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Research Article

Analytical Solution for the Time-Fractional Telegraph Equation

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We discuss and derive the analytical solution for three basic problems of the so-called time-fractional telegraph equation. The Cauchy and Signaling problems are solved by means of juxtaposition of transforms of the Laplace and Fourier transforms in variable t and x, respectively. the appropriate structures and negative prosperities for their Green functions are provided. The boundary problem in a bounded space domain is also solved by the spatial Sine transform and temporal Laplace transform, whose solution is given in the form of a series.

1. Introduction

Fractional differential equations (FDEs) have attracted in the recent years a considerable interest due to their frequent appearance in various fields and their more accurate models of systems under consideration provided by fractional derivatives. For example, fractional derivatives have been used successfully to model frequency dependent damping behavior of many viscoelastic materials. They are also used in modeling of many chemical processed, mathematical biology and many other problems in engineering. The history and a comprehensive treatment of FDEs are provided by Podlubny [1] and a review of some applications of FDEs are given by Mainardi [2].

The fractional telegraph equation has recently been considered by many authors. Cascaval et al. [3] discussed the time-fractional telegraph equations, dealing with well-posedness and presenting a study involving asymptotic by using the Riemann-Liouville approach. Orsingher and Beghin [4] discussed the time-fractional telegraph equation and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations. Chen et al. [5] also discussed and derived the

solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, by the method of separating variables. Orsingher and Zhao [6] considered the space-fractional telegraph equations, obtaining the Fourier transform of its fundamental solution and presenting a symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation. Momani [7] discussed analytic and approximate solutions of the space- and time-fractional telegraph differential equations by means of the so-called Adomian decomposition method. Camargo et al. [8] discussed the so-called general space-time fractional telegraph equations by the methods of differential and integral calculus, discussing the solution by means of the Laplace and Fourier transforms in variables t and x, respectively.

In this paper, we consider the following time-fractional telegraph equation (TFTE)

$$D_t^{2\alpha}u(x,t) + 2aD_t^{\alpha}u(x,t) = d\frac{\partial^2}{\partial x^2}u(x,t) + f(x,t), \quad t \in \mathbb{R}^+, \tag{1.1}$$

where a, d are positive constants, $1/2 < \alpha \leq 1$, D_t^{β} is the fractional derivative defined in the Caputo sense:

$$D_t^{\beta}f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n}, & \beta = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\beta)}\displaystyle\int_0^t (t-\tau)^{n-\beta-1}\frac{d^n f(\tau)}{d\tau^n}d\tau, & n-1 < \beta < n, \end{cases} \tag{1.2}$$

where $f(t)$ is a continuous function. Properties and more details about the Caputo's fractional derivative also can be found in [1, 2].

For the TFTE (1.1), we will consider three basic problems with the following three kinds of initial and boundary conditions, respectively.

Problem 1. TFTE in a whole-space domain (Cauchy problem)

$$\begin{gathered} u(x,0) = \phi(x), \quad \frac{\partial}{\partial t}u(x,0) = 0, \quad x \in \mathbb{R}, \\ u(\mp\infty, t) = 0, \quad t > 0. \end{gathered} \tag{1.3}$$

Problem 2. TFTE in a half-space domain (Signaling problem)

$$u(x,0) = \frac{\partial}{\partial t}u(x,0) = 0, \quad x \in \mathbb{R}^+, \tag{1.4}$$

$$u(0,t) = g(t), \quad u(+\infty,t) = 0 \quad t > 0, \tag{1.5}$$

and we set $f(x,t) = 0$ in (1.1).

Problem 3. TFTE in a bounded-space domain

$$u(x,0) = \phi(x), \quad \frac{\partial}{\partial t}u(x,0) = \varphi(x), \quad 0 < x \le L, \tag{1.6}$$

$$u(0,t) = u(L,t) = 0, \quad t > 0, \tag{1.7}$$

here we also set $f(x,t) = 0$ in (1.1).

In this paper, we derive the analytical solutions of the previous three problems for the TFTE. The structure of the paper is as follows. In Section 2, by using the method of Laplace and Fourier transforms, the fundamental solution of Problem 1 is derived. In Section 3, by investigating the explicit relationships of the Laplace Transforms to the Green functions between Problems 1 and 2, the fundamental solution of the Problem 2 is also derived. The analytical solution of Problem 3 is presented in Section 4. Some conclusions are drawn in Section 5.

