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Review Article

Mittag-Leffler Functions and Their Applications

H. J. Haubold,[1] A. M. Mathai,[2,3] and R. K. Saxena[4]

[1] *Office for Outer Space Affairs, United Nations, Vienna International Centre, P.O. Box 500, 1400 Vienna, Austria*
[2] *Department of Mathematics and Statistics, McGill University, Montreal, QC, Canada H3A 2K6*
[3] *Centre for Mathematical Sciences, Pala Campus, Arunapuram P.O., Pala, Kerala 686574, India*
[4] *Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur 342005, India*

Correspondence should be addressed to H. J. Haubold, hans.haubold@unvienna.org

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Motivated essentially by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, the authors present, in a unified manner, a detailed account or rather a brief survey of the Mittag-Leffler function, generalized Mittag-Leffler functions, Mittag-Leffler type functions, and their interesting and useful properties. Applications of G. M. Mittag-Leffler functions in certain areas of physical and applied sciences are also demonstrated. During the last two decades this function has come into prominence after about nine decades of its discovery by a Swedish Mathematician Mittag-Leffler, due to the vast potential of its applications in solving the problems of physical, biological, engineering, and earth sciences, and so forth. In this survey paper, nearly all types of Mittag-Leffler type functions existing in the literature are presented. An attempt is made to present nearly an exhaustive list of references concerning the Mittag-Leffler functions to make the reader familiar with the present trend of research in Mittag-Leffler type functions and their applications.

1. Introduction

The special function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad \alpha \in C,\ \Re(\alpha) > 0,\ z \in C \tag{1.1}$$

and its general form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \quad \alpha,\beta \in C,\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ z \in C \tag{1.2}$$

with C being the set of complex numbers are called Mittag-Leffler functions [1, Section 18.1]. The former was introduced by Mittag-Leffler [2, 3], in connection with his method of summation of some divergent series. In his papers [2–4], he investigated certain properties of this function. Function defined by (1.2) first appeared in the work of Wiman [5, 6]. The function (1.2) is studied, among others, by Wiman [5, 6], Agarwal [7], Humbert [8], and Humbert and Agarwal [9] and others. The main properties of these functions are given in the book by Erdélyi et al. [1, Section 18.1], and a more comprehensive and a detailed account of Mittag-Leffler functions is presented in Dzherbashyan [10, Chapter 2]. In particular, functions (1.1) and (1.2) are entire functions of order $\rho = 1/\alpha$ and type $\sigma = 1$; see, for example, [1, page 118].

The Mittag-Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Lévy flights, superdiffusive transport and in the study of complex systems. The ordinary and generalized Mittag-Leffler functions interpolate between a purely exponential law and power-law-like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts, see Lang [11, 12], Hilfer [13, 14], and Saxena [15].

The Mittag-Leffler function is not given in the tables of Laplace transforms, where it naturally occurs in the derivation of the inverse Laplace transform of the functions of the type $p^{\alpha}(a + bp^{\beta})$, where p is the Laplace transform parameter and a and b are constants. This function also occurs in the solution of certain boundary value problems involving fractional integrodifferential equations of Volterra type [16]. During the various developments of fractional calculus in the last four decades this function has gained importance and popularity on account of its vast applications in the fields of science and engineering. Hille and Tamarkin [17] have presented a solution of the Abel-Volterra type equation in terms of Mittag-Leffler function. During the last 15 years the interest in Mittag-Leffler function and Mittag-Leffler type functions is considerably increased among engineers and scientists due to their vast potential of applications in several applied problems, such as fluid flow, rheology, diffusive transport akin to diffusion, electric networks, probability, and statistical distribution theory. For a detailed account of various properties, generalizations, and application of this function, the reader may refer to earlier important works of Blair [18], Torvik and Bagley [19], Caputo and Mainardi [20], Dzherbashyan [10], Gorenflo and Vessella [21], Gorenflo and Rutman [22], Kilbas and Saigo [23], Gorenflo and Luchko [24], Gorenflo and Mainardi [25, 26], Mainardi and Gorenflo [27, 28], Gorenflo et al. [29], Gorenflo et al. [30], Luchko [31], Luchko and Srivastava [32], Kilbas et al. [33, 34], Saxena and Saigo [35], Kiryakova [36, 37], Saxena et al. [38], Saxena et al. [39–43], Saxena and Kalla [44], Mathai et al. [45], Haubold and Mathai [46], Haubold et al. [47], Srivastava and Saxena [48], and others.

This paper is organized as follows: Section 2 deals with special cases of $E_{\alpha}(z)$. Functional relations of Mittag-Leffler functions are presented in Section 3. Section 4 gives the basic properties. Section 5 is devoted to the derivation of recurrence relations for Mittag-Leffler functions. In Section 6, asymptotic expansions of the Mittag-Leffler functions are given. Integral representations of Mittag-Leffler functions are given in Section 7. Section 8 deals with the H-function and its special cases. The Melllin-Barnes integrals for the Mittag-Leffler functions are established in Section 9. Relations of Mittag-Leffler functions with Riemann-Liouville fractional calculus operators are derived in Section 10. Generalized Mittag-Leffler functions and some of their properties are given in Section 11. Laplace transform, Fourier transform, and fractional integrals and derivatives are discussed in Section 12. Section 13 is devoted to the application of Mittag-Leffler function in fractional

kinetic equations. In Section 14, time-fractional diffusion equation is solved. Solution of space-fractional diffusion equation is discussed in Section 15. In Section 16, solution of a fractional reaction-diffusion equation is investigated in terms of the H-function. Section 17 is devoted to the application of generalized Mittag-Leffler functions in nonlinear waves. Recent generalizations of Mittag-Leffler functions are discussed in Section 18.

2. Some Special Cases

We begin our study by giving the special cases of the Mittag-Leffler function $E_\alpha(z)$.

(i)

$$E_0(z) = \frac{1}{1-z}, \quad |z| < 1, \tag{2.1}$$

(ii)

$$E_1(z) = e^z, \tag{2.2}$$

(iii)

$$E_2(z) = \cosh\left(\sqrt{z}\right), \quad z \in C, \tag{2.3}$$

(iv)

$$E_2\left(-z^2\right) = \cos z, \quad z \in C, \tag{2.4}$$

(v)

$$E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-(1/2)z^{1/3}} \cos\left(\frac{\sqrt{3}}{2} z^{1/3}\right)\right], \quad z \in C, \tag{2.5}$$

(vi)

$$E_4(z) = \frac{1}{2}\left[\cos\left(z^{1/4}\right) + \cosh\left(z^{1/4}\right)\right], \quad z \in C, \tag{2.6}$$

(vii)

$$E_{1/2}\left(\pm z^{1/2}\right) = e^z\left[1 + \operatorname{erf}\left(\pm z^{1/2}\right)\right] = e^z \operatorname{erfc}\left(\mp z^{1/2}\right), \quad z \in C, \tag{2.7}$$

where erfc denotes the complimentary error function and the error function is defines as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(-t^2\right) dt, \quad \operatorname{erfc}(z) = 1 - \operatorname{erf}(z), \quad z \in C. \tag{2.8}$$

