Research Article

An Analytic Solution for a Vasicek Interest Rate Convertible Bond Model

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This paper provides the analytic solution to the partial differential equation for the value of a convertible bond. The equation assumes a Vasicek model for the interest rate and a geometric Brownian motion model for the stock price. The solution is obtained using integral transforms.

This work corrects errors in the original paper by Mallier and Deakin [1] on the Green's function for the Vasicek convertible bond equation. One error involves subtle points of the inverse Laplace transform. We show that the solution of

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma cS \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}c^2 \frac{\partial^2 V}{\partial r^2} + rS \frac{\partial V}{\partial S} + (a - br) \frac{\partial V}{\partial r} - rV \quad (1)$$

in the log stock variables $x = \log S$ and $\tilde{x} = \log \tilde{S}$ is

$$V(S, r, \tau) = \iint_{-\infty}^{\infty} V_0(e^{\tilde{x}}, \tilde{r}) G(r, \tilde{r}, x - \tilde{x}) d\tilde{r} d\tilde{x}, \quad (2)$$

where $V = V_0(S, r)$ at $\tau = 0$ and the Green's function (GF) is

$$G(r, \tilde{r}, x - \tilde{x}) = \exp(F) N(w, \Xi) N(\alpha, \Phi). \quad (3)$$

The normal distribution with variance w and argument Ξ is here denoted by

$$N(w, \Xi) = (2\pi w)^{-1/2} \exp\left[-\frac{\Xi^2}{(2w)}\right], \quad (4)$$

and the coefficients are

$$w = \frac{(1 - e^{-2b\tau})c^2}{2b}, \quad \Xi = \tilde{r} - re^{-b\tau} - B\left(a - \frac{Bc^2}{2}\right), \quad (5)$$

$$F = A - Br, \quad \Phi = \tilde{x} - x - D - \frac{\Xi(2\rho\sigma/c + B)}{1 + e^{-b\tau}}, \quad (6)$$

$$\alpha = \tau\sigma^2(1 - \rho^2) + \left(\frac{c}{b} + \rho\sigma\right)^2\left(\tau - \frac{2}{b}\tanh\left(\frac{b\tau}{2}\right)\right), \quad (7)$$

$$F + D + \frac{v}{2} = 0, \quad A = \frac{(B - \tau)(2ab - c^2)}{2b^2} - \frac{c^2B^2}{4b}, \quad (8)$$

$$B = \frac{1 - e^{-b\tau}}{b}, \quad v = \tau\sigma^2 + \frac{(\tau - B)(2\rho\sigma b + c)c}{b^2} - \frac{c^2B^2}{2b}. \quad (9)$$

In the case of the convertible bond, the initial condition V_0 in (2) is independent of $\tilde{r}$. Integrating (2) in $\tilde{r}$, we obtain the simpler Green's function

$$G(r, \tau, x - \tilde{x}) = \exp(F(r, \tau))N(v(\tau), \tilde{x} - x - D(r, \tau)). \quad (10)$$

The parameters in the solution have the range of values: $\sigma > 0$, $c > 0$, $|\rho| < 1$, while a and b are arbitrary since the solutions are analytic in a and b .

To prove (3), we assume V to be bounded as $S \rightarrow 0$ and $S^{c_0}V$, where c_0 is a positive constant, is bounded as $S \rightarrow \infty$ so that the Mellin transform of V exists. Once the solution is determined, the initial condition may be extended to include the more general case where the integral (2) exists (e.g., $V_0 = \max(S, 1)$). In the derivation of the solution, the condition $b > 0$ is assumed in (1).

To solve for V in (1), the Mellin and Laplace transform $\hat{V}(p) := \mathcal{M}[V]$ and $\bar{V}(z) := \mathcal{L}[\hat{V}]$ (equations (2.6), (2.7) in [1]) are applied to obtain the ODE

$$\left(\frac{c^2}{2}\right)\bar{V}_{rr} + (a - \rho c \sigma p - br)\bar{V}_r + \left[(2^{-1}\sigma^2 p - r)(1 + p) - z\right]\bar{V} = -\mathcal{M}[V_0(S, r)]. \quad (11)$$

