Research Article

An Optimal Double Inequality between Seiffert and Geometric Means

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Received 30 June 2011; Accepted 14 October 2011

Academic Editor: J. C. Butcher

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For $\alpha, \beta \in (0, 1/2)$ we prove that the double inequality $G(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < P(a, b) < G(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 - \sqrt{1 - 4/\pi^2})/2$ and $\beta \geq (3 - \sqrt{3})/6$. Here, $G(a, b)$ and $P(a, b)$ denote the geometric and Seiffert means of two positive numbers a and b , respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$ the Seiffert mean $P(a, b)$ was introduced by Seiffert [1] as follows:

$$P(a, b) = \frac{a - b}{4 \arctan \sqrt{a/b} - \pi}. \quad (1.1)$$

Recently, the bivariate mean values have been the subject of intensive research. In particular, many remarkable inequalities for the Seiffert mean can be found in the literature [1–9].

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (a - b)/(\log a - \log b)$, $I(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a, b) = (a + b)/2$, $C(a, b) = (a^2 + b^2)/(a + b)$, and $M_p(a, b) = [(a^p + b^p)/2]^{1/p}$ ($p \neq 0$) and $M_0(a, b) = \sqrt{ab}$ be the harmonic, geometric, logarithmic, identric, arithmetic, contraharmonic, and p th power means of two different positive numbers a and b ,

respectively. Then it is well known that

$$\begin{aligned} \min\{a, b\} < H(a, b) = M_{-1}(a, b) < G(a, b) = M_0(a, b) < L(a, b) \\ < I(a, b) < A(a, b) = M_1(a, b) < C(a, b) < \max\{a, b\} \end{aligned} \quad (1.2)$$

for all $a, b > 0$ with $a \neq b$.

For all $a, b > 0$ with $a \neq b$, Seiffert [1] established that $L(a, b) < P(a, b) < I(a, b)$; Jagers [4] proved that $M_{1/2}(a, b) < P(a, b) < M_{2/3}(a, b)$ and $M_{2/3}(a, b)$ is the best possible upper power mean bound for the Seiffert mean $P(a, b)$; Seiffert [7] established that $P(a, b) > A(a, b)G(a, b)/L(a, b)$ and $P(a, b) > 2A(a, b)/\pi$; Sándor [6] presented that $(A(a, b) + G(a, b))/2 < P(a, b) < \sqrt{A(a, b)(A(a, b) + G(a, b))/2}$ and $\sqrt[3]{A^2(a, b)G(a, b)} < P(a, b) < (G(a, b) + 2A(a, b))/3$; Hästö [3] proved that $P(a, b) > M_{\log 2 / \log \pi}(a, b)$ and $M_{\log 2 / \log \pi}(a, b)$ is the best possible lower power mean bound for the Seiffert mean $P(a, b)$.

Very recently, Wang and Chu [8] found the greatest value α and the least value β such that the double inequality $A^\alpha(a, b)H^{1-\alpha}(a, b) < P(a, b) < A^\beta(a, b)H^{1-\beta}(a, b)$ holds for $a, b > 0$ with $a \neq b$; For any $\alpha \in (0, 1)$, Chu et al. [10] presented the best possible bounds for $P^\alpha(a, b)G^{1-\alpha}(a, b)$ in terms of the power mean; In [2] the authors proved that the double inequality $\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$; Liu and Meng [5] proved that the inequalities

$$\begin{aligned} \alpha_1 C(a, b) + (1 - \alpha_1)G(a, b) < P(a, b) < \beta_1 C(a, b) + (1 - \beta_1)G(a, b), \\ \alpha_2 C(a, b) + (1 - \alpha_2)H(a, b) < P(a, b) < \beta_2 C(a, b) + (1 - \beta_2)H(a, b) \end{aligned} \quad (1.3)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/9$, $\beta_1 \geq 1/\pi$, $\alpha_2 \leq 1/\pi$ and $\beta_2 \geq 5/12$.

For fixed $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$, let

$$g(x) = G(xa + (1 - x)b, xb + (1 - x)a). \quad (1.4)$$

Then it is not difficult to verify that $g(x)$ is continuous and strictly increasing in $[0, 1/2]$. Note that $g(0) = G(a, b) < P(a, b)$ and $g(1/2) = A(a, b) > P(a, b)$. Therefore, it is natural to ask what are the greatest value α and least value β in $(0, 1/2)$ such that the double inequality $G(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < P(a, b) < G(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)$ holds for all $a, b > 0$ with $a \neq b$. The main purpose of this paper is to answer these questions. Our main result is the following Theorem 1.1.

