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Research Article

The Robustness of Strong Stability of Positive Homogeneous Difference Equations

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We study the robustness of strong stability of the homogeneous difference equation via the concept of strong stability radii: complex, real and positive radii in this paper. We also show that in the case of positive systems, these radii coincide. Finally, a simple example is given.

1. Introduction

Motivated by many applications in control engineering, problems of robust stability of dynamical systems have attracted a lot of attention of researchers during the last twenty years. In the study of these problems, the notion of stability radius was proved to be an effective tool, see [1–5]. In this paper, we study the robustness of strong stability of the homogeneous difference equation under parameter perturbations.

The organization of this paper is as follows. In Section 2, we recall some results on nonnegative matrices and present preliminary results on homogeneous equations for later use. In Section 3, we study a complex strong stability radius under multiperturbations. Next, we present some results on strong stability radii of the positive class equations under parameter perturbations. It is shown that complex, real, and positive strong stability radii of positive systems coincide. More important, estimates and computable formulas of these stability radii are also derived. Finally, a simple example is given.

2. Preliminaries

2.1. Nonnegative matrices

We first introduce some notations. Let n, l, q be positive integers, a matrix $P = [p_{ij}] \in \mathbb{R}^{l\times q}$ is said to be nonnegative ($P \geq 0$) if all its entries p_{ij} are nonnegative. It is said to be positive

$(P > 0)$ if all its entries p_{ij} are positive. For $P, Q \in \mathbb{R}^{l\times q}$, $P > Q$ means that $P - Q > 0$. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}_+^{l\times q}$. A similar notation will be used for vectors. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, then for any $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$, we define $|x| \in \mathbb{R}_+^n$ and $|P| \in \mathbb{R}_+^{l\times q}$ by $|x| = (|x_i|)$, $|P| = [|p_{ij}|]$. For any matrix $A \in \mathbb{K}^{n\times n}$ the spectral radius and the spectral abscissa of A is defined by $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ and $\mu(A) = \max\{\Re\lambda : \lambda \in \sigma(A)\}$, respectively, where $\sigma(A)$ is the spectrum of A. We recall some useful results, see [6].

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be monotonic if it satisfies

$$|x| \le |y| \Longrightarrow \|x\| \le \|y\|, \quad \forall x, y \in \mathbb{K}^n. \tag{2.1}$$

It can be shown that a vector norm $\|\cdot\|$ on $\mathbb{K}^n$ is monotonic if and only if $\|x\| = \||x|\|$ for all $x \in \mathbb{K}^n$, see [7]. All norms on $\mathbb{K}^n$ we use in this paper are assumed to be monotonic.

Theorem 2.1 (Perron-Frobenius). *Suppose that $A \in \mathbb{R}_+^{n\times n}$. Then*

(i) *$r(A)$ is an eigenvalue of A and there is a nonnegative eigenvector $x \ge 0$, $x \ne 0$ such that $Ax = r(A)x$.*

(ii) *If $\lambda \in \sigma(A)$ and $|\lambda| = r(A)$ then the algebraic multiplicity of λ is not greater than the algebraic multiplicity of the eigenvalue $r(A)$.*

(iii) *Given $\alpha > 0$, there exists a nonzero vector $x \ge 0$ such that $Ax \ge \alpha x$ if and only if $r(A) \ge \alpha$.*

(iv) *$(tI - A)^{-1}$ exists and is nonnegative if and only if $t > r(A)$.*

Theorem 2.2. *Let $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{R}_+^{n\times n}$. If $|A| \le B$ then*

$$r(A) \le r(|A|) \le r(B). \tag{2.2}$$

2.2. Homogeneous difference equations

Consider the neutral differential difference equation of the following form:

$$\frac{d}{dt}\left[D(r, A)y_t\right] = f(t, y_t), \tag{2.3}$$

where $D(r, A) : C([-h; 0], \mathbb{R}^n) \to \mathbb{R}^n$ is linear continuous defined by

$$D(r, A)\phi = \phi(0) - \sum_{i=1}^{N} A_i\phi(-r_i), \quad \phi \in C([-h; 0], \mathbb{R}^n). \tag{2.4}$$

