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*Research Article*

# Lie Group Analysis of a Flow with Contaminant-Modified Viscosity

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A class of coupled system of diffusion equations is considered. Lie group techniques resulted in a rich array of admitted point symmetries for special cases of the source term. We also employ potential symmetry methods for chosen cases of concentration and a zero source term. Some invariant solutions are constructed using both classical Lie point and potential symmetries.



## 1. Introduction

There has been a great interest in studying the fluid dynamics of pollutants in rivers, due to increasingly important environmental and engineering problems, and more and more refined models have been proposed [1–6]. Shulka [4] obtained analytical solutions by the Fourier transform method for the case of unsteady transport dispersion of nonconservative pollutant/biochemical oxygen demand with first-order decay under each of the sine and cosine variations of waste discharge concentration at upstream boundary and nonzero initial condition through the river.

In this paper, we focus on a system of partial differential equations derived from both Navier-Stokes and concentration equations for fluid flow. We employ the methods of group of transformations for differential equations to analyze the system. In particular, we make use of Lie point and potential symmetry techniques. Over the past 120 years, the use of groups based on local symmetries originally due to Lie [7] has played an important role in obtaining invariant/similarity solutions of differential equations (see, e.g., [8–13]). Among other types, classical Lie point symmetries are classified as local symmetries and potential symmetries as nonlocal symmetries.

Calculations of symmetries are usually very long and tedious. However, we use the free available program Dimsym [14], which is written as a subprogram for the computer algebra package Reduce [15] to construct the admitted symmetries. This paper is structured as follows; In Section 2, we discuss the derivation of the governing equation. In Section 3, we use classical Lie point symmetry techniques to analyze the system of equations for a river pollution. Some exotic cases for the source term led to extra symmetries being admitted. In Section 4, we discuss the potential symmetries admitted by the system and construct some invariant solutions. Lastly we have conclusion in Section 5.

## 2. Governing equations

The equations governing the water pollution problem is derived from both navier-stokes and concentration equations for fluid flow. The basic assumption in the derivation of such equations is initially the river is incompressible, fully developed with constant viscosity, then a given pollutant is injected into the river and the fluid viscosity then changes due to the concentration of the pollutant, that is, the fluid dynamic viscosity is now pollutant-concentration-dependent. The problem now is to determine the diffusion of pollutant with time and space in the river and the effect of pollutant on the river velocity profiles.

The basic one-dimensional equations in original variables are

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad \text{(equation for fluid motion)}, \tag{2.1}$$

$$\rho\frac{\partial c}{\partial t} = \frac{\partial}{\partial y}\left(D\frac{\partial c}{\partial y}\right) + S(y,t) \quad \text{(pollutant concentration equation)}, \tag{2.2}$$

where

(i) $\rho$ = constant river density,
(ii) $D$ = mass diffusion,
(iii) $S(y,t)$ = external source of pollutant,
(iv) $\mu$ = river dynamic viscosity,
(v) $u$ = river velocity,
(vi) $c$ = pollutant concentration,
(vii) $P$ = river pressure,
(viii) $t$ = time,
(ix) $(x,y)$ = the axial and transverse coordinate, respectively.

At time $t > 0$, the fully developed stream is disturbed by injection of pollutant from an external source. The river viscosity and the mass diffusivity is assumed to vary as follows:

$$\frac{\mu}{\mu_0} = \left(\frac{c}{c_0}\right)^{\lambda}, \qquad \frac{D}{D_0} = \left(\frac{c}{c_0}\right)^{\lambda}, \tag{2.3}$$

where $\mu_0$, $c_0$, $D_0$, and $\lambda$ are the river viscosity coefficient, characteristic concentration, mass diffusivity coefficient, and constant exponent, respectively. The following dimensionless variables are introduced:

$$u' = \frac{uL}{v_0}, \qquad y' = \frac{y}{L}, \qquad x' = \frac{x}{L}, \qquad t' = \frac{tv_0}{L^2}, \qquad P' = \frac{L^2 P}{\rho v_0^2}, \qquad v_0 = \frac{\mu_0}{\rho},$$
$$R = \frac{v_0}{D_0}, \qquad K = -\frac{\partial P'}{\partial x'}, \qquad S' = \frac{SL^2}{v_0 c_0}, \qquad c' = \frac{c}{c_0}. \tag{2.4}$$