Theorem 1.1. *If $\alpha, \beta \in (0, 1/2)$, then the double inequality*

$$G(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < P(a, b) < G(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) \quad (1.5)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 - \sqrt{1 - 4/\pi^2})/2$ and $\beta \geq (3 - \sqrt{3})/6$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (1 - \sqrt{1 - 4/\pi^2})/2$ and $\mu = (3 - \sqrt{3})/6$. We first prove that inequalities

$$P(a, b) > G(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a), \quad (2.1)$$

$$P(a, b) < G(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) \quad (2.2)$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in (0, 1/2)$, then from (1.1) one has

$$\begin{aligned} & \log G(pa + (1 - p)b, pb + (1 - p)a) - \log P(a, b) \\ &= \frac{1}{2} \log \left[(pt^2 + (1 - p))((1 - p)t^2 + p) \right] - \log \frac{t^2 - 1}{4 \arctan t - \pi}. \end{aligned} \quad (2.3)$$

Let

$$f(t) = \frac{1}{2} \log \left[(pt^2 + (1 - p))((1 - p)t^2 + p) \right] - \log \frac{t^2 - 1}{4 \arctan t - \pi}, \quad (2.4)$$

then simple computations lead to

$$f(1) = 0, \quad (2.5)$$

$$\lim_{t \rightarrow +\infty} f(t) = \frac{1}{2} \log [p(1 - p)] + \log \pi, \quad (2.6)$$

$$f'(t) = \frac{t(t^2 + 1)}{(t^2 - 1)(4 \arctan t - \pi)(pt^2 + (1 - p))((1 - p)t^2 + p)} f_1(t), \quad (2.7)$$

where

$$f_1(t) = \frac{4(t^2 - 1)(pt^2 + 1 - p)[(1 - p)t^2 + p]}{t(t^2 + 1)^2} - 4 \arctan t + \pi. \quad (2.8)$$

$$f_1(1) = 0, \quad (2.9)$$

$$\lim_{t \rightarrow +\infty} f_1(t) = +\infty, \quad (2.10)$$

$$f_1'(t) = \frac{4f_2(t^2)}{t^2(t^2 + 1)^4}, \quad (2.11)$$

where $f_2(t) = p(1 - p)t^5 - (3p - 2)(3p - 1)t^4 + 2(5p^2 - 5p + 1)t^3 + 2(5p^2 - 5p + 1)t^2 - (3p - 2)(3p - 1)t + p(1 - p)$.

Note that

$$f_2(1) = 0, \quad (2.12)$$

$$\lim_{t \rightarrow +\infty} f_2(t) = +\infty, \quad (2.13)$$

$$\begin{aligned} f_2'(t) = & 5p(1-p)t^4 - 4(3p-2)(3p-1)t^3 + 6(5p^2-5p+1)t^2 \\ & + 4(5p^2-5p+1)t - (3p-2)(3p-1), \end{aligned} \quad (2.14)$$

$$f_2'(1) = 0, \quad (2.15)$$

$$\lim_{t \rightarrow +\infty} f_2'(t) = +\infty, \quad (2.16)$$

$$f_2''(t) = 20p(1-p)t^3 - 12(3p-2)(3p-1)t^2 + 12(5p^2-5p+1)t + 4(5p^2-5p+1), \quad (2.17)$$

$$f_2''(t) = -8(6p^2-6p+1), \quad (2.18)$$

$$\lim_{t \rightarrow +\infty} f_2''(t) = +\infty, \quad (2.19)$$

$$f_2'''(t) = 60p(1-p)t^2 - 24(3p-2)(3p-1)t + 12(5p^2-5p+1), \quad (2.20)$$

$$f_2'''(1) = -36(6p^2-6p+1), \quad (2.21)$$

$$\lim_{t \rightarrow +\infty} f_2'''(t) = +\infty, \quad (2.22)$$

$$f_2^{(4)}(t) = 120p(1-p)t - 24(3p-2)(3p-1), \quad (2.23)$$

$$f_2^{(4)}(1) = -48(7p^2-7p+1), \quad (2.24)$$

$$\lim_{t \rightarrow +\infty} f_2^{(4)}(t) = +\infty. \quad (2.25)$$

We divide the proof into two cases.

Case 1 ($p = \lambda = (1 - \sqrt{1 - 4/\pi^2})/2$). Then (2.6), (2.18), (2.21), and (2.24) become

$$\lim_{t \rightarrow +\infty} f(t) = 0, \quad (2.26)$$

$$f_2''(1) = -\frac{8(\pi^2-6)}{\pi^2} < 0, \quad (2.27)$$

$$f_2'''(1) = -\frac{36(\pi^2-6)}{\pi^2} < 0, \quad (2.28)$$

$$f_2^{(4)}(1) = -\frac{48(\pi^2-7)}{\pi^2} < 0. \quad (2.29)$$

From (2.23) we clearly see that $f_2^{(4)}(t)$ is strictly increasing in $[1, +\infty)$, then (2.25) and inequality (2.29) lead to the conclusion that there exists $\lambda_1 > 1$ such that $f_2^{(4)}(t) < 0$ for $t \in [1, \lambda_1)$ and $f_2^{(4)}(t) > 0$ for $t \in (\lambda_1, +\infty)$. Thus, $f_2'''(t)$ is strictly decreasing in $[1, \lambda_1]$ and strictly increasing in $[\lambda_1, +\infty)$.