For half-integer $n/2$ the function can be written explicitly as

(viii)

$$E_{n/2}(z) = {}_0F_{n-1}\left(:\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n};\frac{z^2}{n^n}\right) + \frac{2^{(n+1)/2}z}{n!\sqrt{\pi}}\,{}_1F_{2n-1}\left(1;\frac{n+2}{2n},\frac{n+3}{2n},\ldots,\frac{3n}{2n};\frac{z^2}{n^n}\right), \tag{2.9}$$

(ix)

$$E_{1,2}(z) = \frac{e^z-1}{z}, \qquad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}. \tag{2.10}$$

3. Functional Relations for the Mittag-Leffler Functions

In this section, we discuss the Mittag-Leffler functions of rational order $\alpha = m/n$, with $m, n \in N$ relatively prime. The differential and other properties of these functions are described in Erdélyi et al. [1] and Dzherbashyan [10].

Theorem 3.1. *The following results hold:*

$$\frac{d^m}{dz^m}E_m(z^m) = E_m(z^m), \tag{3.1}$$

$$\frac{d^m}{dz^m}E_{m/n}\left(z^{m/n}\right) = E_{m/n}\left(z^{m/n}\right) + \sum_{r=1}^{n-1}\frac{z^{-rm/n}}{\Gamma(1-rm/n)}, \quad n = 2,3,\ldots, \tag{3.2}$$

$$E_{m/n}(z) = \frac{1}{m}\sum_{r=1}^{m-1}E_{1/n}\left(z^{1/m}\exp\left(\frac{i2\pi r}{m}\right)\right), \tag{3.3}$$

$$E_{1/n}\left(z^{1/n}\right) = e^z\left[1 + \sum_{r=1}^{n-1}\frac{\gamma(1-r/n,z)}{\Gamma(1-r/n)}\right], \quad n = 2,3,\ldots, \tag{3.4}$$

where $\gamma(a,z)$ denotes the incomplete gamma function, defined by

$$\gamma(a,z) = \int_0^z e^{-t}t^{a-1}dt. \tag{3.5}$$

In order to establish the above formulas, we observe that (3.1) and (3.2) readily follow from definition (1.2). For proving formula (3.3), we recall the identity

$$\sum_{r=0}^{m-1}\exp\left[\frac{i2\pi kr}{m}\right] = \begin{cases} m & \text{if } k = 0 \pmod{m}, \\ 0 & \text{if } k \neq 0 \pmod{m}. \end{cases} \tag{3.6}$$

By virtue of the results (1.1) and (3.6), we find that

$$\sum_{r=0}^{m-1} E_\alpha\left(z e^{i2\pi r/m}\right) = m E_{\alpha m}(z^m), \quad m \in N \tag{3.7}$$

which can be written as

$$E_\alpha(z) = \frac{1}{m}\sum_{r=0}^{m-1} E_{\alpha/m}\left(z^{1/m} e^{i2\pi r/m}\right), \quad m \in N \tag{3.8}$$

and result (3.3) now follows by taking $\alpha = m/n$. To prove relation (3.4), we set $m = 1$ in (3.1) and multiply it by $\exp(-z)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[e^{-z} E_{1/n}\left(z^{1/n}\right)\right] = e^{-z}\sum_{r=1}^{m-1}\frac{z^{-r/m}}{\Gamma(1-r/m)}. \tag{3.9}$$

On integrating both sides of the above equation with respect to z and using the definition of incomplete gamma function (3.5), we obtain the desired result (3.4). An interesting case of (3.8) is given by

$$E_{2\alpha}\left(z^2\right) = \frac{1}{2}\left[E_\alpha(z) + E_\alpha(-z)\right]. \tag{3.10}$$

4. Basic Properties

This section is based on the paper of Berberan-Santos [49]. From (1.1) and (1.2) it is not difficult to prove that

$$E_\alpha(-x) = E_{2\alpha}\left(x^2\right) - x E_{2\alpha,1+\alpha}\left(x^2\right), \tag{4.1}$$

$$E_\alpha(-ix) = E_{2x}\left(-x^2\right) - ix E_{2\alpha,1+\alpha}\left(-x^2\right), \quad i = \sqrt{-1}. \tag{4.2}$$

It is shown in Berberan-Santos [49, page 631] that the following three equations can be used for the direct inversion of a function $I(x)$ to obtain its inverse $H(k)$:

$$H(k) = \frac{e^{ck}}{\pi}\int_0^\infty \Re[I(c+i\omega)]\cos(k\omega) - \Im[I(c+i\omega)]\sin(k\omega)\mathrm{d}\omega \tag{4.3}$$

$$= \frac{2e^{ck}}{\pi}\int_0^\infty \Re[I(c+i\omega)]\cos(k\omega)\mathrm{d}\omega, \quad k > 0 \tag{4.4}$$

$$= -\frac{2e^{ck}}{\pi}\int_0^\infty \Im[I(c+i\omega)]\sin(k\omega)\mathrm{d}\omega, \quad k > 0. \tag{4.5}$$

With help of the results (4.2) and (4.4), it yields the following formula for the inverse Laplace transform $H(k)$ of the function $E_\alpha(-x)$:

$$H_\alpha(k) = \frac{2}{\pi}\int_0^\infty E_{2\alpha}\left(-t^2\right)\cos(kt)\mathrm{d}t, \quad k > 0,\ 0 \le \alpha \le 1. \tag{4.6}$$

In particular, the following interesting results can be derived from the above result:

$$\begin{gathered} H_1(k) = \frac{2}{\pi}\int_0^\infty \cosh(it)\cos(kt)\mathrm{d}t = \frac{2}{\pi}\int_0^\infty \cos(t)\cos(kt)\mathrm{d}t = \delta(k-1), \quad i = \sqrt{-1}, \\ H_{1/2}(k) = \frac{2}{\pi}\int_0^\infty \mathrm{e}^{-t^2}\cos(kt)\mathrm{d}t = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-k^2/4}, \\ H_{1/4}(k) = \frac{2}{\pi}\int_0^\infty \mathrm{e}^{t^2}\operatorname{erfc}\left(t^2\right)\cos(kt)\mathrm{d}t. \end{gathered} \tag{4.7}$$

Another integral representation of $H_\alpha(k)$ in terms of the Lévy one-sided stable distribution $L_\alpha(k)$ was given by Pollard [50] in the form

$$H_\alpha(k) = \frac{1}{\alpha}k^{-(1+1/\alpha)}L_\alpha\left(k^{-1/\alpha}\right). \tag{4.8}$$

