Research Article

The Robustness of Strong Stability of Positive Homogeneous Difference Equations

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We study the robustness of strong stability of the homogeneous difference equation via the concept of strong stability radii: complex, real and positive radii in this paper. We also show that in the case of positive systems, these radii coincide. Finally, a simple example is given.

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1. Introduction

Motivated by many applications in control engineering, problems of robust stability of dynamical systems have attracted a lot of attention of researchers during the last twenty years. In the study of these problems, the notion of stability radius was proved to be an effective tool, see [1–5]. In this paper, we study the robustness of strong stability of the homogeneous difference equation under parameter perturbations.

The organization of this paper is as follows. In Section 2, we recall some results on nonnegative matrices and present preliminary results on homogeneous equations for later use. In Section 3, we study a complex strong stability radius under multiperturbations. Next, we present some results on strong stability radii of the positive class equations under parameter perturbations. It is shown that complex, real, and positive strong stability radii of positive systems coincide. More important, estimates and computable formulas of these stability radii are also derived. Finally, a simple example is given.

2. Preliminaries

2.1. Nonnegative matrices

We first introduce some notations. Let $n$, $l$, $q$ be positive integers, a matrix $P = [p_{ij}] \in \mathbb{R}^{l \times q}$ is said to be nonnegative ($P \geq 0$) if all its entries $p_{ij}$ are nonnegative. It is said to be positive
(P > 0) if all its entries $p_{ij}$ are positive. For $P, Q \in \mathbb{R}^{l \times q}$, $P > Q$ means that $P - Q > 0$. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}^{l \times q}_+$. A similar notation will be used for vectors. Let $K = \mathbb{C}$ or $\mathbb{R}$, then for any $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l \times q}$, we define $|x| \in \mathbb{K}^n_+$ and $|P| \in \mathbb{K}^{l \times q}_+$ by $|x| = (|x_i|), \ |P| = [|p_{ij}|]$. For any matrix $A \in \mathbb{K}^{n \times n}_+$ the spectral radius and the spectral abscissa of $A$ is defined by $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ and $\mu(A) = \max\{|\Re \lambda : \lambda \in \sigma(A)\}$, respectively, where $\sigma(A)$ is the spectrum of $A$. We recall some useful results, see [6].

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be monotonic if it satisfies

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|, \ \forall x, y \in \mathbb{K}^n. \quad (2.1)$$

It can be shown that a vector norm $\|\cdot\|$ on $\mathbb{K}^n$ is monotonic if and only if $\|x\| = \||x|\|$ for all $x \in \mathbb{K}^n$, see [7]. All norms on $\mathbb{K}^n$ we use in this paper are assumed to be monotonic.

**Theorem 2.1** (Perron-Frobenius). Suppose that $A \in \mathbb{R}^{n \times n}_+$. Then

(i) $r(A)$ is an eigenvalue of $A$ and there is a nonnegative eigenvector $x \geq 0, x \neq 0$ such that $Ax = r(A)x$.

(ii) If $\lambda \in \sigma(A)$ and $|\lambda| = r(A)$ then the algebraic multiplicity of $\lambda$ is not greater than the algebraic multiplicity of the eigenvalue $r(A)$.

(iii) Given $\alpha > 0$, there exists a nonzero vector $x \geq 0$ such that $Ax \geq \alpha x$ if and only if $r(A) \geq \alpha$.

(iv) $(I - A)^{-1}$ exists and is nonnegative if and only if $t > r(A)$.