Here each A_i is an $n \times n$-matrix, each r_i is a constant satisfying $r_i > 0$ and $h = \max\{r_i : i \in \overline{N}\}$, $\overline{N} = \{1, 2, \ldots, N\}$ and $y_t \in C([-h; 0], \mathbb{R}^n)$ is defined by $y_t(s) = y(t + s)$, $s \in [-h; 0]$, $t \ge 0$. Recall that there is a strictly close relation between the asymptotic behavior of solutions of (2.3) and that of associated linear homogeneous difference equations

$$D(r, A)y_t = 0, \quad t \ge 0, \tag{2.5}$$

or equivalently,

$$y(t) - \sum_{i=1}^{N} A_i y(t - r_i) = 0. \tag{2.6}$$

A study of the asymptotic behavior of solutions of system (2.6) plays a fundamental role in understanding the asymptotic behavior of solutions of linear neutral differential equations of the form (2.3), see [8].

We recall the definition in [8]: *the operator* $D(r, A)$ *or system* (2.6) *is called stable if the zero solution of* (2.6) *with* $y_0 \in C_D(r, A) = \{\phi \in C([-h, 0], \mathbb{R}^n) : D(r, A)\phi = 0\}$ *is uniformly asymptotically stable.*

Associated with system (2.6) we define the quasipolynomial

$$H(s) = I - \sum_{k=1}^{N} e^{-sr_k} A_k. \tag{2.7}$$

For $s \in \mathbb{C}$, if $\det H(s) = 0$, then s is called a characteristic root of the quasipolynomial matrix (2.7). Then, a nonzero vector $x \in \mathbb{C}^n$ satisfying $H(s)x = 0$ is called an eigenvector of $H(\cdot)$ corresponding to the characteristic root s. We set $\sigma(H(\cdot)) = \{\lambda \in \mathbb{C} : \det H(\lambda) = 0\}$, the spectral set of (2.7), and $a_H = \sup\{\Re\lambda : \lambda \in \sigma(H(\cdot))\}$, the spectral abscissa of (2.7). The following lemma is a well-known result in [8].

Theorem 2.3. *System* (2.6) *is stable if and only if* $a_H < 0$.

It is well known that a_H is not continuous in the delays $(r_1, \ldots, r_N)$, see [9]. One consequence of the noncontinuity is that arbitrarily small perturbations on the delays may destroy stability of the difference equation. This has led to the introduction of the concept of strong stability in Hale and Verduyn Lunel [10].

Definition 2.4. System (2.6) is strongly stable in the delays if it is stable for each $(r_i)_{i\in\overline{N}} \in \mathbb{R}_+^N$.

The concept of strong stability has interested many researchers as in [8–13] and references therein. Now we recall a result in [10].

Theorem 2.5. *The following statements are equivalent:*

(i) *system* (2.6) *is strongly stable,*

(ii) $\sup\{r(\sum_{i=1}^{N} z_i A_i) : |z_i| = 1,\ i \in \overline{N}\} < 1$.

We set $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$. Since $r(\cdot)$ is continuous in $\mathbb{C}^{n\times n}$, we imply the continuity of the following function $g : (\partial\mathbb{C}_1)^N \to \mathbb{R}$ defined by

$$g(z_1, \ldots, z_n) = r\left(\sum_{i=1}^{N} z_i A_i\right). \tag{2.8}$$

Moreover, by the compactness of the set $(\partial\mathbb{C}_1)^N$, there exists $z^* = (z_1^*, \ldots, z_N^*)$ such that

$$r\left(\sum_{i=1}^{N} z_i^* A_i\right) = \sup\left\{r\left(\sum_{i=1}^{N} z_i A_i\right) : |z_i| = 1,\ i \in \overline{N}\right\}. \tag{2.9}$$

By the above result, we can get the following statement: system (2.6) is strongly stable if and only if

$$\max\left\{r\left(\sum_{i=1}^{N} z_i A_i\right) : (z_i)_{i\in\overline{N}} \in (\partial\mathbb{C}_1)^N\right\} < 1. \tag{2.10}$$