The inverse Laplace transform of $E_\alpha(-x^\beta)$, denoted by $H_\alpha^\beta(k)$ with $0 < \alpha \le 1$, is obtained as

$$H_\alpha^\beta(k) = \int_0^\infty t^{\alpha/\beta}L_\alpha(t)L_\beta\left(kt^{\alpha/\beta}\right)\mathrm{d}t, \tag{4.9}$$

where $L_\alpha(t)$ is the one-sided Lévy probability density function. From Berberan-Santos [49, page 432] we have

$$H_\alpha(k) = \frac{1}{\pi}\int_0^\infty \left[E_{2\alpha}\left(-\omega^2\right)\cos(k\omega) + \omega E_{2\alpha,1+\alpha}\left(-\omega^2\right)\sin(k\omega)\right]\mathrm{d}\omega, \quad 0 < \alpha \le 1. \tag{4.10}$$

Expanding the above equation in a power series, it gives

$$H_\alpha(k) = \frac{1}{\pi}\sum_{n=0}^\infty b_n(\alpha)k^n, \quad 0 \le \alpha < 1 \tag{4.11}$$

with

$$b_0(\alpha) = \int_0^\infty E_{2\alpha}\left(-t^2\right)\mathrm{d}t. \tag{4.12}$$

The Laplace transform of (4.11) is the asymptotic expansion of $E_\alpha(-x)$ as

$$E_\alpha(-x) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{b_n(\alpha)}{x^{n+1}}, \quad 0 \le \alpha < 1. \tag{4.13}$$

5. Recurrence Relations

By virtue of definition (1.2), the following relations are obtained in the following form:

Theorem 5.1. *One has*

$$\begin{gathered} E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \\ E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}(z), \\ \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left[z^{\beta-1}E_{\alpha,\beta}(z^\alpha)\right] = z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha), \quad \Re(\beta-m) > 0,\ m \in N, \\ \frac{\mathrm{d}}{\mathrm{d}z}E_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z) - (\beta-1)E_{\alpha,\beta}(z)}{\alpha z}. \end{gathered} \tag{5.1}$$

The above formulae are useful in computing the derivative of the Mittag-Leffler function $E_{\alpha,\beta}(z)$. The following theorem has been established by Saxena [15].

Theorem 5.2. *If $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $r \in N$ then there holds the formula*

$$z^r E_{\alpha,\beta+r\alpha}(z) = E_{\alpha,\beta}(z) - \sum_{n=0}^{r-1}\frac{z^n}{\Gamma(\beta+n\alpha)}. \tag{5.2}$$

Proof. We have from the right side of (5.2)

$$E_{\alpha,\beta}(z) - \sum_{n=0}^{r-1}\frac{z^n}{\Gamma(\beta+n\alpha)} = \sum_{n=r}^{\infty}\frac{z^n}{\Gamma(\beta+n\alpha)}. \tag{5.3}$$

Put $n - r = k$ or $n = k + r$. Then,

$$\begin{aligned} \sum_{n=r}^{\infty}\frac{z^n}{\Gamma(\beta+n\alpha)} &= \sum_{k=0}^{\infty}\frac{z^{k+r}}{\Gamma(\beta+r\alpha+k\alpha)} \\ &= z^r E_{\alpha,\beta+r\alpha}(z). \end{aligned} \tag{5.4}$$

For $r = 2, 3, 4$ we obtain the following corollaries.

Corollary 5.3. *If $\Re(\alpha) > 0$, $\Re(\beta) > 0$ then there holds the formula*

$$z^2 E_{\alpha,\beta+2\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)}. \tag{5.5}$$

Corollary 5.4. *If $\Re(\alpha) > 0$, $\Re(\beta) > 0$ then there holds the formula*

$$z^3 E_{\alpha,\beta+3\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)} - \frac{z^2}{\Gamma(2\alpha+\beta)}. \tag{5.6}$$

Corollary 5.5. *If $\Re(\alpha) > 0$, $\Re(\beta) > 0$ then there holds the formula*

$$z^4 E_{\alpha,\beta+4\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)} - \frac{z^2}{\Gamma(2\alpha+\beta)} - \frac{z^3}{\Gamma(3\alpha+\beta)}. \tag{5.7}$$

□

Remark 5.6. For a generalization of result (5.2), see Saxena et al. [38].

6. Asymptotic Expansions

The asymptotic behavior of Mittag-Leffler functions plays a very important role in the interpretation of the solution of various problems of physics connected with fractional reaction, fractional relaxation, fractional diffusion, and fractional reaction-diffusion, and so forth, in complex systems. The asymptotic expansion of $E_\alpha(z)$ is based on the integral representation of the Mittag-Leffler function in the form

$$E_\alpha(z) = \frac{1}{2\pi i}\int_\Omega \frac{t^{\alpha-1}\exp(t)}{t^\alpha - z}\mathrm{d}t, \quad \Re(\alpha) > 0,\ \alpha, z \in C, \tag{6.1}$$

where the path of integration Ω is a loop starting and ending at $-\infty$ and encircling the circular disk $|t| \le |z|^{1/\alpha}$ in the positive sense, $|\arg t| < \pi$ on Ω. The integrand has a branch point at $t = 0$. The complex t-plane is cut along the negative real axis and in the cut plane the integrand is single-valued the principal branch of t^α is taken in the cut plane. Equation (6.1) can be proved by expanding the integrand in powers of t and integrating term by term by making use of the well-known Hankel's integral for the reciprocal of the gamma function, namely

$$\frac{1}{\Gamma(\beta)} = \frac{1}{2\pi i}\int_{H_a} \frac{e^\zeta}{\zeta^\beta}\mathrm{d}\zeta. \tag{6.2}$$

The integral representation (6.1) can be used to obtain the asymptotic expansion of the Mittag-Leffler function at infinity [1]. Accordingly, the following cases are mentioned.

(i) If $0 < \alpha < 2$ and μ is a real number such that

$$\frac{\pi\alpha}{2} < \mu < \min[\pi, \pi\alpha], \tag{6.3}$$

then for $N^* \in N$, $N^* \neq 1$ there holds the following asymptotic expansion:

$$E_\alpha(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right) - \sum_{r=1}^{N^*} \frac{1}{\Gamma(1-\alpha r)} \frac{1}{z^r} + O\left[\frac{1}{z^{N^*+1}}\right], \tag{6.4}$$

as $|z| \to \infty$, $|\arg z| \leq \mu$ and

$$E_\alpha(z) = -\sum_{r=1}^{N^*} \frac{1}{\Gamma(1-\alpha r)} \frac{1}{z^r} + O\left[\frac{1}{z^{N^*+1}}\right], \tag{6.5}$$

as $|z| \to \infty$, $\mu \leq |\arg z| \leq \pi$.