**Theorem 2.2.** Let $A \in \mathbb{K}^{n \times n}, B \in \mathbb{R}^{n \times n}$. If $|A| \leq B$ then

$$r(A) \leq r(|A|) \leq r(B). \quad (2.2)$$

### 2.2. Homogeneous difference equations

Consider the neutral differential difference equation of the following form:

$$\frac{d}{dt}[D(r, A)y_1] = f(t, y_1), \quad (2.3)$$

where $D(r, A) : C([-h; 0], \mathbb{R}^n) \to \mathbb{R}^n$ is linear continuous defined by

$$D(r, A)\phi = \phi(0) - \sum_{i=1}^{N} A_i \phi(-r_i), \quad \phi \in C([-h; 0], \mathbb{R}^n). \quad (2.4)$$

Here each $A_i$ is an $n \times n$-matrix, each $r_i$ is a constant satisfying $r_i > 0$ and $h = \max\{r_i : i \in \overline{N}\}, \ \overline{N} = \{1, 2, \ldots, N\}$ and $y_i \in C([-h; 0], \mathbb{R}^n)$ is defined by $y_i(s) = y(t + s), s \in [-h; 0], t \geq 0$. Recall that there is a strictly close relation between the asymptotic behavior of solutions of (2.3) and that of associated linear homogeneous difference equations

$$D(r, A)y_t = 0, \quad t \geq 0, \quad (2.5)$$
or equivalently,

\[ y(t) - \sum_{i=1}^{N} A_i y(t - r_i) = 0. \]  

(2.6)

A study of the asymptotic behavior of solutions of system (2.6) plays a fundamental role in understanding the asymptotic behavior of solutions of linear neutral differential equations of the form (2.3), see [8].

We recall the definition in [8]: the operator \( D(r, A) \) or system (2.6) is called stable if the zero solution of (2.6) with \( y_0 \in C_D(r, A) = \{ \phi \in C([-h, 0], \mathbb{R}^n) : D(r, A)\phi = 0 \} \) is uniformly asymptotically stable.

Associated with system (2.6) we define the quasipolynomial

\[ H(s) = I - \sum_{k=1}^{N} e^{-\sigma_k s} A_k. \]  

(2.7)

For \( s \in \mathbb{C} \), if \( \det H(s) = 0 \), then \( s \) is called a characteristic root of the quasipolynomial matrix (2.7). Then, a nonzero vector \( x \in \mathbb{C}^n \) satisfying \( H(s)x = 0 \) is called an eigenvector of \( H(\cdot) \) corresponding to the characteristic root \( s \). We set \( \sigma(H(\cdot)) = \{ \lambda \in \mathbb{C} : \det H(\lambda) = 0 \} \), the spectral set of (2.7), and \( a_H = \sup \{ \Re \lambda : \lambda \in \sigma(H(\cdot)) \} \), the spectral abscissa of (2.7). The following lemma is a well-known result in [8].

**Theorem 2.3.** System (2.6) is stable if and only if \( a_H < 0 \).

It is well known that \( a_H \) is not continuous in the delays \( (r_1, \ldots, r_N) \), see [9]. One consequence of the noncontinuity is that arbitrarily small perturbations on the delays may destroy stability of the difference equation. This has led to the introduction of the concept of strong stability in Hale and Verduyn Lunel [10].

**Definition 2.4.** System (2.6) is strongly stable in the delays if it is stable for each \( (r_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^N \).

The concept of strong stability has interested many researchers as in [8–13] and references therein. Now we recall a result in [10].

**Theorem 2.5.** The following statements are equivalent:

(i) system (2.6) is strongly stable,

(ii) \( \sup \{ r(\sum_{i=1}^{N} z_i A_i) : |z_i| = 1, i \in \mathbb{N} \} < 1. \)

We set \( \mathbb{C}_1 = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \partial \mathbb{C}_1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Since \( r(\cdot) \) is continuous in \( \mathbb{C}^{\times n} \), we imply the continuity of the following function \( g : (\partial \mathbb{C}_1)^N \to \mathbb{R} \) defined by

\[ g(z_1, \ldots, z_n) = r \left( \sum_{i=1}^{N} z_i A_i \right). \]  

(2.8)
Moreover, by the compactness of the set \((\partial C_1)^N\), there exists \(z^* = (z_1^*, \ldots, z_N^*)\) such that
\[
\rho \left( \sum_{i=1}^N z_i^* A_i \right) = \sup \left\{ \rho \left( \sum_{i=1}^N z_i A_i \right) : |z_i| = 1, i \in \mathbb{N} \right\}. \tag{2.9}
\]