2. The Cauchy Problem for the TFTE

We first focus our attention on (1.1) in a whole-space domain, that is to say, Problem 1 will to be considered, which we refer to as the so-called Cauchy problem.

Applying temporal Laplace and spatial Fourier transforms to (1.1) and using the initial boundary conditions (1.3), we obtain the following nonhomogeneous differential equation:

$$\begin{aligned} P^{2\alpha}\tilde{u}(x,p) - p^{2\alpha-1}\phi(x) + 2ap^{\alpha}\tilde{u}(x,p) - 2ap^{\alpha-1}\phi(x) &= d\frac{\partial^2}{\partial x^2}\tilde{u}(x,p) + \tilde{f}(x,p), \\ P^{2\alpha}\hat{\tilde{u}}(k,p) - p^{2\alpha-1}\hat{\phi}(k) + 2ap^{\alpha}\hat{\tilde{u}}(k,p) - 2ap^{\alpha-1}\hat{\phi}(k) &= -dk^2\hat{\tilde{u}}(k,p) + \hat{\tilde{f}}(k,p). \end{aligned} \tag{2.1}$$

Then we derive

$$\begin{aligned} \hat{\tilde{u}}(k,p) &= \frac{p^{2\alpha-1} + 2ap^{\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + dk^2}\hat{\phi}(k) + \frac{1}{p^{2\alpha} + 2ap^{\alpha} + dk^2}\hat{\tilde{f}}(k,p) \\ &:= \hat{\tilde{G}}_1(k,p)\hat{\phi}(k) + \hat{\tilde{G}}_2(k,p)\hat{\tilde{f}}(k,p), \end{aligned} \tag{2.2}$$

where

$$\hat{\tilde{G}}_2(k,p) = \frac{1}{p^{2\alpha} + 2ap^{\alpha} + dk^2}, \tag{2.3}$$

$$\begin{aligned} \hat{\tilde{G}}_1(k,p) &= \frac{p^{2\alpha-1} + 2ap^{\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + dk^2} := \hat{\tilde{G}}_{1,1}(k,p) + \hat{\tilde{G}}_{1,2}(k,p), \\ \hat{\tilde{G}}_{1,1}(k,p) &= \frac{p^{2\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + dk^2}, \qquad \hat{\tilde{G}}_{1,2}(k,p) = \frac{2ap^{\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + dk^2}. \end{aligned} \tag{2.4}$$

By the Fourier transform pair

$$e^{-c|x|} \overset{\mathcal{F}}{\leftrightarrow} \frac{2c}{c^2+k^2}, \tag{2.5}$$

we also have

$$\widetilde{G}_{1,1}(x,p) = \frac{p^{2\alpha-1}}{2\sqrt{d(p^{2\alpha}+2ap^{\alpha})}} e^{-\sqrt{((p^{2\alpha}+2ap^{\alpha})/d)}|x|}, \tag{2.6}$$

$$\widetilde{G}_{1,2}(x,p) = \frac{2ap^{\alpha-1}}{2\sqrt{d(p^{2\alpha}+2ap^{\alpha})}} e^{-\sqrt{((p^{2\alpha}+2ap^{\alpha})/d)}|x|}. \tag{2.7}$$

We invert the Fourier transform in (2.2) to obtain

$$u(x,t) = \int_{-\infty}^{+\infty} G_1(x-y,t)\phi(y)dy + \int_{-\infty}^{+\infty} dy \int_0^t d\tau G_2(x-y,t-\tau) f(y,\tau), \tag{2.8}$$

where $G_1(x,t)$, $G_2(x,t)$ is the corresponding Green function or fundamental solution obtained when $\phi(x) = \delta(x)$, $f(x) = 0$ and $\phi(x) = 0$, $f(x,t) = \delta(x)\delta(t)$ respectively, which is characterized by (2.4) or (2.3).