(ii) When $\alpha \geq 2$ then there holds the following asymptotic expansion:

$$E_\alpha(z) = \frac{1}{\alpha} \sum_n z^{1/n} \exp\left[\exp\left(\frac{2n\pi i}{\alpha}\right) z^{1/\alpha}\right] - \sum_{r=1}^{N^*} \frac{1}{\Gamma(1-\alpha r)} \frac{1}{z^r} + O\left[\frac{1}{z^{N^*+1}}\right] \tag{6.6}$$

as $|z| \to \infty$, $|\arg z| \leq \alpha\pi/2$ and where the first sum is taken over all integers n such that

$$\left|\arg(z) + 2\pi n\right| \leq \frac{\alpha\pi}{2}. \tag{6.7}$$

The asymptotic expansion of $E_{\alpha,\beta}(z)$ is based on the integral representation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ in the form

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_\Omega \frac{t^{\alpha-\beta} \exp(t)}{t^\alpha - z} \mathrm{d}t, \quad \Re(\alpha) > 0,\ \Re(\beta) > 0,\ z, \alpha, \beta \in C, \tag{6.8}$$

which is an extension of (6.1) with the same path. As in the previous case, the Mittag-Leffler function has the following asymptotic estimates.

(iii) If $0 < \alpha < 2$ and μ is a real number such that

$$\frac{\pi\alpha}{2} < \mu < \min[\pi, \pi\alpha], \tag{6.9}$$

then there holds the following asymptotic expansion:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right) - \sum_{r=1}^{N^*} \frac{1}{\Gamma(\beta-\alpha r)} \frac{1}{z^r} + O\left[\frac{1}{z^{N^*+1}}\right] \tag{6.10}$$

as $|z| \to \infty$, $|\arg z| \le \mu$ and

$$E_{\alpha,\beta}(z) = -\sum_{r=1}^{N^*} \frac{1}{\Gamma(\beta - \alpha r)} \frac{1}{z^r} + O\left[\frac{1}{z^{N^*+1}}\right], \tag{6.11}$$

as $|z| \to \infty$, $\mu \le |\arg z| \le \pi$.

(iv) When $\alpha \ge 2$ then there holds the following asymptotic expansion:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_n z^{1/n} \exp\left[\exp\left(\frac{2n\pi i}{\alpha}\right) z^{1/\alpha}\right]^{1-\beta} - \sum_{r=1}^{N^*} \frac{1}{\Gamma(\beta - \alpha r)} \frac{1}{z^r} + O\left[\frac{1}{z^{N^*+1}}\right], \tag{6.12}$$

as $|z| \to \infty$, $|\arg z| \le \alpha\pi/2$ and where the first sum is taken over all integers n such that

$$\left|\arg(z) + 2\pi n\right| \le \frac{\alpha\pi}{2}. \tag{6.13}$$

7. Integral Representations

In this section several integrals associated with Mittag-Leffler functions are presented, which can be easily established by the application by means of beta and gamma function formulas and other techniques, see Erdélyi et al. [1], Gorenflo et al. [29, 51, 52],

$$\begin{gathered}
\int_0^\infty e^{-\zeta} E_\alpha(\zeta^\alpha z) d\zeta = \frac{1}{1-z}, \quad |z| < 1,\ \alpha \in C,\ \Re(\alpha) > 0, \\
\int_0^\infty e^{-x} x^{\beta-1} E_{\alpha,\beta}(x^\alpha z) dx = \frac{1}{1-z}, \quad |z| < 1,\ \alpha, \beta \in C,\ \Re(\alpha) > 0,\ \Re(\beta) > 0, \\
\int_0^x (x-\zeta)^{\beta-1} E_\alpha(\zeta^\alpha) d\zeta = \Gamma(\beta) x^\beta E_{\alpha,\beta+1}(x^\alpha), \quad \Re(\beta) > 0, \\
\int_0^\infty e^{-s\zeta} E_\alpha(-\zeta^\alpha) d\zeta = \frac{s^{\alpha-1}}{1+s^\alpha}, \quad \Re(s) > 0, \\
\int_0^\infty e^{-s\zeta} \zeta^{m\alpha+\beta-1} E_{\alpha,\beta}^{(m)}(\pm a\zeta^\alpha) d\zeta = \frac{m! s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}, \quad \Re(s) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0,
\end{gathered} \tag{7.1}$$

where $\alpha, \beta \in C$ and

$$E_{\alpha,\beta}^{(m)}(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m} E_{\alpha,\beta}(z),$$
$$E_\alpha(-x^\alpha) = \frac{2}{\pi}\sin\left(\frac{\alpha\pi}{2}\right)\int_0^\infty \frac{\zeta^{\alpha-1}\cos(x\zeta)}{1+2\zeta^\alpha\cos(\alpha\pi/2)+\zeta^{2\alpha}}\mathrm{d}\zeta, \quad \alpha \in C,\ \Re(\alpha) > 0, \tag{7.2}$$

$$E_\alpha(-x) = \frac{1}{\pi}\sin(\alpha\pi)\int_0^\infty \frac{\zeta^{\alpha-1}}{1+2\zeta^\alpha\cos(\alpha\pi)+\zeta^{2\alpha}}\mathrm{e}^{-\zeta x^{1/\alpha}}\zeta\,\mathrm{d}\zeta, \quad \alpha \in C,\ \Re(\alpha) > 0, \tag{7.3}$$

$$E_\alpha(-x) = 1 - \frac{1}{2\alpha} + \frac{x^{1/\alpha}}{\pi}\int_0^\infty \arctan\left[\frac{\zeta^\alpha+\cos(\alpha\pi)}{\sin(\alpha\pi)}\right]\mathrm{e}^{-\zeta x^{1/\alpha}}\zeta\,\mathrm{d}\zeta, \quad \alpha \in C,\ \Re(\alpha) > 0. \tag{7.4}$$

Note 1. Equation (7.3) can be employed to compute the numerical coefficients of the leading term of the asymptotic expansion of $E_\alpha(-x)$. Equation (7.4) yields

$$b_0(\alpha) = \frac{\alpha}{\pi}\Gamma(\alpha)\sin(2\alpha\pi)\int_0^\infty \frac{\zeta^{\alpha-1}}{1+2\zeta^{2\alpha}\cos(2\alpha\pi)+\zeta^{4\alpha}}\mathrm{d}\zeta, \quad \alpha < \frac{1}{2}. \tag{7.5}$$

From Berberan-Santos [49] and Gorenflo et al. [29, 51] the following results hold:

$$E_\alpha(-x) = \frac{2x}{\pi}\int_0^\infty \frac{E_{2\alpha}(-t^2)}{x^2+t^2}\mathrm{d}t, \quad 0 \le \alpha \le 1,\ \alpha \in C. \tag{7.6}$$

In particular, the following cases are of importance:

$$E_1(-x) = \frac{2x}{\pi}\int_0^\infty \frac{\cosh(it)}{x^2+t^2}\mathrm{d}t = \exp(-x),$$
$$E_{1/2}(-x) = \frac{2x}{\pi}\int_0^\infty \frac{\exp(-t^2)}{x^2+t^2}\mathrm{d}t = \mathrm{e}^{x^2}\,\mathrm{erfc}(x), \tag{7.7}$$
$$E_{1/4}(-x) = \frac{2x}{\pi}\int_0^\infty \frac{\mathrm{e}^{t^2}\,\mathrm{erfc}(-t^2)}{x^2+t^2}\mathrm{d}t.$$

Note 2. Some new properties of the Mittag-Leffler functions are recently obtained by Gupta and Debnath [53].