By the above result, we can get the following statement: system (2.6) is strongly stable if and only if
\[
\max \left\{ \rho \left( \sum_{i=1}^N z_i A_i \right) : (z_i)_{i \in \mathbb{N}} \in (\partial C_1)^N \right\} < 1. \tag{2.10}
\]

3. Main results

3.1. Complex strong stability radius

Suppose that system (2.6) is strongly stable. Now we assume that each matrix \(A_i\) is subjected to the perturbation of the form
\[
A_i \rightarrow A_i + D_i \Delta_i E_i, \quad i \in \mathbb{N}, \tag{3.1}
\]
where \(D_i \in \mathbb{C}^{n \times p}, E_i \in \mathbb{C}^{p \times n}\) are given matrices defining the structure of the perturbations and \(\Delta_i \in \mathbb{C}^{p \times q_i}\) are unknown matrices, \(i \in \mathbb{N}\). We write the perturbed system
\[
y(t) - \sum_{i=1}^N (A_i + D_i \Delta_i E_i)y(t - r_i) = 0. \tag{3.2}
\]

Definition 3.1. Let system (2.6) be strongly stable. The complex, real, and positive strong stability radii of system (2.6) under perturbations of the form (3.1) are defined by
\[
\begin{align*}
\rho_C & = \inf \left\{ \sum_{i=1}^N \| \Delta_i \| : \Delta_i \in \mathbb{C}^{p \times q_i}, i \in \mathbb{N}, \text{system (3.2) is not strongly stable} \right\}, \\
\rho_R & = \inf \left\{ \sum_{i=1}^N \| \Delta_i \| : \Delta_i \in \mathbb{R}^{p \times q_i}, i \in \mathbb{N}, \text{system (3.2) is not strongly stable} \right\}, \\
\rho_+ & = \inf \left\{ \sum_{i=1}^N \| \Delta_i \| : \Delta_i \in \mathbb{R}^{p \times q_i}, i \in \mathbb{N}, \text{system (3.2) is not strongly stable} \right\},
\end{align*}
\tag{3.3}
\]
respectively, we set \(\inf \emptyset = +\infty\).

If system (2.6) is strongly stable, we define a function \(H(\cdot, \cdot) : \mathbb{C} \setminus C_1 \times (\partial C_1)^N \rightarrow \mathbb{C}^{n \times n}\) by
\[
H(\lambda, z) = (I - \sum_{i=1}^N z_i A_i)^{-1}, \quad \lambda \in \mathbb{C} \setminus C_1, z = (z_i)_{i \in \mathbb{N}} \in (\partial C_1)^N. \tag{3.4}
\]
It is easy to see that \(H(\cdot, \cdot)\) is well-defined. For any \(\lambda \in \mathbb{C} \setminus C_1, z = (z_i)_{i \in \mathbb{N}} \in (\partial C_1)^N\), we set
\[
G_{ij}(\lambda, z) = E_i H(\lambda, z) D_j \in \mathbb{C}^{p \times p}. \tag{3.4}
\]
Theorem 3.2. Let system (2.6) be strongly stable. Then we have

\[
\max_{i,j} \sup_{|\lambda| \geq 1, z \in \partial C_1} \|G_{ij}(\lambda, z)\| \leq r_C \leq \frac{1}{\max_{i \in N} \sup_{|\lambda| \geq 1, z \in \partial C_1} \|G_{ii}(\lambda, z)\|}. \tag{3.5}
\]

(i) in particular, if \(D_i = D_j\) (or \(E_i = E_j\)) for all \(i, j \in \mathbb{N}\), then we have

\[
r_C = \frac{1}{\max_{i \in N} \sup_{|\lambda| \geq 1, z \in \partial C_1} \|G_{ii}(\lambda, z)\|}. \tag{3.6}
\]

