce^{-(\sigma/2)t} \quad \forall t \geq 1,\ x \in \Omega, \tag{A.12}$$

where

$$N = C\left(\frac{3}{4},n\right)M\left(\frac{3}{4},n\right)(1+6(\hat{\lambda}\psi(K)+\sigma))\{([\varphi(\eta_0)]^n + 2[1+\delta(K+K^2)])|\Omega|\}^{1/n}. \tag{A.13}$$

Hence Lemma 2.3 is proved.

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