8. The H-Function and Its Special Cases

The H-function is defined by means of a Mellin-Barnes type integral in the following manner [54]:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] = \frac{1}{2\pi i}\int_{\Omega}\Theta(\zeta)z^{-\zeta}\mathrm{d}\zeta, \tag{8.1}$$

where $i = \sqrt{(-1)}$ and

$$\Theta(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j + sB_j)\right\}\left\{\prod_{j=1}^{n}\Gamma(1 - a_j - sA_j)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1 - b_j - sB_j)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j + sA_j)\right\}}, \tag{8.2}$$

and an empty product is interpreted as unity, $m, n, p, q \in N_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_i, B_j \in R_+$, $a_i, b_j \in C$, $i = 1, \ldots, p$; $j = 1, \ldots, q$ such that

$$A_i(b_j + k) \neq B_j(a_i - \lambda - 1), \quad k, \lambda \in N_0;\ i = 1, \ldots, n;\ j = 1, \ldots, m, \tag{8.3}$$

where we employ the usual notations: $N_0 = 0, 1, 2, \ldots$, $R = (-\infty, \infty)$, $R_+ = (0, \infty)$, and C being the complex number field. The contour Ω is the infinite contour which separates all the poles of $\Gamma(b_j + sB_j)$, $j = 1, \ldots, m$ from all the poles of $\Gamma(1 - a_i + sA_i)$, $i = 1, \ldots, n$. The contour Ω could be $\Omega = L_{-\infty}$ or $\Omega = L_{+\infty}$ or $\Omega = L_{i\gamma\infty}$, where $L_{-\infty}$ is a loop starting at $-\infty$ encircling all the poles of $\Gamma(b_j + sB_j)$, $j = 1, \ldots, m$ and ending at $-\infty$. $L_{+\infty}$ is a loop starting at $+\infty$, encircling all the poles of $\Gamma(1 - a_i - sA_i)$, $i = 1, \ldots, n$ and ending at $+\infty$. $L_{i\gamma\infty}$ is the infinite semicircle starting at $\gamma - i\infty$ and going to $\gamma + i\infty$. A detailed and comprehensive account of the H-function is available from the monographs of Mathai and Saxena [54], Prudnikov et al. [55] and Kilbas and Saigo [56]. The relation connecting the Wright's function ${}_p\psi_q(z)$ and the H-function is given for the first time in the monograph of Mathai and Saxena [54, page 11, equation (1.7.8)] as

$${}_p\psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\middle| z\right] = H_{p,q+1}^{1,p}\left[-z\Big|_{(0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)}^{(1-a_1\cdot A_1),\ldots,(1-a_p,A_p)}\right], \tag{8.4}$$

where ${}_p\psi_q(z)$ is the Wright's generalized hypergeometric function [57, 58]; also see Erdélyi et al. [59, Section 4.1], defined by means of the series representation in the form

$${}_p\psi_q(z) = \sum_{r=0}^{\infty}\frac{\left\{\prod_{j=1}^{p}\Gamma(a_j + A_j r)\right\}}{\left\{\prod_{j=1}^{q}\Gamma(b_j + B_j r)\right\}}\frac{z^r}{r!}, \tag{8.5}$$

where $z \in C$, $a_i, b_j \in C$, $A_i, B_j \in R_+$, $A_i \neq 0$, $B_j \neq 0$; $i = 1, \ldots, p$; $j = 1, \ldots, q$,

$$\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i = \Delta > -1. \tag{8.6}$$

The Mellin-Barnes contour integral for the generalized Wright function is given by

$${}_p\psi_q\left[\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\middle| z\right] = \frac{1}{2\pi i}\int_{\Omega} \frac{\Gamma(s)\prod_{j=1}^{p}\Gamma(a_j - A_j s)}{\prod_{j=1}^{q}\Gamma(b_j - sB_j)}(-z)^{-s}\mathrm{d}s, \tag{8.7}$$

where the path of integration separates all the poles of $\Gamma(s)$ at the points $s = -\nu$, $\nu \in N_0$ lying to the left and all the poles of $\prod_{j=1}^{p}\Gamma(a_j - sA_j)$, $j = 1, \ldots, p$ at the points $s = (A_j + \nu_j)/A_j$, $\nu_j \in N_0$, $j = 1, \ldots, p$ lying to the right. If $\Omega = (\gamma - i\infty, \gamma + i\infty)$, then the above representation is valid if either of the conditions are satisfied:

(i)

$$\Delta < 1, \quad |\arg(-z)| < \frac{(1-\Delta)\pi}{2}, \quad z \neq 0, \tag{8.8}$$

(ii)

$$\Delta = 1, \quad (1+\Delta)\gamma + \frac{1}{2} < \Re(\delta), \quad \arg(-z) = 0, \quad z \neq 0, \quad \delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{(p-q)}{2}. \tag{8.9}$$

This result was proved by Kilbas et al. [60].

The generalized Wright function includes many special functions besides the Mittag-Leffler functions defined by equations (1.1) and (1.2). It is interesting to observe that for $A_i = B_j = 1$, $i = 1, \ldots, p$; $j = 1, \ldots, q$, (8.5) reduces to a generalized hypergeometric function ${}_pF_q(z)$. Thus

$${}_p\psi_q\left[\begin{matrix}(a_p, 1)\\(b_q, 1)\end{matrix}\middle| z\right] = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}\, {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z), \tag{8.10}$$

where $a_j \neq -v$, $j = 1, \ldots, p$, $v = 0, 1, \ldots$, $b_j \neq -\lambda$, $j = 1, \ldots, q$, $\lambda = 0, 1, \ldots$, $p \leq q$ or $p = q + 1$, $|z| < 1$. Wright [61] introduced a special case of (8.5) in the form

$$\phi(a, b; z) = {}_0\psi_1[(b, a) \mid z] = \sum_{r=0}^{\infty} \frac{1}{\Gamma(ar + b)}\frac{z^r}{r!}, \tag{8.11}$$

which widely occurs in problems of fractional diffusion. It has been shown by Saxena et al. [41], also see Kiryakova [62], that

$$E_{\alpha,\beta}(z) = {}_1\psi_1\left[\begin{array}{c}(1,1)\\(\beta,\alpha)\end{array}\middle| z\right] = H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(0,1)}\right]. \tag{8.12}$$

If we further take $\beta = 1$ in (8.12) we find that

$$E_{\alpha,1}(z) = E_\alpha(z) = {}_1\psi_1\left[\begin{array}{c}(1,1)\\(1,\alpha)\end{array}\middle| z\right] = H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right], \tag{8.13}$$

where $\alpha \in C$, $\Re(\alpha) > 0$.