To express the Green function, we recall two Laplace transform pairs and one Fourier transform pair,

$$\begin{gathered} F_1^{(\beta)}(ct) := t^{-\beta} M_\beta\left(ct^{-\beta}\right) \overset{\mathcal{L}}{\leftrightarrow} p^{\beta-1} e^{-cp^\beta}, \\ F_2^{(\beta)}(ct) := c w_\beta(ct) \overset{\mathcal{L}}{\leftrightarrow} e^{-(p/c)^\beta}, \\ F_3(ct) := \frac{1}{2\sqrt{\pi}} c^{-1/2} e^{-x^2/4c} \overset{\mathcal{F}}{\leftrightarrow} e^{-ck^2}, \end{gathered} \tag{2.9}$$

where M_β denotes the so-called M function (of the Wright type) of order β, which is defined

$$M_\beta(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\beta n + (1-\beta)]}, \quad 0 < \beta < 1. \tag{2.10}$$

Mainardi, see, for example, [9] has showed that $M_\beta(z)$ is positive for $z > 0$, the other general properties can be found in some references (see [1, 9–11] e.g,).

w_β $(0 < \beta < 1)$ denotes the one-sided stable (or Lévy) probability density which can be explicitly expressed by Fox function [12]

$$w_\beta(t) = \beta^{-1} t^{-2} H_{11}^{10}\left(t^{-1} \middle| \begin{matrix} (-1,1) \\ (-1/\beta, 1/\beta) \end{matrix}\right). \tag{2.11}$$

Then the Fourier-Laplace transform of the Green function (2.4) can be rewritten in integral form

$$\begin{aligned}
\widehat{\widetilde{G}}_1(k,p) &= \left(p^{2\alpha-1}+2ap^{\alpha-1}\right)\int_0^{+\infty} e^{-u\left(p^{2\alpha}+2ap^{\alpha}+dk^2\right)}du \\
&= \int_0^{+\infty}\left(p^{2\alpha-1}e^{-up^{2\alpha}}\right)e^{-2ap^{\alpha}u}e^{-dk^2u}du + 2a\int_0^{+\infty}\left(p^{\alpha-1}e^{-2ap^{\alpha}u}\right)e^{-p^{2\alpha}u}e^{-dk^2u}du \\
&= \int_0^{+\infty}\mathcal{L}\left\{F_1^{(2\alpha)}(ut)\right\}\cdot\mathcal{L}\left\{F_2^{(\alpha)}\left[(2au)^{-1/\alpha}t\right]\right\}\cdot\mathcal{F}\{F_3(dut)\}du \\
&\quad + 2a\int_0^{+\infty}\mathcal{L}\left\{F_1^{(\alpha)}(ut)\right\}\cdot\mathcal{L}\left\{F_2^{(2\alpha)}\left[u^{-1/2\alpha}t\right]\right\}\cdot\mathcal{F}\{F_3(dut)\}du \\
&= \int_0^{+\infty}\mathcal{L}\left\{F_1^{(2\alpha)}(ut)*F_2^{(\alpha)}\left[(2au)^{-1/\alpha}t\right]\right\}\cdot\mathcal{F}\{F_3(dut)\}du \\
&\quad + 2a\int_0^{+\infty}\mathcal{L}\left\{F_1^{(\alpha)}(ut)*F_2^{(2\alpha)}\left[u^{-1/2\alpha}t\right]\right\}\cdot\mathcal{F}\{F_3(dut)\}du.
\end{aligned} \tag{2.12}$$

Going back to the space-time domain we obtain the relation

$$\begin{aligned}
G_1(x,t) &= \int_0^{+\infty}\left\{F_1^{(2\alpha)}(ut)*F_2^{(\alpha)}\left[(2au)^{-1/\alpha}t\right]\right\}F_3(dut)du \\
&\quad + 2a\int_0^{+\infty}\left\{F_1^{(\alpha)}(ut)*F_2^{(2\alpha)}\left[u^{-1/2\alpha}t\right]\right\}F_3(dut)du \\
&= \int_0^{+\infty}F_3(dut)\left(\int_0^t F_1^{(2\alpha)}[u(t-\tau)]F_2^{(\alpha)}\left[(2au)^{-1/\alpha}\tau\right]d\tau\right)du \\
&\quad + 2a\int_0^{+\infty}F_3(dut)\left(\int_0^t F_1^{(\alpha)}[u(t-\tau)]F_2^{(2\alpha)}\left[u^{-1/2\alpha}\tau\right]d\tau\right)du \\
&:= G_{1,1}(x,t)+G_{1,2}(x,t).
\end{aligned} \tag{2.13}$$