Proof. Let \(\Delta = (\Delta_i)_{i \in \mathbb{N}}\) be a destabilizing disturbance. Then there exists \((\lambda, z) \in \mathbb{C} \setminus \mathbb{C}_1 \times (\partial \mathbb{C}_1)^N\) such that \(\lambda \in \sigma(\sum_{i=1}^N z_i(A_i + D_i \Delta_i E_i))\). This means that there exists a nonzero vector \(x\) satisfying

\[
\left( \sum_{i=1}^N z_i(A_i + D_i \Delta_i E_i) \right) x = \lambda x. \tag{3.7}
\]

This follows that

\[
\left( \sum_{i=1}^N z_i D_i \Delta_i E_i \right) x = \left( \lambda I - \sum_{i=1}^N z_i A_i \right)x, \tag{3.8}
\]

or equivalently,

\[
x = \sum_{i=1}^N z_i E_i x. \tag{3.9}
\]

Choose \(q \in \mathbb{N}\) such that \(\|E_q x\| = \max\{\|E_i x\| : i \in \mathbb{N}\}\). Multiplying the above equation with \(E_q\), we obtain

\[
E_q x = \sum_{i=1}^N z_i E_q H(\lambda, z) D_i \Delta_i E_i x
\]

\[
= \sum_{i=1}^N z_i G_{qi}(\lambda, z) \Delta_i E_i x. \tag{3.10}
\]

This implies that

\[
\|E_q x\| \leq \sum_{i=1}^N z_i \|G_{qi}(\lambda, z)\| \|\Delta_i\| \|E_i x\|
\]

\[
\leq \sum_{i=1}^N \max_{i,j} \sup_{|\lambda| \geq 1, z \in \partial C_1} \|G_{ij}(\lambda, z)\| \|\Delta_i\| \|E_q x\|. \tag{3.11}
\]
From this inequality and the definition of $r_c$, the left-hand inequality of (i) follows:

$$r_c \geq \max_{i,j \in \mathbb{N}} \sup_{|\lambda| \geq 1, z \in (\partial C_i)^N} \| G_{ij}(\lambda, z) \|.$$  

(3.12)

Now it remains to prove the right-hand inequality of (i):

$$r_c \leq \max_{i \in \mathbb{N}} \sup_{|\lambda| \geq 1, z \in (\partial C_i)^N} \| G_{ii}(\lambda, z) \|.$$  

(3.13)

Indeed, for any $(\lambda, z) \in C \setminus C_1 \times (\partial C_1)^N$, and $i \in \overline{N}$, there exists nonzero vector $x \in \mathbb{C}^n$ such that $\|x\| = 1$ and $\|G_{ii}(\lambda, z)x\| = \|G_{ii}(\lambda, z)\|$. By Hahn-Banach theorem, there exists $y^* \in (\mathbb{C}^p)^*$ satisfying $\|y^*\| = 1$ and $y^*(G_{ii}(\lambda, z)x) = \|G_{ii}(\lambda, z)x\|$. We define a matrix $\bar{\Delta} \in \mathbb{C}^{n \times \phi}$ by setting

$$\bar{\Delta} = \frac{1}{\| G_{ii}(\lambda, z) \|} xy^* \in \mathbb{C}^{n \times \phi}.$$  

(3.14)

Now we construct the disturbance $\Delta = (\Delta_1, \ldots, \Delta_N)$ defined by

$$\Delta_j = \begin{cases} z_i^* \bar{\Delta}, & j = i, \\ 0, & j \neq i. \end{cases}$$  

(3.15)

It is easy to check that $\sum_{i=1}^N \| \Delta_i \| = \| \bar{\Delta} \| = 1/\| G_{ii}(\lambda, z) \|$. Moreover, we have

$$\lambda \hat{x} = \left( \sum_{i=1}^N z_i(A_i + D_i \Delta_i E_i) \right) \hat{x},$$  

(3.16)

where $\hat{x} = H(\lambda, z)D_i x$. This means that $\Delta$ is a destabilizing disturbance. Thus,

$$r_c \leq \max_{i \in \mathbb{N}} \sup_{|\lambda| \geq 1, z \in (\partial C_i)^N} \| G_{ii}(\lambda, z) \|.$$  

(3.17)

The proof of (i) is complete, and (ii) can be obtained directly from (i).