It follows from (2.22) and inequality (2.28) together with the piecewise monotonicity of $f_2'''(t)$ that there exists $\lambda_2 > \lambda_1 > 1$ such that $f_2''(t)$ is strictly decreasing in $[1, \lambda_2]$ and strictly increasing in $[\lambda_2, +\infty)$. Then (2.19) and inequality (2.27) lead to the conclusion that there exists $\lambda_3 > \lambda_2 > 1$ such that $f_2'(t)$ is strictly decreasing in $[1, \lambda_3]$ and strictly increasing in $[\lambda_3, +\infty)$.

From (2.15) and (2.16) together with the piecewise monotonicity of $f_2'(t)$ we know that there exists $\lambda_4 > \lambda_3 > 1$ such that $f_2(t)$ is strictly decreasing in $[1, \lambda_4]$ and strictly increasing in $[\lambda_4, +\infty)$. Then (2.11)–(2.13) lead to the conclusion that there exists $\lambda_5 > \lambda_4 > 1$ such that $f_1(t)$ is strictly decreasing in $[1, \sqrt{\lambda_5}]$ and strictly increasing in $[\sqrt{\lambda_5}, +\infty)$.

It follows from (2.7)–(2.10) and the piecewise monotonicity of $f_1(t)$ that there exists $\lambda_6 > \sqrt{\lambda_5} > 1$ such that $f(t)$ is strictly decreasing in $[1, \lambda_6]$ and strictly increasing in $[\lambda_6, +\infty)$.

Therefore, inequality (2.1) follows from (2.3)–(2.5) and the piecewise monotonicity of $f(t)$.

Case 2 ($p = \mu = (3 - \sqrt{3})/6$). Then (2.18), (2.21) and (2.24) become

$$f_2''(1) = 0, \quad (2.30)$$

$$f_2'''(1) = 0, \quad (2.31)$$

$$f_2^{(4)}(1) = 8 > 0. \quad (2.32)$$

From (2.23) we clearly see that $f_2^{(4)}(t)$ is strictly increasing in $[1, +\infty)$, then inequality (2.32) leads to the conclusion that $f_2'''(t)$ is strictly increasing in $[1, +\infty)$.

Therefore, inequality (2.2) follows from (2.3)–(2.5), (2.7)–(2.9), (2.11), (2.12), (2.15), and inequalities (2.30) and (2.31) together with the monotonicity of $f_2'''(t)$.

Next, we prove that $\lambda = (1 - \sqrt{1 - 4/\pi^2})/2$ is the best possible parameter such that inequality (2.1) holds for all $a, b > 0$ with $a \neq b$. In fact, if $(1 - \sqrt{1 - 4/\pi^2})/2 = \lambda < p < 1/2$, then (2.6) leads to

$$\lim_{t \rightarrow +\infty} f(t) = \frac{1}{2} \log [p(1-p)] + \log \pi > 0. \quad (2.33)$$

Inequality (2.33) implies that there exists $T = T(p) > 1$ such that

$$f(t) > 0 \quad (2.34)$$

for $t \in (T, +\infty)$.

It follows from (2.3) and (2.4) together with inequality (2.34) that $P(a, b) < G(pa + (1-p)b, pb + (1-p)a)$ for $a/b \in (T^2, +\infty)$.

Finally, we prove that $\mu = (3 - \sqrt{3})/6$ is the best possible parameter such that inequality (2.2) holds for all $a, b > 0$ with $a \neq b$. In fact, if $0 < p < \mu = (3 - \sqrt{3})/6$, then from (2.18) we get $f_2''(1) < 0$, which implies that there exists $\delta > 0$ such that

$$f_2''(t) < 0 \quad (2.35)$$

for $t \in [1, 1 + \delta)$.

Therefore, $P(a, b) > G(pa + (1 - p)b, pb + (1 - p)a)$ for $a/b \in (1, (1 + \delta)^2)$ follows from (2.3)–(2.5), (2.7)–(2.9), (2.11), (2.12), and (2.15) together with inequality (2.35). $\square$

Acknowledgments

This research was supported by the Natural Science Foundation of China under Grant 11071069 and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

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