Remark 8.1. A series of papers are devoted to the application of the Wright function in partial differential equation of fractional order extending the classical diffusion and wave equations. Mainardi [63] has obtained the result for a fractional diffusion wave equation in terms of the fractional Green function involving the Wright function. The scale-variant solutions of some partial differential equations of fractional order were obtained in terms of special cases of the generalized Wright function by Buckwar and Luchko [64] and Luchko and Gorenflo [65].

9. Mellin-Barnes Integrals for Mittag-Leffler Functions

These integrals can be obtained from identities (8.12) and (8.13).

Lemma 9.1. *If $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $z \in C$ the following representations are obtained:*

$$\begin{aligned} E_\alpha(z) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}\mathrm{d}s, \\ E_{\alpha,\beta}(z) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s, \end{aligned} \tag{9.1}$$

where the path of integration separates all the poles of $\Gamma(s)$ at the points $s = -\nu, \nu = 0, 1, \ldots$ from those of $\Gamma(1-s)$ at the points $s = 1 + v$, $v = 0, 1, \ldots$.

On evaluating the residues at the poles of the gamma function $\Gamma(1-s)$ we obtain the following analytic continuation formulas for the Mittag-Leffler functions:

$$\begin{aligned} E_\alpha(z) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}\mathrm{d}s = -\sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)}, \\ E_{\alpha,\beta}(z) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s = -\sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(\beta-\alpha k)}. \end{aligned} \tag{9.2}$$

10. Relation with Riemann-Liouville Fractional Calculus Operators

In this section, we present the relations of Mittag-Leffler functions with the left- and right-sided operators of Riemann-Liouville fractional calculus, which are defined

$$\left(I_{0_+}^{\alpha}\phi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\phi(t)\mathrm{d}t, \quad \Re(\alpha) > 0, \tag{10.1}$$

$$(I_{-}^{\alpha})(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1}\phi(t)\mathrm{d}t, \quad \Re(\alpha) > 0, \tag{10.2}$$

$$\begin{aligned}\left(D_{0_+}^{\alpha}\phi\right)(x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\left[I_{0_+}^{1-\{\alpha\}}\phi\right](x)\\ &= \frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\int_0^x (x-t)^{\alpha-1}\phi(t)\mathrm{d}t, \quad \Re(\alpha) > 0\end{aligned} \tag{10.3}$$

$$\begin{aligned}(D_{-}^{\alpha}\phi)(x) &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\left[I_{-}^{1-\{\alpha\}}\phi\right](x)\\ &= \frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\int_x^{\infty} (t-x)^{-\{\alpha\}}\phi(t)\mathrm{d}t, \quad \Re(\alpha) > 0,\end{aligned} \tag{10.4}$$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

Note 3. The fractional integrals (10.1) and (10.2) are connected by the relation [66, page 118]

$$\left[I_{-}^{\alpha}\phi\left(\frac{1}{t}\right)\right](x) = x^{\alpha-1}\left(I_{0_+}^{\alpha}\left[t^{-\alpha-1}\phi(t)\right]\right)\left(\frac{1}{x}\right). \tag{10.5}$$

Theorem 10.1. *Let $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ then there holds the formulas*

$$\begin{aligned}\left(I_{0_+}^{\alpha}\left[t^{\alpha-1}E_{\alpha,\beta}(at^{\alpha})\right]\right)(x) &= \frac{x^{\beta-1}}{a}\left[E_{\alpha,\beta}(ax^{\alpha}) - \frac{1}{\Gamma(\beta)}\right], \quad a \neq 0,\\ \left(I_{0_+}^{\alpha}\left[E_{\alpha}(at^{\alpha})\right]\right)(x) &= \frac{1}{a}\left[E_{\alpha}(ax^{\alpha}) - 1\right], \quad a \neq 0\end{aligned} \tag{10.6}$$

which by virtue of definitions (1.1) *and* (1.2) *can be written as*

$$\begin{aligned}\left(I_{0_+}^{\alpha}\left[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\right]\right)(x) &= x^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(ax^{\alpha}),\\ \left(I_{0_+}^{\alpha}\left[E_{\alpha}(at^{\alpha})\right]\right)(x) &= x^{\alpha}E_{\alpha,\alpha+1}(ax^{\alpha}).\end{aligned} \tag{10.7}$$

Theorem 10.2. *Let $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ then there holds the formulas*

$$\begin{gathered}\left(I_{-}^{\alpha}\left[t^{-\alpha-\beta}E_{\alpha,\beta}\left(at^{-\alpha}\right)\right]\right)(x) = \frac{x^{\alpha-\beta}}{a}\left[E_{\alpha,\beta}\left(ax^{-\alpha}\right) - \frac{1}{\Gamma(\beta)}\right], \quad a \neq 0, \\ \left(I_{-}^{\alpha}\left[t^{-\alpha-1}E_{\alpha}\left(at^{-\alpha}\right)\right]\right)(x) = \frac{1}{a}x^{\alpha-1}\left[E_{\alpha}\left(ax^{-\alpha}\right) - 1\right], \quad a \neq 0.\end{gathered} \tag{10.8}$$

Theorem 10.3. *Let $0 < \Re(\alpha) < 1$ and $\Re(\beta) > R(\alpha)$ then there holds the formulas*

$$\begin{gathered}\left(D_{0_+}^{\alpha}\left[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\right]\right)(x) = \frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} + ax^{\beta-1}E_{\alpha,\beta}(ax^{\alpha}), \\ \left(D_{0_+}^{\alpha}\left[E_{\alpha}(at^{\alpha})\right]\right)(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + aE_{\alpha}(ax^{\alpha}).\end{gathered} \tag{10.9}$$

Theorem 10.4. *Let $\Re(\alpha) > 0$ and $\Re(\beta) > [R(\alpha)] + 1$ then there holds the formula*

$$\left(D_{-}^{\alpha}\left[t^{\alpha-\beta}E_{\alpha,\beta}\left(at^{-\alpha}\right)\right]\right)(x) = \frac{x^{-\beta}}{\Gamma(\beta-\alpha)} + ax^{-\alpha-\beta}E_{\alpha,\beta}\left(ax^{-\alpha}\right). \tag{10.10}$$