The general homogeneous solution ([2, 3] Section V.I, page 249) of (11) is

$$V_h = \exp\left(-\frac{(1+p)r}{b}\right) \mathcal{F}\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{u^2}{2}\right), \quad (12)$$

$$-\nu = \frac{z}{b} + 2E, \quad u(r) = \sqrt{\frac{2}{b^3 c^2}} \left(r b^2 - a b + c^2 + p(c b \sigma \rho + c^2) \right), \quad (13)$$

$$E = \frac{(1+p)(2ab - c^2 - p\Lambda)b^{-3}}{4}, \quad \Lambda = (c + b\sigma\rho)^2 + (b\sigma)^2(1 - \rho^2), \quad (14)$$

and $\mathcal{F}$ is the general solution of the confluent hypergeometric equation ([2, 3] Section V.I). The general solution (12) in terms of the parabolic cylinder function $D_\nu(u)$ ([2, 3] Section V.II, page 117), with arbitrary constants C_1 and C_2 ($\nu \neq 0, 1, \dots$), is

$$V_h = \exp\left(-\frac{(1+p)r}{b}\right) 2^{-\nu/2} e^{(u^2/4)} (C_1 D_\nu(u) + C_2 D_\nu(-u)). \quad (15)$$

Replacing $\mathcal{M}[V_0(S, r)]$ in (11) by the delta function $\delta(r - \tilde{r})$ (c.f., (20) for details), the GF for (11) has the form

$$G_1(r, \tilde{r}) = 2c^{-2} h_1(r) h_2(\tilde{r}) W^{-1}[h_1(\tilde{r}), h_2(\tilde{r})], \quad r > \tilde{r}, \quad (16)$$

where h_j are appropriate homogeneous solutions in (15), W is the Wronskian, and G_1 for $r < \tilde{r}$ is defined by interchanging r and $\tilde{r}$ in h_j , but not in W .

For the existence and the evaluation of the inverse Laplace transform (ILT) of G_1 , the asymptotic expansion, valid for large $(-\nu)$ in the sector $|\arg(-\nu)| < \pi$,

$$\Gamma(-\nu) D_\nu(v(r)) D_\nu(-w(\tilde{r})) \sim \left(-\frac{\nu^2}{\pi}\right)^{-1/2} \exp\left(-(-\nu)^{1/2}(v(r) - w(\tilde{r}))\right) \quad (17)$$

is required where $v(r) = \pm u(r)$ and $w(\tilde{r}) = \pm u(\tilde{r})$. The expansion for the Gamma function is given in ([2, 3] Section V.I, page 47). The expansion with a restricted domain for the parabolic cylinder function appears in [2, 3] (Section V.I, page 249 (8)) and the general case is proved by applying the Method of Steepest Descent to the integral representation ([4, 5], page 349). The solutions h_i in (16) must be chosen such that G_1 has an ILT that exists for all r and $\tilde{r}$. For the general case, we define h_i in (15) by replacing C_j by C_{ij} . There are four terms in (16), only one for which the ILT exists: $C_{12} = C_{21} = 0$, $v - w = (2b)^{1/2}|r - \tilde{r}|/c$ in (17). Thus,

$$G_1 = g_1(r) g_2(\tilde{r}) c^{-1} (b\pi)^{-1/2} \Gamma(-\nu) D_\nu(u(r)) D_\nu(-u(\tilde{r})), \quad r > \tilde{r}, \quad (18)$$

where $g_j(r) = \exp[(-1)^j((1+p)r/b - u^2(r)/4)]$. For $r < \tilde{r}$, G_1 is defined by interchanging r and $\tilde{r}$ in D_ν . However, to explain the results in [1], we compare (2.16) to (16, 20) so that $h_1 \propto \mathcal{U}_2$ and $h_2 \propto \mathcal{U}_1$ in (2.13) (change sign on RHS of (2.14), (2.16)). Consequently, h_1 and h_2 are defined in (15) by taking $(C_1 = 0, C_2 = 1)$ and $(C_1 = -1, C_2 = 1)$, respectively. The modified

GF is $G_1^m := -G_1^* + G_1^s$ where G_1^* and G_1^s are defined from G_1 by changing u to $-u$ and $u(r)$ to $-u(r)$, respectively.