3. Main results

3.1. Complex strong stability radius

Suppose that system (2.6) is strongly stable. Now we assume that each matrix A_i is subjected to the perturbation of the form

$$A_i \longrightarrow A_i + D_i \Delta_i E_i, \quad i \in \overline{N}, \tag{3.1}$$

where $D_i \in \mathbb{C}^{n\times p_i}$, $E_i \in \mathbb{C}^{q_i\times n}$ are given matrices defining the structure of the perturbations and $\Delta_i \in \mathbb{C}^{p_i\times q_i}$ are unknown matrices, $i \in \overline{N}$. We write the perturbed system

$$y(t) - \sum_{i=1}^{N}(A_i + D_i\Delta_i E_i)y(t - r_i) = 0. \tag{3.2}$$

Definition 3.1. Let system (2.6) be strongly stable. The complex, real, and positive strong stability radii of system (2.6) under perturbations of the form (3.1) are defined by

$$\begin{aligned}
r_{\mathbb{C}} &= \inf\left\{\sum_{i=1}^{N}\|\Delta_i\| : \Delta_i \in \mathbb{C}^{p_i\times q_i},\ i \in \overline{N},\ \text{system (3.2) is not strongly stable}\right\},\\
r_{\mathbb{R}} &= \inf\left\{\sum_{i=1}^{N}\|\Delta_i\| : \Delta_i \in \mathbb{R}^{p_i\times q_i},\ i \in \overline{N},\ \text{system (3.2) is not strongly stable}\right\},\\
r_{+} &= \inf\left\{\sum_{i=1}^{N}\|\Delta_i\| : \Delta_i \in \mathbb{R}_{+}^{p_i\times q_i},\ i \in \overline{N},\ \text{system (3.2) is not strongly stable}\right\},
\end{aligned} \tag{3.3}$$

respectively, we set $\inf\varnothing = +\infty$.

If system (2.6) is strongly stable, we define a function $H(\cdot,\cdot) : \mathbb{C}\setminus\mathbb{C}_1 \times (\partial\mathbb{C}_1)^N \to \mathbb{C}^{n\times n}$ by $H(\lambda, z) = (\lambda I - \sum_{i=1}^{N} z_i A_i)^{-1}$, $\lambda \in \mathbb{C}\setminus\mathbb{C}_1$, $z = (z_i)_{i\in\overline{N}} \in (\partial\mathbb{C}_1)^N$. It is easy to see that $H(\cdot,\cdot)$ is well-defined. For any $\lambda \in \mathbb{C}\setminus\mathbb{C}_1$, $z = (z_i)_{i\in\overline{N}} \in (\partial\mathbb{C}_1)^N$, we set

$$G_{ij}(\lambda, z) = E_i H(\lambda, z) D_j \in \mathbb{C}^{q_i\times p_j}. \tag{3.4}$$

Theorem 3.2. *Let system* (2.6) *be strongly stable. Then we have*

(i)

$$\frac{1}{\max_{i,j\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \|G_{ij}(\lambda,z)\|} \leq r_{\mathbb{C}} \leq \frac{1}{\max_{i\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \|G_{ii}(\lambda,z)\|}, \tag{3.5}$$

(ii) *in particular, if $D_i = D_j$ (or $E_i = E_j$) for all $i, j \in \overline{N}$, then we have*

$$r_{\mathbb{C}} = \frac{1}{\max_{i\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \|G_{ii}(\lambda,z)\|}. \tag{3.6}$$

Proof. Let $\Delta = (\Delta_i)_{i\in\overline{N}}$ be a destabilizing disturbance. Then there exists $(\lambda, z) \in \mathbb{C}\setminus\mathbb{C}_1 \times (\partial\mathbb{C}_1)^N$ such that $\lambda \in \sigma(\sum_{i=1}^N z_i(A_i + D_i\Delta_i E_i))$. This means that there exists a nonzero vector x satisfying

$$\left(\sum_{i=1}^N z_i\left(A_i + D_i\Delta_i E_i\right)\right)x = \lambda x. \tag{3.7}$$