11. Generalized Mittag-Leffler Type Functions

By means of the series representation a generalization of (1.1) and (1.2) is introduced by Prabhakar [67] as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(\beta+\alpha n)}, \quad \alpha,\beta,\gamma \in C,\ \Re(\alpha) > 0,\ \Re(\beta) > 0, \tag{11.1}$$

where

$$(\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \tag{11.2}$$

whenever $\Gamma(\gamma)$ is defined, $(\gamma)_0 = 1, \gamma \neq 0$. It is an entire function of order $\rho = [\Re(\alpha)]^{-1}$ and type $\sigma = (1/\rho)[\{\Re(\alpha)\}^{\Re(\alpha)}]^{-\rho}$. It is a special case of Wright's generalized hypergeometric function, Wright [57, 68] as well as the H-function [54]. For various properties of this function with applications, see Prabhakar [67]. Some special cases of this function are enumerated below

$$\begin{gathered}E_{\alpha}(z) = E_{\alpha,1}^{1}(z), \\ E_{\alpha,\beta}(z) = E_{\alpha,\beta}^{1}(z), \\ \alpha\gamma E_{\alpha,\beta}^{\gamma+1}(z) = (1+\alpha\gamma-\beta)E_{\alpha,\beta}^{\gamma}(z) + E_{\alpha,\beta-1}^{\gamma}(z), \\ \phi(\alpha,\beta;z) = \Gamma(\beta)E_{1,\beta}^{\alpha}(z),\end{gathered} \tag{11.3}$$

where $\phi(\alpha,\beta;z)$ is the Kummer's confluent hypergeometric function. $E^{\gamma}_{\alpha,\beta}(z)$ has the following representations in terms of the Wright's function and H-function:

$$\begin{aligned} E^{\gamma}_{\alpha,\beta}(z) &= \frac{1}{\Gamma(\gamma)}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\ z\right] \\ &= \frac{1}{\Gamma(\gamma)}H^{1,1}_{1,2}\left[-z\Big|^{(1-\gamma,1)}_{(0,1),(1-\beta,\alpha)}\right] \\ &= \frac{1}{2\pi\omega\Gamma(\gamma)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s, \quad \Re(\gamma)>0, \end{aligned} \tag{11.4}$$

where ${}_1\psi_1(\cdot)$ and $H^{1,1}_{1,2}(\cdot)$ are, respectively, Wright generalized hypergeometric function and the H-function. In the Mellin-Barnes integral representation, $\omega=\sqrt{-1}$ and the c in the contour is such that $0<c<\Re(\gamma)$, and it is assumed that the poles of $\Gamma(s)$ and $\Gamma(\gamma-s)$ are separated by the contour. The following two theorems are given by Kilbas et al. [34].

Theorem 11.1. *If $\alpha,\beta,\gamma,a\in C$, $\Re(\alpha)>0$, $\Re(\beta)>n$, $\Re(\gamma)>0$ then for $n\in N$ the following results hold:*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[z^{\beta-1}E^{\gamma}_{\alpha,\beta}(az^{\alpha})\right]=z^{\beta-n-1}E^{\gamma}_{\alpha,\beta-n}(az^{\alpha}). \tag{11.5}$$

In particular,

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[z^{\beta-1}E_{\alpha,\beta}(az^{\alpha})\right] &= z^{\beta-n-1}E_{\alpha,\beta-n}(az^{\alpha}), \\ \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[z^{\beta-1}\phi(\gamma,\beta;az)\right] &= \frac{\Gamma(\beta)}{\Gamma(\beta-n)}z^{\beta-n-1}\phi(\gamma;\beta-n;az). \end{aligned} \tag{11.6}$$

Theorem 11.2. *If $\alpha,\beta,\gamma,a,\nu,\sigma\in C$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$, $\Re(\sigma)>0$ then,*

$$\int_0^x (x-t)^{\beta-1}E^{\gamma}_{\alpha,\beta}\left[a(x-t)^{\alpha}\right]t^{\nu-1}E^{\sigma}_{\alpha,\nu}(at^{\alpha})\mathrm{d}t = x^{\beta+\nu-1}E^{\gamma+\sigma}_{\alpha,\beta+\nu}(ax^{\alpha}). \tag{11.7}$$

The proof of (11.7) can be developed with the help of the Laplace transform formula

$$L\left[x^{\beta-1}E^{\gamma}_{\alpha,\beta}(ax^{\alpha})\right](s)=s^{-\beta}\left(1-as^{-\alpha}\right)^{-\gamma}, \tag{11.8}$$

where $\alpha,\beta,\gamma,a\in C$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $s\in C$, $\Re(s)>0$, $|as^{-\alpha}|<1$. For $\gamma=1$, (11.8) reduces to

$$L\left[x^{\beta-1}E_{\alpha,\beta}(ax^{\alpha})\right](s)=s^{-\beta}\left(1-as^{-\alpha}\right)^{-1}. \tag{11.9}$$

Generalization of the above two results is given by Saxena [15]

$$L\left[t^{\rho-1}E^{\delta}_{\beta,\gamma}(at^{\alpha})\right](s) = \frac{s^{-\rho}}{\Gamma(\delta)}\,{}_2\psi_1\left[\begin{matrix}(\delta,1),(\rho,\alpha)\\(\gamma,\beta)\end{matrix};\ \frac{a}{s^{\alpha}}\right], \tag{11.10}$$

where $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(s) > 0, \Re(\rho) > 0, s > |a|^{1/\Re(\alpha)}, \Re(\delta) > 0$.

Relations connecting the function defined by (11.1) and the Riemann-Liouville fractional integrals and derivatives are given by Saxena and Saigo [35] in the form of nine theorems. Some of the interesting theorems are given below.

Theorem 11.3. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $a \in R$. Let $I^{\alpha}_{0_+}$ be the left-sided operator of Riemann-Liouville fractional integral. Then there holds the formula*

$$\left(I^{\alpha}_{0_+}\left[t^{\gamma-1}E^{\delta}_{\beta,\gamma}\left(at^{\beta}\right)\right]\right)(x) = x^{\alpha+\gamma-1}E^{\delta}_{\beta,\alpha+\gamma}\left(ax^{\beta}\right). \tag{11.11}$$

Theorem 11.4. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $a \in R$. Let I^{α}_{-} be the right-sided operator of Riemann-Liouville fractional integral. Then there holds the formula*

$$\left(I^{\alpha}_{-}\left[t^{-\alpha-\gamma}E^{\delta}_{\beta,\gamma}\left(at^{-\beta}\right)\right]\right)(x) = x^{-\gamma}E^{\delta}_{\beta,\alpha+\gamma}\left(ax^{-\beta}\right). \tag{11.12}$$

Theorem 11.5. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $a \in R$. Let $D^{\alpha}_{0_+}$ be the left-sided operator of Riemann-Liouville fractional derivative. Then there holds the formula*

$$\left(D^{\alpha}_{0_+}\left[t^{\gamma-1}E^{\delta}_{\beta,\gamma}\left(at^{\beta}\right)\right]\right)(x) = x^{\gamma-\alpha-1}E^{\delta}_{\beta,\gamma-\alpha}\left(ax^{\beta}\right). \tag{11.13}$$

Theorem 11.6. *Let $\alpha > 0$, $\beta > 0$, $\gamma - \alpha + \{\alpha\} > 1$, and $a \in R$. Let D^{α}_{-} be the right-sided operator of Riemann-Liouville fractional derivative. Then there holds the formula*

$$\left(D^{\alpha}_{-}\left[t^{\alpha-\gamma}E^{\delta}_{\beta,\gamma}\left(at^{-\beta}\right)\right]\right)(x) = x^{-\gamma}E^{\delta}_{\beta,\gamma-\alpha}\left(ax^{-\beta}\right). \tag{11.14}$$