Neglecting the prime symbol for clarity, we obtain a dimensionless system of partial differential equations (PDEs):

$$\frac{\partial u}{\partial t} = K + \frac{\partial}{\partial y}\left(c^{\lambda}\frac{\partial u}{\partial y}\right), \tag{2.5}$$

$$\frac{\partial c}{\partial t} = \frac{1}{R}\frac{\partial}{\partial y}\left(c^{\lambda}\frac{\partial c}{\partial y}\right) + S(y,t), \tag{2.6}$$

where $R$ is the Schmidt number and $K$ is the imposed constant pressure axial gradient.

## 3. Point symmetry reductions of the system of (2.5) and (2.6)

In the initial Lie point symmetry analysis of the system of PDEs (2.5) and (2.6), with $S$ being an arbitrary function of $y$ and $t$, the admitted generic point symmetries or the principal Lie algebra are two dimensional and spanned by the base vectors

$$\Gamma_1 = \frac{\partial}{\partial u}, \qquad \Gamma_2 = (u - Kt)\frac{\partial}{\partial u}. \tag{3.1}$$

The principal Lie algebra extends for the cases listed in Table 3.1. Wherever they appear $A$, $\alpha$, $\beta$, $m$, $p$, and $w$ are arbitrary constants. Note that, following multiplication by the constant $R$ and then letting it vanish, the resulting equation (2.6) becomes an ordinary differential equation (ODE). This case may not be physically realistic but leads to extra admitted symmetries. The exercise of searching for the forms of arbitrary functions that extend the principal Lie algebra is called *group classification*. The problem of group classification was introduced by Ovsiannikov [16] and recent accounts on this topic may be found for example in [9, 17–19]. In this section we adopt methods in [9] to perform group classification of the system of (2.5) and (2.6).

*Invariant solutions.* Here we consider only two examples for the cases which are more realistic. Note that reduction by other symmetries leads to reduced nonlinear ordinary differential equations.

*Example 3.1.* If we consider from Table 3.1 the case $S = 0$ with arbitrary $R$ and $\lambda$, then $\Gamma_3$-invariant solution is given by

$$u = Kt + c_1, \quad c = y^{2/\lambda}\left\{-\frac{1}{R}\left(2 + \frac{4}{\lambda}\right)t + k_1\right\}^{-1/\lambda}, \tag{3.2}$$

where $c_1$, and $k_1$ are arbitrary constants of integration.

Table 3.1. Special cases of $S(y,t)$ and extension of the principal Lie algebra.