This follows that

$$\left(\sum_{i=1}^N z_i D_i \Delta_i E_i\right)x = \left(\lambda I - \sum_{i=1}^N z_i A_i\right)x, \tag{3.8}$$

or equivalently,

$$x = \sum_{i=1}^N z_i H(\lambda,z) D_i \Delta_i E_i x. \tag{3.9}$$

Choose $q \in \overline{N}$ such that $\|E_q x\| = \max\{\|E_i x\| : i \in \overline{N}\}$. Multiplying the above equation with E_q, we obtain

$$\begin{aligned} E_q x &= \sum_{i=1}^N z_i E_q H(\lambda,z) D_i \Delta_i E_i x \\ &= \sum_{i=1}^N z_i G_{qi}(\lambda,z) \Delta_i E_i x. \end{aligned} \tag{3.10}$$

This implies that

$$\begin{aligned} \|E_q x\| &\leq \sum_{i=1}^N \|z_i\| \|G_{qi}(\lambda,z)\| \|\Delta_i\| \|E_i x\| \\ &\leq \sum_{i=1}^N \max_{i,j\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \|G_{ij}(\lambda,z)\| \|\Delta_i\| \|E_q x\|. \end{aligned} \tag{3.11}$$

From this inequality and the definition of $r_{\mathbb{C}}$, the left-hand inequality of (i) follows:

$$r_{\mathbb{C}} \geq \frac{1}{\max_{i,j\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \|G_{ij}(\lambda,z)\|}. \tag{3.12}$$

Now it remains to prove the right-hand inequality of (i):

$$r_{\mathbb{C}} \leq \frac{1}{\max_{i\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \|G_{ii}(\lambda,z)\|}. \tag{3.13}$$

Indeed, for any $(\lambda, z) \in \mathbb{C} \setminus \mathbb{C}_1 \times (\partial\mathbb{C}_1)^N$, and $i \in \overline{N}$, there exists nonzero vector $x \in \mathbb{C}^{p_i}$ such that $\|x\| = 1$ and $\|G_{ii}(\lambda, z)x\| = \|G_{ii}(\lambda, z)\|$. By Hahn-Banach theorem, there exists $y^* \in (\mathbb{C}^{q_i})^*$ satisfying $\|y^*\| = 1$ and $y^*(G_{ii}(\lambda, z)x) = \|G_{ii}(\lambda, z)x\|$. We define a matrix $\overline{\Delta} \in \mathbb{C}^{p_i\times q_i}$ by setting

$$\overline{\Delta} = \frac{1}{\|G_{ii}(\lambda,z)\|} xy^* \in \mathbb{C}^{p_i\times q_i}. \tag{3.14}$$

Now we construct the disturbance $\Delta = (\Delta_1, \ldots, \Delta_N)$ defined by

$$\Delta_j = \begin{cases} z_i^{-1}\overline{\Delta}, & j = i, \\ 0, & j \neq i. \end{cases} \tag{3.15}$$

It is easy to check that $\sum_{i=1}^N \|\Delta_i\| = \|\overline{\Delta}\| = 1/\|G_{ii}(\lambda, z)\|$. Moreover, we have

$$\lambda\hat{x} = \left(\sum_{i=1}^N z_i\left(A_i + D_i\Delta_i E_i\right)\right)\hat{x}, \tag{3.16}$$

where $\hat{x} = H(\lambda, z)D_i x$. This means that Δ is a destabilizing disturbance. Thus,

$$r_{\mathbb{C}} \leq \frac{1}{\max_{i\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \|G_{ii}(\lambda,z)\|}. \tag{3.17}$$

The proof of (i) is complete, and (ii) can be obtained directly from (i). □

In general, the complex, real, and positive radius are distinct, see [4, 5]. Theorem 3.2 reduces the computation of the complex strong stability radius to a global optimization problem with many variations while the problem for the real stability radius is much more difficult, see [5]. It is therefore natural to investigate for which kind of systems these three radii coincide. The answer will be found in the next section.