As outlined in [1], the ILT $G_2 := \mathcal{L}^{-1}(G_1)$ ((2.17), [1]) is equal to the contributions from the simple poles of $\Gamma(-\nu)$ at $\nu = n$ ($n = 0, 1 \dots$). G_2 is equal to a sum involving Hermite polynomials ([2, 3] Section V.II, page 194 (22)) so that

$$G_2 = N(\eta, \tilde{r} - r) \exp \left[\frac{\sqrt{2b}}{4c} (r - \tilde{r}) s_1 - s_2 \frac{\lambda b \tau}{8} + \frac{b\tau}{2} - 2bE\tau - \frac{(1+p)}{b} (r - \tilde{r}) \right], \quad (19)$$

where $s_m = u^m(r) + u^m(\tilde{r})$, $\eta = \tau c^2 \sinh(b\tau)/(b\tau)$, $\lambda = (2/(b\tau)) \tanh(b\tau/2)$. The semicircle's contribution to G_2 goes to zero as the radius goes to infinity follows from the approximation of G_1 in (18) via (17). For the modified GF, $G_2^m := -G_2^* + G_2^s$ where G_2^* and G_2^s are formally defined by the contributions from the poles: $G_2^* = G_2$, $G_2^s = G_2 \exp(-u(r)u(\tilde{r})/\sinh(b\tau))$.

The last step is to evaluate the inverse Mellin transforms (IMT; (2.18), [1]) $G_3 := \mathcal{M}^{-1}G_2$ and, for the modified GF, $G_3^m := -G_3 + G_3^s$, where $G_3^s := \mathcal{M}^{-1}G_2^s$. To do this, the argument of the exponential in G_2 and G_2^s is expressed in the form $\alpha p^2/2 + \beta p + \gamma$, and formula (2.29) in [1] is applied. For G_2 , α is given by (7). For G_2^s , $\alpha := \alpha^s$ is given by (7) where $\tanh$ is replaced by $\coth$. Correcting the error in [1] (page 228, L.-4, (+) to (-)), then $2\alpha_+ = \alpha$ and $2\alpha_- = \alpha^s$, where $\alpha_{\pm}$ appear in (2.27) and (2.33). Assuming that $(c/b + \rho\sigma) \neq 0$, then there is a positive number τ_0 such that $\alpha_- < 0$ for $0 < \tau < \tau_0$. Thus the IMT of G_2^s does not exist for $0 < \tau < \tau_0$, and G_1 in (18) is the correct Green's function. For G_3 , we have $G_3 = \exp \gamma N(\eta, \tilde{r} - r) N(\alpha, \beta - \log S)$. The variables $(\bar{V}, V_0, G_1)$ and (V, V_0, G_3) are connected by

$$\bar{V} = \int_{-\infty}^{\infty} \mathcal{M}[V_0(S, \tilde{r})] G_1 d\tilde{r}, \quad V = \int_{-\infty}^{\infty} \mathcal{M}^{-1}[\mathcal{M}[V_0] \mathcal{M}[G_3]] d\tilde{r}. \quad (20)$$

Using the convolution theorem ((2.30), [1]), the solution is (2), where

$$G(r, \tilde{r}, x - \tilde{x}) = \exp(\gamma) N(\eta, \tilde{r} - r) N(\alpha, \tilde{x} - x + \beta), \quad (21)$$

$$\alpha = \tau \left\{ \sigma^2 (1 - \rho^2) + \left(\frac{c}{b} \right)^2 (1 - \lambda) \phi^2 \right\}, \quad \phi = 1 + \frac{\rho\sigma b}{c}, \quad (22)$$

$$2\beta = \frac{2(r - \tilde{r})\rho\sigma}{c} + \tau \left\{ \left(\frac{c}{b} \right)^2 (1 - \lambda) d_1 \phi + \sigma^2 - (r + \tilde{r})\phi + \frac{2\sigma a \rho}{c} \right\} \quad (23)$$

$$2\gamma = \frac{(r - \tilde{r})((r + \tilde{r})b - 2a)}{c^2} + \tau \left\{ b - \left(\frac{a}{c} \right)^2 + \frac{(c/b)^2 (1 - \lambda) d_2}{2} - (r + \tilde{r}) \left(1 - \frac{ab}{c^2} \right) - \frac{(r^2 + \tilde{r}^2)(b/c)^2}{2} \right\}, \quad (24)$$

$d_m = q^m(r) + q^m(\tilde{r})$, and $q(r) = (rb^2 - ab + c^2)/c^2$. Extensive algebraic manipulations are required to express G in (21) in the final form (3). The Green's function in (3) has the expected property: $G \rightarrow \delta(r - \tilde{r})\delta(x - \tilde{x})$ and $V(S, r, \tau) \rightarrow V_0(S, r)$ as $\tau \rightarrow 0$ in (2).

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