In general, the complex, real, and positive radius are distinct, see [4, 5]. Theorem 3.2 reduces the computation of the complex strong stability radius to a global optimization problem with many variations while the problem for the real stability radius is much more difficult, see [5]. It is therefore natural to investigate for which kind of systems these three radii coincide. The answer will be found in the next section.

### 3.2. Strong stability radii of positive systems

In this section, we restrict system (2.6) to be positive, that is, $A_i$ are nonnegative for all $i \in \overline{N}$.

**Lemma 3.3.** Let $A_i \in \mathbb{R}_+^{n \times n}$. Then we have

(i) $r(\sum_{i=1}^N A_i) = \sup_{z \in (\partial C_i)^N} r(\sum_{i=1}^N z_i A_i)$;

(ii) $r(\sum_{i=1}^N A_i) < t_1 \leq t_2 \Rightarrow 0 \leq (t_2 I - \sum_{i=1}^N A_i)^{-1} \leq (t_1 I - \sum_{i=1}^N A_i)^{-1}$,
Proof. (i) By Theorem 2.2, we have
\[
  r\left(\sum_{i=1}^{N} z_i A_i \right) \leq r\left(\sum_{i=1}^{N} |z_i A_i| \right) = r\left(\sum_{i=1}^{N} A_i \right); \tag{3.18}
\]
(ii) the positivity of \((t_1 I - \sum_{i=1}^{N} A_i)^{-1}\), \((t_2 I - \sum_{i=1}^{N} A_i)^{-1}\) can be implied by Theorem 2.1. The right-hand inequality can be obtained by the following formula:
\[
  \left( t_2 I - \sum_{i=1}^{N} A_i \right)^{-1} - \left( t_1 I - \sum_{i=1}^{N} A_i \right)^{-1} = -(t_2 - t_1) \left( t_1 I - \sum_{i=1}^{N} A_i \right)^{-1} \left( t_2 I - \sum_{i=1}^{N} A_i \right)^{-1}. \tag{3.19}
\]
This completes the proof.

It is important to note from above lemma that under positivity assumptions, system (2.6) is strongly stable if and only if \(r(A_1 + \cdots + A_N) < 1\).

Lemma 3.4. Suppose that system (2.6) is positive and strongly stable. Then, for any \(D \in \mathbb{R}^{n \times p}, E \in \mathbb{R}^{p \times n}, \lambda \in \mathbb{C} \setminus \mathbb{C}_1, z \in (\partial \mathbb{C}_1)^N\), we have
\[
  \left\| E \left( \lambda I - \sum_{i=1}^{N} z_i A_i \right)^{-1} D \right\| \leq \left\| E \left( I - \sum_{i=1}^{N} A_i \right)^{-1} D \right\|. \tag{3.20}
\]
Proof. For any \(\lambda \in \mathbb{C} \setminus \mathbb{C}_1, z \in (\partial \mathbb{C}_1)^N\), we have \(r(\sum_{i=1}^{N} z_i A_i) < 1 \leq |\lambda|\). Thus, for an arbitrary vector \(x \in \mathbb{R}^p\),
\[
  \left| E \left( \lambda I - \sum_{i=1}^{N} z_i A_i \right)^{-1} D x \right| \leq E \left( \lambda I - \sum_{i=1}^{N} z_i A_i \right)^{-1} \left| D x \right| \leq E \left( \lambda I - \sum_{i=1}^{N} z_i A_i \right)^{-1} \left| D x \right| \\
  \leq E \left( \sum_{n=0}^{\infty} \frac{\left( \sum_{i=1}^{N} z_i A_i \right)^n}{\lambda^n} \right) D |x| \\
  \leq E \left( \sum_{n=0}^{\infty} \frac{\left( \sum_{i=1}^{N} |z_i A_i| \right)^n}{|\lambda|^n} \right) D |x| \\
  \leq E \left( \sum_{n=0}^{\infty} \left( \sum_{i=1}^{N} A_i \right)^n \right) D |x| \\
  \leq E \left( I - \sum_{i=1}^{N} A_i \right)^{-1} \left| D x \right|. \tag{3.21}
\]
By Lemma 3.3, we have \((I - \sum_{i=1}^{N} A_i)^{-1} \geq 0\). Thus, we imply
\[
  \left\| E \left( \lambda I - \sum_{i=1}^{N} z_i A_i \right)^{-1} D \right\| \leq \left\| E \left( I - \sum_{i=1}^{N} A_i \right)^{-1} D \right\|. \tag{3.22}
\]
Theorem 3.5. Let system (2.6) be strongly stable and positive. Assume that all \( D_i, E_i, i \in \overline{N} \) are nonnegative matrices. If \( D_i = D_j \) or \( E_i = E_j \), \( \forall i, j \in \overline{N} \), then