3.2. Strong stability radii of positive systems

In this section, we restrict system (2.6) to be positive, that is, A_i are nonnegative for all $i \in \overline{N}$.

Lemma 3.3. *Let $A_i \in \mathbb{R}_+^{n\times n}$. Then we have*

(i) $r(\sum_{i=1}^N A_i) = \sup_{z\in(\partial\mathbb{C}_1)^N} r(\sum_{i=1}^N z_i A_i)$;

(ii) $r(\sum_{i=1}^N A_i) < t_1 \leq t_2 \Rightarrow 0 \leq (t_2 I - \sum_{i=1}^N A_i)^{-1} \leq (t_1 I - \sum_{i=1}^N A_i)^{-1}$,

Proof. (i) By Theorem 2.2, we have

$$r\left(\sum_{i=1}^{N} z_i A_i\right) \le r\left(\sum_{i=1}^{N} |z_i A_i|\right) = r\left(\sum_{i=1}^{N} A_i\right); \tag{3.18}$$

(ii) the positivity of $(t_1 I - \sum_{i=1}^{N} A_i)^{-1}$, $(t_2 I - \sum_{i=1}^{N} A_i)^{-1}$ can be implied by Theorem 2.1. The right-hand inequality can be obtained by the following formula:

$$\left(t_2 I - \sum_{i=1}^{N} A_i\right)^{-1} - \left(t_1 I - \sum_{i=1}^{N} A_i\right)^{-1} = -(t_2 - t_1)\left(t_1 I - \sum_{i=1}^{N} A_i\right)^{-1}\left(t_2 I - \sum_{i=1}^{N} A_i\right)^{-1}. \tag{3.19}$$

This completes the proof. □

It is important to note from above lemma that under positivity assumptions, system (2.6) is strongly stable if and only if $r(A_1 + \cdots + A_N) < 1$.

Lemma 3.4. *Suppose that system* (2.6) *is positive and strongly stable. Then, for any* $D \in \mathbb{R}_+^{n\times p}$, $E \in \mathbb{R}_+^{q\times n}$, $\lambda \in \mathbb{C} \setminus \mathbb{C}_1$, $z \in (\partial\mathbb{C}_1)^N$, *we have*

$$\left\| E\left(\lambda I - \sum_{i=1}^{N} z_i A_i\right)^{-1} D \right\| \le \left\| E\left(I - \sum_{i=1}^{N} A_i\right)^{-1} D \right\|. \tag{3.20}$$

Proof. For any $\lambda \in \mathbb{C} \setminus \mathbb{C}_1, z \in (\partial\mathbb{C}_1)^N$, we have $r(\sum_{i=1}^{N} z_i A_i) < 1 \le |\lambda|$. Thus, for an arbitrary vector $x \in \mathbb{R}^p$,

$$\begin{aligned}
\left| E\left(\lambda I - \sum_{i=1}^{N} z_i A_i\right)^{-1} Dx \right| &\le E\left| \left(\lambda I - \sum_{i=1}^{N} z_i A_i\right)^{-1} \right| D|x| \\
&\le E\frac{1}{|\lambda|}\left(\sum_{n=0}^{\infty}\left|\frac{\sum_{i=1}^{N} z_i A_i}{\lambda}\right|^n\right) D|x| \\
&\le E\frac{1}{|\lambda|}\left(\sum_{n=0}^{\infty}\left(\frac{\sum_{i=1}^{N} |z_i A_i|}{|\lambda|}\right)^n\right) D|x| \\
&\le E\left(\sum_{n=0}^{\infty}\left(\sum_{i=1}^{N} A_i\right)^n\right) D|x| \\
&\le E\left(I - \sum_{i=1}^{N} A_i\right)^{-1} D|x|.
\end{aligned} \tag{3.21}$$

By Lemma 3.3, we have $(I - \sum_{i=1}^{N} A_i)^{-1} \ge 0$. Thus, we imply

$$\left\| E\left(\lambda I - \sum_{i=1}^{N} z_i A_i\right)^{-1} D \right\| \le \left\| E\left(I - \sum_{i=1}^{N} A_i\right)^{-1} D \right\|. \tag{3.22}$$

□

Theorem 3.5. *Let system* (2.6) *be strongly stable and positive. Assume that all* D_i, E_i, $i \in \overline{N}$ *are nonnegative matrices. If* $D_i = D_j$ *or* $E_i = E_j$, $\forall i, j \in \overline{N}$, *then*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_{+} = \frac{1}{\max_{i\in\overline{N}} \left\| G_{ii}(1,1) \right\|}, \tag{3.23}$$

where $G_{ii}(1,1) = E_i\left(I - \sum_{i=1}^{N} A_i\right)^{-1} D_i$.