| $S(y,t)$ | Constants | Symmetries |
|---|---|---|
| Arbitrary | $R=0$, $\lambda$ arbitrary | $\Gamma_3=\partial_y$; $\Gamma_4=\frac{2c}{\lambda}\partial_c+y\partial_y$<br>$\Gamma_5=\frac{H'(t)}{\lambda}c\partial_c+KH(t)\partial_u+H(t)\partial_t$ |
| 0 | $R,\lambda$ arbitrary | $\Gamma_3=2Kt\partial_u+y\partial_y+2t\partial_t$; $\Gamma_4=\partial_t$;<br>$\Gamma_5=2c\partial_c+\lambda y\partial_y$; $\Gamma_6=\partial_y$ |
| | $R=1$, $\lambda$ arbitrary | $\Gamma_7=c\partial_u$ |
| | $R=1$, $\lambda=-2$ | $\Gamma_8=-c\partial_c+y\partial_y$ |
| $A(\alpha y+\beta)^m e^{wt}$ | $R,\lambda$ arbitrary | $\Gamma_3=c\partial_c+\frac{\lambda}{2}\left(y+\frac{\beta}{\alpha}\right)\partial_y+\frac{1}{2w}(2-\lambda m)\partial_t$ |
| $e^{ny}t^m$ | $R,\lambda$ arbitrary | $\Gamma_3=\frac{1}{\lambda(m+1)+1}\{-K\lambda nt\partial_u+nc\partial_c$<br>$+[\lambda(m+1)+1]\partial_y-\lambda nt\partial_t\}$ |
| $y^n t^m$ | $R,\lambda$ arbitrary | $\Gamma_3=\frac{1}{\lambda(m+1)+1}\{-K(\lambda n+2)t\partial_u+[2(m+1)+n]c\partial_c$<br>$+[\lambda(m+1)+1]y\partial_y-\lambda nt\partial_t\}$ |
| | $\lambda=\frac{-1}{m+1}$, $m=\frac{n+2}{-2}$ | $\Gamma_3=2Kt\partial_u-nc\partial_c+2t\partial_t$<br>$\Gamma_4=nc\partial_c+y\partial_y$ |
| $e^{my+nt}$ | $\lambda$, $R$ arbitrary | $\Gamma_3=\frac{-Km}{n}\partial_u+\partial_y-\frac{m}{n}\partial_t$ |
| $Ae^{wt}$ | $R,\lambda$ arbitrary | $\Gamma_3=2Kt\partial_u+2wc\partial_c+\lambda wy\partial_y+2\partial_t$;<br>$\Gamma_4=\partial_y$ |
| $y^m f(t)$ | $\lambda=2/m$ | $\Gamma_3=mc\partial_c+y\partial_y$ |
| $f(t)$ | $R,\lambda$ arbitrary | $\Gamma_3=\partial_y$ |
| $t^m$ | $R,\lambda$ arbitrary | $\Gamma_4=2(m+1)c\partial_c+2Kt\partial_u+2t\partial_t+[1+(m+1)\lambda]y\partial_y$ |
| $f(y)$ | $R,\lambda$ arbitrary | $\Gamma_3=K\partial_u$ |
| $e^{my}$ | $R,\lambda$ arbitrary | $\Gamma_4=c\partial_c-K\lambda t\partial_u+\frac{\lambda+1}{m}\partial_y-\lambda t\partial_t$ |
| $y^m$ | $R,\lambda$ arbitrary | $\Gamma_4=c\partial_c+\frac{2-\lambda m}{m+2}Kt\partial_u+\frac{\lambda+1}{m+2}y\partial_y+\frac{2-\lambda m}{m+2}t\partial_t$ |
| $m(ny-pt)$ | $R,\lambda$ arbitrary | $\Gamma_3=\frac{p}{n}\partial_y+\partial_t$ |

*Example 3.2.* For the case $S=A(\alpha y+\beta)^m e^{wt}$, we have $\Gamma_3$ leads to the invariant solution in functional form

$$c=\exp\left(\frac{2wt}{2-\lambda m}\right)G(\eta),\quad u=H(\eta),\quad \eta=\left(y+\frac{\beta}{\alpha}\right)\exp\left(\frac{-\lambda wt}{2-\lambda m}\right),\tag{3.3}$$

with $G$ and $H$ satisfying the system of coupled ODEs

$$\begin{gathered}(2-\lambda m)G^{\lambda}G''+\lambda(2-\lambda m)G^{\lambda-1}(G')^2+\lambda Rw\eta G'+\lambda^m AR(2-\lambda m)\eta^m-2Rw=0,\\ \lambda(2-\lambda m)G^{\lambda-1}G'H'+(2-\lambda m)G^{\lambda}H''+w\lambda\eta H'+K(2-\lambda m)=0,\end{gathered} \tag{3.4}$$

where the prime indicate differentiation with respect to $\eta$. This system of ODE has no exact solution for arbitrary values of $\lambda$. If $\lambda=0$, then we obtain invariant solutions

$$c=\exp\left(\frac{2wt}{2-\lambda m}\right)\left(\frac{Rw}{2}y^2+c_1y+c_2\right),\qquad u=-\frac{Ky^2}{2}+c_3y+c_4, \tag{3.5}$$

wherein $c_i's$ are constants. However, in this case the system is now linear and uncoupled. Symmetry reductions by other admitted symmetries lead to highly nonlinear ODEs. We herein omit those reductions.