\[
r_C = r_R = r_* = \frac{1}{\max_{i \in \overline{N}} \|G_{ii}(1, 1)\|^*},
\]

where \( G_{ii}(1, 1) = E_i(I - \sum_{i=1}^{N} A_i)^{-1} D_i \).

Proof. By Theorem 3.2, we have

\[
r_C = \frac{1}{\max_{i \in \overline{N}} \sup_{|\lambda| \leq 1, z \in \partial C} \|G_{ii}(\lambda, z)\|}.
\]

Moreover, using Lemma 3.4, we get

\[
r_C = \frac{1}{\max_{i \in \overline{N}} \|G_{ii}(1, 1)\|^*}.
\]

Since \( r_C \leq r_R \leq r_* \), we only need to prove that

\[
r_* \leq \frac{1}{\max_{i \in \overline{N}} \|G_{ii}(1, 1)\|^*}.
\]

Indeed, for any \( i \in \overline{N} \), since \( G_{ii}(1, 1) \) is a nonnegative matrix, there exists nonnegative vector \( x \in \mathbb{R}^n \) such that \( \|x\| = 1 \) and \( \|G_{ii}(1, 1)x\| = \|G_{ii}(1, 1)\| \). Using Krein-Rutman theorem, see [14], there exists \( y^* \in (\mathbb{R}_+^q)^* \) satisfying \( \|y^*\| = 1 \) and \( y^*(G_{ii}(1, 1)x) = \|G_{ii}(1, 1)x\| \). We define a nonnegative matrix \( \overline{\Delta} \in \mathbb{R}_+^{p \times q} \) by setting

\[
\overline{\Delta} = \frac{1}{\|G_{ii}(1, 1)\|^*} x y^* \in \mathbb{R}_+^{p \times q}.
\]

Now we construct the positive disturbance \( \Delta = (\Delta_1, \ldots, \Delta_N) \) defined by

\[
\Delta_j = \begin{cases} 
\overline{\Delta}, & j = i, \\
0, & j \neq i.
\end{cases}
\]

It is easy to check that \( \sum_{i=1}^{N} \|\Delta_i\| = \|\overline{\Delta}\| = 1/\|G_{ii}(1, 1)\|. \) Moreover, we have

\[
\lambda \tilde{x} = \left( \sum_{i=1}^{N} (A_i + D_i \Delta_i E_i) \right) \tilde{x},
\]

where \( \tilde{x} = H(1, 1)D_i x \). It means that \( \Delta \) is a destabilizing disturbance. Thus

\[
r_* \leq \frac{1}{\max_{i \in \overline{N}} \|G_{ii}(1, 1)\|^*}.
\]

The proof is complete. \( \square \)
Now we turn to a different perturbation structure and assume that each matrix $A_i$ is subjected to perturbations of the form

$$A_i \mapsto A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij},$$

(3.31)

where $B_{ij}$ are given matrices defining the structure of the perturbations and $\delta_{ij}$ are unknown scalars representing parameter uncertainties. So we can write the perturbed system

$$y(t) - \sum_{i=1}^{N} \left( A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij} \right) y(t - r_i) = 0. \tag{3.32}$$