Proof. By Theorem 3.2, we have

$$r_{\mathbb{C}} = \frac{1}{\max_{i\in\overline{N}} \sup_{|\lambda|\geq 1, z\in(\partial\mathbb{C}_1)^N} \left\| G_{ii}(\lambda, z) \right\|}. \tag{3.24}$$

Moreover, using Lemma 3.4, we get

$$r_{\mathbb{C}} = \frac{1}{\max_{i\in\overline{N}} \left\| G_{ii}(1,1) \right\|}. \tag{3.25}$$

Since $r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_{+}$, we only need to prove that

$$r_{+} \leq \frac{1}{\max_{i\in\overline{N}} \left\| G_{ii}(1,1) \right\|}. \tag{3.26}$$

Indeed, for any $i \in \overline{N}$, since $G_{ii}(1,1)$ is a nonnegative matrix, there exists nonnegative vector $x \in \mathbb{R}_{+}^{p_i}$ such that $\|x\| = 1$ and $\|G_{ii}(1,1)x\| = \|G_{ii}(1,1)\|$. Using Krein-Rutman theorem, see [14], there exists $y^* \in (\mathbb{R}_{+}^{q_i})^*$ satisfying $\|y^*\| = 1$ and $y^*(G_{ii}(1,1)x) = \|G_{ii}(1,1)x\|$. We define a nonnegative matrix $\overline{\Delta} \in \mathbb{R}_{+}^{p_i\times q_i}$ by setting

$$\overline{\Delta} = \frac{1}{\left\| G_{ii}(1,1) \right\|} x y^* \in \mathbb{R}_{+}^{p_i\times q_i}. \tag{3.27}$$

Now we construct the positive disturbance $\Delta = (\Delta_1, \ldots, \Delta_N)$ defined by

$$\Delta_j = \begin{cases} \overline{\Delta}, & j = i, \\ 0, & j \neq i. \end{cases} \tag{3.28}$$

It is easy to check that $\sum_{i=1}^{N} \|\Delta_i\| = \|\overline{\Delta}\| = 1/\|G_{ii}(1,1)\|$. Moreover, we have

$$\lambda \widehat{x} = \left(\sum_{i=1}^{N} \left(A_i + D_i \Delta_i E_i \right) \right) \widehat{x}, \tag{3.29}$$

where $\widehat{x} = H(1,1) D_i x$. It means that Δ is a destabilizing disturbance. Thus

$$r_{+} \leq \frac{1}{\max_{i\in\overline{N}} \left\| G_{ii}(1,1) \right\|}. \tag{3.30}$$

The proof is complete. □

Now we turn to a different perturbation structure and assume that each matrix A_i is subjected to perturbations of the form

$$A_i \longrightarrow A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij}, \tag{3.31}$$

where B_{ij} are given matrices defining the structure of the perturbations and δ_{ij} are unknown scalars representing parameter uncertainties. So we can write the perturbed system

$$y(t) - \sum_{i=1}^{N} \left(A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij} \right) y(t - r_i) = 0. \tag{3.32}$$