By the same technique, we can obtain the expression of $G_2(x,t)$:

$$\begin{aligned}
\widehat{\widetilde{G}}_2(k,p) &= \int_0^{+\infty} e^{-u\left(p^{2\alpha}+2ap^{\alpha}+dk^2\right)}du \\
&= \int_0^{+\infty} e^{-up^{2\alpha}}e^{-2ap^{\alpha}u}e^{-dk^2u}du \\
&= \int_0^{+\infty}\mathcal{L}\left\{F_2^{(2\alpha)}\left(u^{-1/2\alpha}t\right)*F_2^{(\alpha)}\left[(2au)^{-1/\alpha}t\right]\right\}\cdot\mathcal{F}\{F_3(dut)\}du.
\end{aligned} \tag{2.14}$$

Going back to the space-time domain we obtain the relation

$$G_2(x,t) = \int_0^{+\infty} F_3(dut)\left(\int_0^t F_2^{(2\alpha)}\left[u^{-1/2\alpha}(t-\tau)\right]F_2^{(\alpha)}\left[(2au)^{-1/\alpha}\tau\right]d\tau\right)du. \tag{2.15}$$

We can ensure that the green functions are nonnegative by the nonnegative prosperities of $F_1^{(\beta)}, F_2^{(\beta)}, F_3$.

3. The Solution for the TFTE in Half-Space Domain (Signaling Problems)

In this section, we considered Problem 2, defined in a half-space domain, which we refer to as the so-called Signaling problem.

By the application of the Laplace transform to (1.1) and (1.5) with $f \equiv 0$ and the initial condition (1.4), we get

$$\begin{gathered}\frac{\partial^2 \tilde{u}(x,p)}{\partial x^2} = \frac{p^{2\alpha} + 2ap^{\alpha}}{d}\tilde{u}(x,p), \\ \tilde{u}(0,p) = \tilde{g}(p), \qquad \tilde{u}(+\infty,p) = 0\end{gathered} \tag{3.1}$$

with the solution

$$\tilde{u}(x,p) = \tilde{g}(p)e^{-\sqrt{\left(\left(p^{2\alpha}+2ap^{\alpha}\right)/d\right)}x} = \mathcal{L}\{G_s(x,t) * g(t)\}, \tag{3.2}$$

where $G_s(x,t)$ is the Green function or fundamental solution of the Signaling problem obtained when $g(x) = \delta(x)$, which is characterized by

$$\tilde{G}_s(x,p) = e^{-\sqrt{\left(\left(p^{2\alpha}+2ap^{\alpha}\right)/d\right)}x}. \tag{3.3}$$

The inverse Laplace transform of (3.2) gives the solution of Problem 2

$$u(x,t) = G_s(x,t) * g(t) = \int_0^t G_s(x,t-\tau)g(\tau)d\tau. \tag{3.4}$$

From (2.6), (2.7) and (3.3), we recognize the relation

$$\frac{\partial}{\partial p}\tilde{G}_s(x,p) = -2\alpha x\tilde{G}_{1,1}(x,p) - \alpha x\tilde{G}_{1,2}(x,p), \quad x > 0. \tag{3.5}$$

Returning to the space-time domain we obtain the relation

$$tG_s(x,t) = 2\alpha x G_{1,1}(x,t) + \alpha x G_{1,2}(x,t), \quad x,t > 0. \tag{3.6}$$

Then we can obtain a representation for $G_s(x,t)$ and prove the negative prosperities.

4. The Solution of the TFTE in a Bounded-Space Domain

In this section we seek the solution of Problem 3, which is defined in a bounded domain.

Taking the finite Sine transform of (1.1) with $f = 0$, and applying the boundary conditions (1.7), we obtain

$$D_t^{2\alpha}\overline{u}(n,t) + 2aD_t^{\alpha}\overline{u}(n,t) = -\left(\frac{nd\pi}{L}\right)^2 \overline{u}(n,t), \quad t > 0, \tag{4.1}$$

where n is a wave number, and

$$\overline{u}(n,t) = \int_0^L u(y,t)\sin\left(\frac{n\pi y}{L}\right)dy \tag{4.2}$$

is the finite Sine transform of $u(x,t)$.