**Definition 3.6.** Let system (2.6) be strongly stable. The complex, real, and positive strong stability radii of system (2.6) under perturbations of the form (3.31) are defined by

$$

c_{\delta}^\delta = \inf \{ \| \delta \|_\infty : \delta = (\delta_{ij}) \in \mathbb{C}^{NK}, \text{ system (3.32) is not strongly stable} \}, \\
r_{\delta}^R = \inf \{ \| \delta \|_\infty : \delta = (\delta_{ij}) \in \mathbb{R}^{NK}, \text{ system (3.32) is not strongly stable} \}, \\
r_{\delta}^+ = \inf \{ \| \delta \|_\infty : \delta = (\delta_{ij}) \in \mathbb{R}_+^{NK}, \text{ system (3.32) is not strongly stable} \},
$$

respectively, we set $\inf \emptyset = +\infty$, and $\| \delta \|_\infty = \max \{ |\delta_{ij}| : i \in \mathcal{N}, j \in \mathcal{K} \}$, where $\mathcal{K} = \{1, \ldots, K\}$.

**Lemma 3.7.** Suppose system (2.6) is strongly stable, positive and $B_{ij} \in \mathbb{R}^{n \times n}$, $i \in \mathcal{N}, j \in \mathcal{K}$. Then

$$c_{\delta}^\delta = r_{\delta}^R = r_{\delta}^+. \tag{3.34}$$

**Proof.** Because $c_{\delta}^\delta \leq r_{\delta}^R \leq r_{\delta}^+$, we only need to prove that $r_{\delta}^\delta \geq r_{\delta}^+$. Indeed, for a destabilizing disturbance $\delta = (\delta_{ij})_{i \in \mathcal{N}, j \in \mathcal{K}}$, there exist a $\lambda \in \mathbb{C} \setminus \mathbb{C}_1$, $z \in (\delta \mathbb{C}_1)^N$ and a nonzero vector $x \in \mathbb{K}^n$ such that

$$\sum_{i=1}^{N} z_i \left( A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij} \right) x = \lambda x. \tag{3.35}$$

This yields

$$\sum_{i=1}^{N} \left( A_i + \sum_{j=1}^{K} |\delta_{ij}| B_{ij} \right) |x| \geq \left| \sum_{i=1}^{N} z_i \left( A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij} \right) x \right| = |\lambda||x|. \tag{3.36}$$

By Theorem 2.1, we get

$$r \left( \sum_{i=1}^{N} \left( A_i + \sum_{j=1}^{K} \delta_{ij} B_{ij} \right) \right) \geq 1. \tag{3.37}$$

It means that $|\delta| = (|\delta_{ij}|)_{i \in \mathcal{N}, j \in \mathcal{K}}$ is also a destabilizing disturbance. Thus, by the definition of complex and real radii, $c_{\delta}^\delta \geq r_{\delta}^+$. The proof is complete. \qed
Theorem 3.8. Suppose system (2.6) is strongly stable, positive and $B_{ij} \in \mathbb{R}^{n \times n}$, $i \in \mathbb{N}$, $j \in \mathbb{K}$. Then

$$r^\delta_C = r^\delta_R = r^\delta_+ = \frac{1}{r((I - \sum_{i=1}^{N} A_i)\bar{B})},$$

(3.38)

where $B = \sum_{i,j} B_{ij}$.

Proof. By Lemma 3.7, we only need to prove that

$$r^\delta_+ = \frac{1}{r((I - \sum_{i=1}^{N} A_i)\bar{B})}.$$  

(3.39)

To do it, taking arbitrary destabilizing disturbance $\delta = (\delta_{ij})_{i \in \mathbb{N}, j \in \mathbb{K}} \in \mathbb{R}^{NK}$, by Lemma 3.3 and Theorem 2.1, there exist a $\lambda \geq 1$ and a nonzero vector $x \in \mathbb{R}^{n_+}$ such that

$$\left(\sum_{i=1}^{N} A_i + \sum_{j=1}^{K} \delta_{ij}B_{ij}\right)x = \lambda x,$$

(3.40)

or equivalently,

$$\sum_{i,j} \delta_{ij}B_{ij}x = \left(\lambda I - \sum_{i=1}^{N} A_i\right)x.$$  