Definition 3.6. Let system (2.6) be strongly stable. The complex, real, and positive strong stability radii of system (2.6) under perturbations of the form (3.31) are defined by

$$\begin{aligned} r_{\mathbb{C}}^{\delta} &= \inf\left\{ \|\delta\|_\infty : \delta = (\delta_{ij}) \in \mathbb{C}^{NK}, \text{ system (3.32) is not strongly stable} \right\}, \\ r_{\mathbb{R}}^{\delta} &= \inf\left\{ \|\delta\|_\infty : \delta = (\delta_{ij}) \in \mathbb{R}^{NK}, \text{ system (3.32) is not strongly stable} \right\}, \\ r_{+}^{\delta} &= \inf\left\{ \|\delta\|_\infty : \delta = (\delta_{ij}) \in \mathbb{R}_{+}^{NK}, \text{ system (3.32) is not strongly stable} \right\}, \end{aligned} \tag{3.33}$$

respectively, we set $\inf \varnothing = +\infty$, and $\|\delta\|_\infty = \max\{|\delta_{ij}| : i \in \overline{N}, j \in \overline{K}\}$, where $\overline{K} = \{1, \ldots, K\}$.

Lemma 3.7. *Suppose system* (2.6) *is strongly stable, positive and* $B_{ij} \in \mathbb{R}_{+}^{n\times n}$, $i \in \overline{N}, j \in \overline{K}$. *Then*

$$r_{\mathbb{C}}^{\delta} = r_{\mathbb{R}}^{\delta} = r_{+}^{\delta}. \tag{3.34}$$

Proof. Because $r_{\mathbb{C}}^{\delta} \le r_{\mathbb{R}}^{\delta} \le r_{+}^{\delta}$, we only need to prove that $r_{\mathbb{C}}^{\delta} \ge r_{+}^{\delta}$. Indeed, for a destabilizing disturbance $\delta = (\delta_{ij})_{i\in\overline{N}, j\in\overline{K}}$, there exist a $\lambda \in \mathbb{C} \setminus \mathbb{C}_1, z \in (\partial\mathbb{C}_1)^N$ and a nonzero vector $x \in \mathbb{K}^n$ such that

$$\sum_{i=1}^{N} z_i \left(A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij} \right) x = \lambda x. \tag{3.35}$$

This yields

$$\sum_{i=1}^{N} \left(A_i + \sum_{j=1}^{K} |\delta_{ij}| B_{ij} \right) |x| \ge \left| \sum_{i=1}^{N} z_i \left(A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij} \right) x \right| = |\lambda||x|. \tag{3.36}$$

By Theorem 2.1, we get

$$r\left(\sum_{i=1}^{N} \left(A_i + \sum_{j=1}^{K} |\delta_{ij}| B_{ij} \right) \right) \ge 1. \tag{3.37}$$

It means that $|\delta| = (|\delta_{ij}|)_{i\in\overline{N}, j\in\overline{K}}$ is also a destabilizing disturbance. Thus, by the definition of complex and real radii, $r_{\mathbb{C}}^{\delta} \ge r_{+}^{\delta}$. The proof is complete. □

Theorem 3.8. *Suppose system* (2.6) *is strongly stable, positive and* $B_{ij} \in \mathbb{R}_+^{n\times n}$, $i \in \overline{N}$, $j \in \overline{K}$. *Then*

$$r_{\mathbb{C}}^{\delta} = r_{\mathbb{R}}^{\delta} = r_{+}^{\delta} = \frac{1}{r\left(\left(I - \sum_{i=1}^{N} A_i\right)^{-1} B\right)}, \tag{3.38}$$

where $B = \sum_{i,j} B_{ij}$.

Proof. By Lemma 3.7, we only need to prove that

$$r_{+}^{\delta} = \frac{1}{r\left(\left(I - \sum_{i=1}^{N} A_i\right)^{-1} B\right)}. \tag{3.39}$$

To do it, taking arbitrary destabilizing disturbance $\delta = (\delta_{ij})_{i\in\overline{N}, j\in\overline{K}} \in \mathbb{R}_+^{NK}$, by Lemma 3.3 and Theorem 2.1, there exist a $\lambda \geq 1$ and a nonzero vector $x \in \mathbb{R}_+^n$ such that

$$\left(\sum_{i=1}^{N}\left(A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij}\right)\right) x = \lambda x, \tag{3.40}$$

or equivalently,

$$\sum_{i,j} \delta_{ij} B_{ij} x = \left(\lambda I - \sum_{i=1}^{N} A_i\right) x. \tag{3.41}$$