In the next section, we analyze (2.5) and (2.6) using potential symmetry. It is possible when using potential symmetries to construct invariant solutions which cannot be obtained by point symmetries.

## 4. Potential symmetry reductions and invariant solutions

**4.1. Trivial case.** The most trivial case is given when $S=0$ and the solution of (2.2) is a constant, say $\sqrt[\lambda]{\gamma}$, that is, given a constant river pollutant concentration. This trivial case implies that (2.1) becomes a linear diffusion equation with a constant diffusion coefficient and a constant $K$. In other words, the system of (2.1) and (2.2) reduces to a single PDE written in conserved form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial y}\left(Ky+\gamma\frac{\partial u}{\partial y}\right). \tag{4.1}$$

Equation (4.1) may be embedded in a system of first order partial differential equation called auxiliary system [9]. $K$ may be equated to zero by the transformation $\overline{u}=u+Kt$. However, here we allow $K$ to be nonzero. Pucci and Saccomandi [20] provided the necessary condition for a PDE to admit potential symmetries. A PDE may be expressed in an auxiliary system in more than one inequivalent way (see, e.g., [20]). For example [21], the inhomogeneous nonlinear diffusion equation

$$u_t=\left[\left(\frac{y}{u}\right)^2u_y\right]_y \tag{4.2}$$

may be embedded into the auxiliary systems

$$v_y=u,\qquad v_t=\left(\frac{y}{u}\right)^2u_y, \tag{4.3}$$

$$v_y=\frac{u}{y},\qquad v_t=\frac{y}{u^2}u_y-\frac{1}{u}. \tag{4.4}$$

Equation (4.4) admits potential symmetries whereas (4.3) does not. Methods are being developed to find the most general conservation laws or auxiliary systems which admit nonlocal symmetries for the given governing equation (see, e.g., [22, 23]). We observe that (4.1) may be expressed in two distinct systems of first-order PDEs, namely,

$$v_y = u, \qquad v_t = Ky + \gamma u_y, \tag{4.5}$$

$$v_y = u - Kt, \qquad v_t = \gamma u_y, \tag{4.6}$$

where $v$ is the potential or nonlocal variable. The first system arises from a simple and natural split of (4.1) whilst the second was, relatively, not quite obvious to construct. We note that system (4.6) possesses point symmetries which induce potential symmetries for (4.1), whereas system (4.5) does not. One may employ the "direct method of finding auxiliary systems" [23]. If we assume that (4.1) can be embedded in the auxiliary system

$$v_y = f(y,t,u,u_t), \tag{4.7}$$

$$v_t = g(y,t,u,u_y). \tag{4.8}$$

For the compatibility of (4.7) and (4.8) we require

$$f_t + u_t f_u + u_{tt} f_{u_t} = g_y + u_y g_u + u_{yy} g_{u_y} \tag{4.9}$$

on solutions of (4.1). Note that the system of (4.7) and (4.8) is the definition of the conservation law. Hence, by simple calculation, from (4.9) it follows that

$$f = c_1 u + J(y,t), \qquad g = \gamma c_1 u_y + H(y,t), \tag{4.10}$$

with $c_1$ being a constant, $J$ and $H$ being functions of $y$, and $t$ satisfying the equation

$$J_t + Kc_1 = H_y. \tag{4.11}$$

Without loss of generality we let $c_1 = 1$. The auxiliary system (4.5) may be obtained by letting $J = 0$, that way $H = Ky$, wherein the integration constant vanishes. Also, auxiliary system (4.6) may be obtained by letting $H = 0$, that way $J = -Kt$ with the integration constant being zero. Note that if the constant of integration is nonzero, then beyond translations, both the auxiliary systems admit infinite point symmetries. The System (4.6) admits a finite six-dimensional Lie algebra spanned by