(3.41)

This yields

$$\left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1}\left(\sum_{i,j} \delta_{ij}B_{ij}\right)x = x.$$  

(3.42)

Then, we have

$$\|\delta\|_{\infty} \left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1} B x \geq \left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1}\left(\sum_{i,j} \delta_{ij}B_{ij}\right)x = x.$$  

(3.43)

Using Theorem 2.1 again, we obtain

$$r \left(\left(\lambda I - \sum_{i=1}^{N} A_i\right)^{-1} B\right) \geq \frac{1}{\|\delta\|_{\infty}}.$$  

(3.44)

or equivalently,

$$\|\delta\|_{\infty} \geq \frac{1}{r((\lambda I - \sum_{i=1}^{N} A_i)^{-1} B)}.$$  

(3.45)

Thus, from the definition of $r^\delta_+$, one has

$$r^\delta_+ \geq \frac{1}{r((I - \sum_{i=1}^{N} A_i)\bar{B})}.$$  

(3.46)
On the other hand, setting \( \lambda = r((I - \sum_{i=1}^{N} A_i)^{-1}B) > 0 \). Then, by Theorem 2.1, there exists a nonnegative vector \( x \in \mathbb{R}_n^N \) satisfying

\[
\left( \left( I - \sum_{i=1}^{N} A_i \right)^{-1}B \right) x = \lambda x. \tag{3.47}
\]

This is equivalent to

\[
Bx = \lambda \left( I - \sum_{i=1}^{N} A_i \right) x. \tag{3.48}
\]

Hence,

\[
\sum_{i=1}^{N} \left( A_i + \frac{1}{\lambda} B \right) x = x. \tag{3.49}
\]

This means that \( \delta = (1/\lambda) \in \mathbb{R}_n^{NK} \) is a destabilizing disturbance and thus, \( r_\delta^2 \leq 1/r((I - \sum_{i=1}^{N} A_i)^{-1}B) \). The proof is complete. \( \square \)

Now we consider the following example to illustrate the obtained results.

**Example 3.9.** Consider system

\[
y(t) = A_1 y(t-r) + A_2 y(t-s), \tag{3.50}
\]

where

\[
A_1 = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 4 & 3 \end{pmatrix}. \tag{3.51}
\]

Then we have

\[
r(A_1 + A_2) = \frac{7 + \sqrt{145}}{24} < 1. \tag{3.52}
\]

Thus system (3.50) is strongly stable.

Assume that the matrices \( A_1, A_2 \) are subjected to perturbations of the form \( A_1 \rightarrow A_1 + D_1 \Delta_1 E_1, \ A_2 \rightarrow A_2 + D_2 \Delta_2 E_2 \), where

\[
D_1 = D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.53}
\]

Then

\[
\begin{pmatrix} 29/3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 14/3 & 5 \\ 0 & 0 \end{pmatrix}. \tag{3.54}
\]
If $\mathbb{R}^2$ is provided with the norm defined by $\|(x, y)\| = |x| + |y|$, then by Theorem 3.5, we have

$$r_{\mathcal{C}} = r_{\mathbb{R}} = r_+ = \frac{3}{25}. \quad (3.55)$$

Assume that the given two matrices $A_1$, $A_2$ are subjected to perturbations of the form $A_1 \to A_1 + \delta_{11} B_{11} + \delta_{12} B_{12}$, $A_2 \to A_2 + \delta_{21} B_{21} + \delta_{22} B_{22}$, where

$$B_{11} = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{12} & 1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}, \quad B_{22} = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}. \quad (3.56)$$

Then

$$(I - A_1 - A_2)^{-1} B = \begin{pmatrix} 10 & \frac{227}{18} \\ 11 & \frac{163}{12} \end{pmatrix}. \quad (3.57)$$

By Theorem 3.8, we get

$$r_{\mathcal{C}}^p = r_{\mathbb{R}}^p = r_+^p = \frac{24}{283 + \sqrt{81753}}. \quad (3.58)$$

References