This yields

$$\left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1} \left(\sum_{i,j} \delta_{ij} B_{ij}\right) x = x. \tag{3.42}$$

Then, we have

$$\|\delta\|_{\infty} \left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1} B x \geq \left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1} \left(\sum_{i,j} \delta_{ij} B_{ij}\right) x = x. \tag{3.43}$$

Using Theorem 2.1 again, we obtain

$$r\left(\left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1} B\right) \geq \frac{1}{\|\delta\|_{\infty}} \tag{3.44}$$

or equivalently,

$$\|\delta\|_{\infty} \geq \frac{1}{r\left(\left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1} B\right)}. \tag{3.45}$$

Thus, from the definition of r_{+}^{δ}, one has

$$r_{+}^{\delta} \geq \frac{1}{r\left(\left(I - \sum_{i=1}^{N} A_i\right)^{-1} B\right)}. \tag{3.46}$$

On the other hand, setting $\lambda = r((I - \sum_{i=1}^{N} A_i)^{-1}B) > 0$. Then, by Theorem 2.1, there exists a nonnegative vector $x \in \mathbb{R}_+^n$ satisfying

$$\left(\left(I - \sum_{i=1}^{N} A_i\right)^{-1} B\right)x = \lambda x. \tag{3.47}$$

This is equivalent to

$$Bx = \lambda\left(I - \sum_{i=1}^{N} A_i\right)x. \tag{3.48}$$

Hence,

$$\sum_{i=1}^{N}\left(A_i + \frac{1}{\lambda}B\right)x = x. \tag{3.49}$$

This means that $\delta = (1/\lambda) \in \mathbb{R}_+^{NK}$ is a destabilizing disturbance and thus, $r_+^{\delta} \leq 1/r((I - \sum_{i=1}^{N} A_i)^{-1}B)$. The proof is complete. □

Now we consider the following example to illustrate the obtained results.

Example 3.9. Consider system

$$y(t) = A_1 y(t-r) + A_2 y(t-s), \tag{3.50}$$

where

$$A_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{3} \end{pmatrix}. \tag{3.51}$$

Then we have

$$r(A_1 + A_2) = \frac{7 + \sqrt{145}}{24} < 1. \tag{3.52}$$

Thus system (3.50) is strongly stable.

Assume that the matrices A_1, A_2 are subjected to perturbations of the form $A_1 \to A_1 + D_1\Delta_1 E_1$, $A_2 \to A_2 + D_2\Delta_2 E_2$, where

$$D_1 = D_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.53}$$

Then

$$G_{11}(1,1) = \begin{pmatrix} \frac{29}{3} & 0 \\ 0 & 0 \end{pmatrix}, \qquad G_{22}(1,1) = \begin{pmatrix} \frac{14}{3} & 5 \\ 0 & 0 \end{pmatrix}. \tag{3.54}$$

If $\mathbb{R}^2$ is provided with the norm defined by $\|(x,y)\| = |x| + |y|$, then by Theorem 3.5, we have

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_{+} = \frac{3}{29}. \tag{3.55}$$

Assume that the given two matrices A_1, A_2 are subjected to perturbations of the form $A_1 \to A_1 + \delta_{11}B_{11} + \delta_{12}B_{12}$, $A_2 \to A_2 + \delta_{21}B_{21} + \delta_{22}B_{22}$, where

$$B_{11} = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{pmatrix}, \qquad B_{12} = \begin{pmatrix} 1 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{pmatrix}, \qquad B_{21} = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}, \qquad B_{22} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 1 \end{pmatrix}. \tag{3.56}$$

Then

$$(I - A_1 - A_2)^{-1}B = \begin{pmatrix} 10 & \frac{227}{18} \\ 11 & \frac{163}{12} \end{pmatrix}. \tag{3.57}$$

By Theorem 3.8, we get

$$r_{\mathbb{C}}^{\delta} = r_{\mathbb{R}}^{\delta} = r_{+}^{\delta} = \frac{24}{283 + \sqrt{81753}}. \tag{3.58}$$

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