$$\begin{aligned}
&\Gamma_1 = \frac{1}{4\gamma}\left(-y^2u - 6\gamma tu - 2yv + ky^2t + 10K\gamma t^2\right)\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t} - \frac{1}{4\gamma}\left(y^2v + 2\gamma vt\right)\frac{\partial}{\partial v} + yt\frac{\partial}{\partial y},\\
&\Gamma_2 = 2Kt\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t}, \qquad \Gamma_3 = K\frac{\partial}{\partial u} + \frac{\partial}{\partial t},\\
&\Gamma_4 = (u - Kt)\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \qquad \Gamma_5 = \frac{\partial}{\partial y},\\
&\Gamma_6 = \frac{1}{2\gamma}(-yu - v + Kyt)\frac{\partial}{\partial u} - \frac{yv}{2\gamma}\frac{\partial}{\partial v} + t\frac{\partial}{\partial y}, \qquad \Gamma_7 = \frac{\partial G}{\partial y}\frac{\partial}{\partial u} + G\frac{\partial}{\partial v},
\end{aligned} \tag{4.12}$$

where $G = G(y,t)$ is a solution of the linear diffusion equation

$$G_t = \gamma G_{yy}. \tag{4.13}$$

Clearly, $\Gamma_1$ and $\Gamma_6$ induce nonlocal symmetries for the governing equation (4.1).

We construct the invariant solution for the system (4.8) using the potential symmetry operator $\Gamma_1$ and $\Gamma_6$. The $\Gamma_1$-invariant solution is given in the functional form by

$$u = t^{-3/2}\exp\left(\frac{y^2}{4\gamma t}\right)\int \frac{1}{4\gamma t}\exp\left(-\frac{y^2}{4\gamma t}\right)\left[10\gamma K t^{5/2} - 2y\exp\left(-\frac{y^2}{4\gamma t}\right)G(\rho) + K y^2 t^{3/2}\right]dt;$$
$$v = \frac{1}{\sqrt{t}}\exp\left(-\frac{y^2}{4\gamma t}\right)G(\rho), \tag{4.14}$$

where the similarity variable $\rho = y/t$, and the $\Gamma_6$-invariant solutions are

$$u = -\frac{Ay}{2\gamma\sqrt{t^3}}\exp\left(-\frac{y^2}{4\gamma t}\right) + Kt; \qquad v = \frac{A}{\sqrt{t}}\exp\left(-\frac{y^2}{4\gamma t}\right). \tag{4.15}$$

Here, $A$ is an arbitrary constant.

**4.2. Nontrivial case.** One may note that the solution of (2.6) appear as the coefficient in (2.5). Unfortunately, as expected, not all the solutions to (2.6) lead to extra symmetries for (2.5). However, we observe that the invariant solution $c = k_1/y$ to (2.6) with $S = 0$ (see, e.g., [9]) leads to a rich array of symmetries, particularly potential symmetries, being admitted by (2.5). In the classical Lie point symmetry analysis we observe that (2.5) admits beside the infinite symmetry generator, the translation in $t$ and scaling of $y$. The linear combination of these finite point symmetries leads to a highly nonlinear ordinary differential equation which was not analytically easy to solve.

On the other hand, (2.5) with $c = k_1/y$ and $\lambda = -2$, may be embedded in an auxiliary system

$$v_y = u - Kt, \qquad v_t = \left(\frac{k_1}{y}\right)^{-2} u_y, \tag{4.16}$$

where $v$ is a potential variable. Note that we choose the auxiliary system that gives rise to potential symmetries. The case $\lambda = -2$ is not realistic, however it is interesting from the mathematical point of view. The integrated form which is equivalent to the auxiliary system (4.16) and arising from letting the potential variable be given by

$$v = \int (u(y,t) - Kt)\,dy + J(t), \tag{4.17}$$

where $J(t)$ is the integration constant, is given by the nonconstant coefficient linear diffusion equation,

$$v_t = \left(\frac{y}{k_1}\right)^2 v_{yy}. \tag{4.18}$$

Note that (4.18) does not admit potential symmetries, but the auxiliary system (4.16) admits the symmetries