Applying the Laplace transform to (4.1) and using the initial conditions (1.6), we obtain

$$\begin{gathered}
\tilde{\overline{u}}(n,p) = \frac{\left(p^{2\alpha-1} + 2ap^{\alpha-1}\right)\overline{u}(n,0)}{p^{2\alpha} + 2ap^{\alpha} + (nd\pi/L)^2} + \frac{p^{2\alpha-2}\overline{u}_t(n,0)}{p^{2\alpha} + 2ap^{\alpha} + (nd\pi/L)^2}, \\
\overline{u}(n,0) = \int_0^L \phi(y)\sin\left(\frac{n\pi y}{L}\right)dy, \\
\overline{u}_t(n,0) = \int_0^L \varphi(y)\sin\left(\frac{n\pi y}{L}\right)dy.
\end{gathered} \tag{4.3}$$

We set $\lambda_{\pm} = -a \pm \sqrt{a^2 - (nd\pi/L)^2}$, then

$$p^{2\alpha} + 2ap^{\alpha} + \left(\frac{nd\pi}{L}\right)^2 = \left(p^{\alpha} - \lambda_-\right)\left(p^{\alpha} - \lambda_+\right). \tag{4.4}$$

To inverse the Laplace transform for (4.3), we recall the known Laplace transform pair

$$t^{\alpha-\beta}E_{\alpha,\beta}(ct^{\alpha}) \overset{\mathcal{L}}{\Leftrightarrow} \frac{p^{\alpha-\beta}}{p^{\alpha} - c}, \tag{4.5}$$

where $E_{\alpha,\beta}(z)$ is the so-called two-parameter Mittag-Leffler function, which is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta > 0, \tag{4.6}$$

and we note $E_{\alpha,1} = E_{\alpha}$.

Then we obtain the pairs

$$\begin{aligned}
\frac{p^{2\alpha-1}+2ap^{\alpha-1}}{p^{2\alpha}+2ap^{\alpha}+(nd\pi/L)^2} &= \frac{c_1p^{\alpha-1}}{p^{\alpha}-\lambda_-}-\frac{c_2p^{\alpha-1}}{p^{\alpha}-\lambda_+} \overset{\mathcal{L}}{\leftrightarrow} (c_1E_{\alpha}(\lambda_-t^{\alpha})-c_2E_{\alpha}(\lambda_+t^{\alpha})),\\
\frac{p^{2\alpha-2}}{p^{2\alpha}+2ap^{\alpha}+(nd\pi/L)^2} &= \frac{c_1p^{\alpha-2}}{p^{\alpha}-\lambda_+}-\frac{c_2p^{\alpha-2}}{p^{\alpha}-\lambda_-} \overset{\mathcal{L}}{\leftrightarrow} (c_1E_{\alpha,2}(\lambda_+t^{\alpha})-c_2E_{\alpha,2}(\lambda_-t^{\alpha})),
\end{aligned} \tag{4.7}$$

where $c_1=\lambda_+/(\lambda_+-\lambda_-)$, $c_2=\lambda_-/(\lambda_+-\lambda_-)$.

So we inverse Laplace and finite Sine transform for (4.3) to obtain

$$\begin{aligned}
u(x,t) = {} & \frac{2}{L}\sum_{n=1}^{\infty}(c_1E_{\alpha}(\lambda_-t^{\alpha})-c_2E_{\alpha}(\lambda_+t^{\alpha}))\sin\left(\frac{n\pi x}{L}\right)\int_0^L\phi(y)\sin\left(\frac{n\pi y}{L}\right)dy\\
& + \frac{2}{L}\sum_{n=1}^{\infty}(c_1E_{\alpha,2}(\lambda_+t^{\alpha})-c_2E_{\alpha,2}(\lambda_-t^{\alpha}))\sin\left(\frac{n\pi x}{L}\right)\int_0^L\varphi(y)\sin\left(\frac{n\pi y}{L}\right)dy.
\end{aligned} \tag{4.8}$$

5. Conclusions

In this paper we have considered the time-fractional telegraph equation. The fundamental solution for the Cauchy problem in a whole-space domain and Signaling problem in a half-space domain is obtained by using Fourier-Laplace transforms and their inverse transforms. The appropriate structures and negative prosperities for the Green functions are provided. On the other hand, the solution in the form of a series for the boundary problem in a bounded-space domain is derived by the Sine-Laplace transforms method.

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