$$\begin{aligned}\Gamma_1 = {} & \left[\left(-\frac{k_1^2}{4}u+\frac{k_1^2K}{4}t-\frac{1}{2}ut-\frac{k_1^2K}{4y}v+\frac{1}{2}Kt^2\right)\log y\right.\\ & \left.-\frac{1}{4k_1^2}ut^2-\frac{3}{2}ut+\frac{1}{2y}vt+\frac{Kt^3}{4k_1^2}-\frac{5K}{2}t^2\right]\frac{\partial}{\partial u}\\ & +\left[\left(-k_1^2y^2v+2vt\right)\log y+t^2v-2vt\right]\frac{\partial}{\partial v}+4ty\log y\frac{\partial}{\partial y}+t^2\frac{\partial}{\partial t};\end{aligned} \tag{4.19}$$

$$\begin{aligned}\Gamma_2 = {} & \left[\left(\frac{u}{4}+\frac{Kt}{4}\right)\log y-\frac{ut}{4k_1^2}-\frac{u}{2}+\frac{v}{4y}+\frac{Kt^2}{4k_1^2}+\frac{3K}{2}t\right]\frac{\partial}{\partial u}\\ & +t\frac{\partial}{\partial t}+\frac{y}{2}\log y\frac{\partial}{\partial y}+\left[\frac{v}{4}\log y-\frac{vt}{4k_1^2}\right]\frac{\partial}{\partial v};\end{aligned} \tag{4.20}$$

$$\begin{aligned}&\Gamma_3 = \frac{\partial}{\partial u}+K\frac{\partial}{\partial t}; \qquad \Gamma_4 = (u-Kt)\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}; \qquad \Gamma_5 = (u-Kt)\frac{\partial}{\partial u}+y\frac{\partial}{\partial y};\\ &\Gamma_6 = \left[\frac{k_1^2}{2}(u-Kt)\log y+\frac{ut}{2}+\frac{k_1^2v}{2y}-\frac{Kt^2}{2}\right]\frac{\partial}{\partial u}-yt\frac{\partial}{\partial y}+\frac{v}{2}\left(k_1^2\log y-t\right)\frac{\partial}{\partial v};\end{aligned} \tag{4.21}$$

and the infinite symmetry generator

$$\Gamma_\infty = \frac{\partial H(y,t)}{\partial y}\frac{\partial}{\partial u}+H(y,t)\frac{\partial}{\partial v}, \tag{4.22}$$

where $H$ is an arbitrary solution of (4.18). $\Gamma_1$, $\Gamma_2$, and $\Gamma_6$ are potential symmetries. Equation (4.18) may naturally split into another auxiliary system using a different potential variable, say $w$. However, the potential symmetries that may rise in terms of the variable $w$ are not transformable to the symmetries in given (4.20). The genuine potential symmetries lead to the functional form of the invariant solution. As an example, we construct the solution using the $\Gamma_6$ generator. $\Gamma_6$-invariant solution is given by

$$v = \sqrt{y}\exp\left(-\frac{k_1^2}{4t}(\log y)^2\right)G(t), \tag{4.23}$$

$$\begin{aligned}u = {} & \left\{\int\frac{\exp\left(-k_1^2(\log y)^2/4t\right)\left(k_1^2Kty\log y-k_1^2\sqrt{y}\exp\left(-k_1^2(\log y)^2/4t\right)\right)G(t)+Kyt^2}{2y^{5/2}t}\mathrm{d}y+c_1\right\}\\ & \cdot y^{(k_1^2/4t)\log y+1/2},\end{aligned} \tag{4.24}$$

where $G(t)$ is an arbitrary function of $t$ and $c_1$ is a constant.

## 5. Some discussions and concluding remarks

The analysis in this paper refers to flow along a single channel. Even if water viscosity does not vary much, the analysis applies equally well to industrial fluid mixtures, for example water-alcohol mixtures, wherein this effect may be more important.

The problem has been mathematically simplified by assuming both viscosity and diffusivity to vary with concentration in the same way, that is, the power law with the same exponent. Further analysis of the system of (2.5) and (2.6) with distinct exponents for viscosity and diffusivity may be an interesting topic for future study.

Lie group analysis resulted in some exotic admitted symmetries and hence some invariant solution. We observed that the problem is difficult when the source term is nonzero. However, we made use of nonconstant concentration that appeared as coefficient in the river velocity equation. A rich symmetry structure indicated in Table 3.1 may lead to reductions and extra invariant solutions.

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