In a series of papers by Luchko and Yakubovich [69, 70], Luckho and Srivastava [32], Al-Bassam and Luchko [71], Hadid and Luchko [72], Gorenflo and Luchko [24], Gorenflo et al. [73, 74], Luchko and Gorenflo [75], the operational method was developed to solve in closed forms certain classes of differential equations of fractional order and also integral equations. Solutions of the equations and problems considered are obtained in terms of generalized Mittag-Leffler functions. The exact solution of certain differential equation of fractional order is given by Luchko and Srivastava [32] in terms of function (11.1) by using operational method. In other papers, the solutions are established in terms of the following functions of Mittag-Leffler type: if $z, \rho, \beta_j \in C$, $\Re(\alpha_j) > 0$, $j = 1, \ldots, m$ and $m \in N$ then,

$$E_{\rho}\left((\alpha_j,\beta_j)_{1,m};(z)\right) = \sum_{k=0}^{\infty}\frac{(\rho)_k}{\prod_{j=1}^{m}\Gamma(a_jk+\beta_j)}\frac{z^k}{k!}. \tag{11.15}$$

For $m = 1$, (11.15) reduces to (11.1). The Mellin-Barnes integral for this function is given by

$$E_\rho\Big((\alpha_j,\beta_j)_{1,m};(z)\Big) = \frac{1}{2\pi i\Gamma(\rho)}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(\rho-s)}{\prod_{j=1}^{m}\Gamma(\beta_j-\alpha_j s)}(-z)^{-s}\mathrm{d}s, \quad i=\sqrt{-1}$$
$$= \frac{1}{\Gamma(\rho)}H_{1,m+1}^{1,1}\Big[-z\Big|_{(0,1),(1-\beta_j,\alpha_j),j=1,\ldots,m}^{(1-\rho,1)}\Big], \tag{11.16}$$

where $0,\gamma < R(\rho)$, $\Re(\rho) > 0$, and the contour separates the poles of $\Gamma(s)$ from those of $\Gamma(\rho-s)$. $\Re(\alpha_j) > 0$, $j = 1,\ldots,m$, $\arg(-z) < \pi$. The Laplace transform of the function defined by (11.15) is given by

$$L\Big[E_\rho\Big((\alpha_j,\beta_j)_{1,m};-t\Big)\Big](s) = \frac{1}{s\Gamma(\rho)}\,{}_2\psi_m\left[\begin{matrix}(\rho,1),(1,1)\\(\alpha_j,\beta_j)_{1,m}\end{matrix}\middle|\frac{1}{s}\right], \tag{11.17}$$

where $\Re(s) > 0$.

Remark 11.7. In a recent paper, Kilbas et al. [33] obtained a closed form solution of a fractional generalization of a free electron equation of the form

$$D_\tau^\alpha a(\tau) = \lambda\int_0^\tau t^\delta a(\tau-t)E_{\rho,\delta+1}^b(i\nu t^\rho)\mathrm{d}t + \beta\tau^\sigma E_{\rho,\sigma+1}^\gamma(i\nu t^\rho), \quad 0\le\tau\le 1,\ i=\sqrt{-1}, \tag{11.18}$$

where $b,\lambda\in C$, $\nu,\beta\in R_+$, $\alpha>0$, $\rho>0$, $\alpha>-1$, $\rho>-1$, $\delta>-1$, and $E_{\rho,\delta+1}^b(\cdot)$ is the generalized Mittag-Leffler function given by (11.1), and $\alpha(\tau)$ is the unknown function to be determined.

Remark 11.8. The solution of fractional differential equations by the operational methods are also obtained in terms of certain multivariate Mittag-Leffler functions defined below: The multivariate Mittag-Leffler function of n complex variables $z_1,\ldots,z_n$ with complex parameters $a_1,\ldots,a_n,b\in C$ is defined as

$$E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n) = \sum_{k=0}^{\infty}\sum_{L_1,\ldots,L_n\ge 0}^{L_1+\cdots+L_n=k}\binom{k}{L_1,\ldots,L_n}\frac{\prod_{j=1}^n z_j^{L_j}}{\Gamma\Big(b\pm\sum_{j=1}^n a_jL_j\Big)} \tag{11.19}$$

in terms of the multinomial coefficients

$$\binom{k}{L_1,\ldots,L_n} = \frac{k!}{L_1!\cdots L_n!}, \quad k,L_j,\in N_0,\ j=1,\ldots,m. \tag{11.20}$$

Another generalization of the Mittag-Leffler function (1.2) was introduced by Kilbas and Saigo [23, 76] in terms of a special function of the form

$$E_{\alpha,m,\beta}(z) = \sum_{k=0}^{\infty}c_kz^k, \quad c_0=1, \quad c_k = \prod_{i=0}^{k-1}\frac{\Gamma(\alpha(im+\beta)+1)}{\Gamma(\alpha(im+\beta+1)+1)}, \quad k\in N_0=\{0,1,2,\ldots\}, \tag{11.21}$$

where an empty product is to be interpreted as unity; $\alpha, \beta \in C$ are complex numbers and $m \in R$, $\Re(\alpha) > 0$, $m > 0$, $\alpha(im+\beta) \notin Z^- = \{0, -1, -2, \ldots\}$, $i = 0, 1, 2, \ldots$ and for $m = 1$ the above defined function reduces to a constant multiple of the Mittag-Leffler function, namely

$$E_{\alpha,1,\beta}(z) = \Gamma(\alpha\beta + 1) E_{\alpha,\alpha\beta+1}(z), \tag{11.22}$$

where $\Re(\alpha) > 0$ and $\alpha(i+\beta) \notin Z^-$. It is an entire function of z of order $[\Re(\alpha)]^{-1}$ and type $\sigma = 1/m$, see Gorenflo et al. [30]. Certain properties of this function associated with Riemann-Liouville fractional integrals and derivatives are obtained and exact solutions of certain integral equations of Abel-Volterra type are derived by their applications [23, 76, 77]. Its recurrence relations, connection with hypergeometric functions and differential formulas are obtained by Gorenflo et al. [30], also see, Gorenflo and Mainardi [25]. In order to present the applications of Mittag-Leffler functions we give definitions of Laplace transform, Fourier transform, Riemann-Liouville fractional calculus operators, Caputo operator and Weyl fractional operators in the next section.

12. Laplace and Fourier Transforms, Fractional Calculus Operators

We will need the definitions of the well-known Laplace and Fourier transforms of a function $N(x,t)$ and their inverses, which are useful in deriving the solution of fractional differential equations governing certain physical problems. The Laplace transform of a function $N(x,t)$ with respect to t is defined by

$$L\{N(x,t)\} = \widetilde{N}(x,s) = \int_0^\infty \mathrm{e}^{-st} N(x,t)\mathrm{d}t, \quad t > 0, \ x \in R, \tag{12.1}$$

where $\Re(s) > 0$ and its inverse transform with respect to s is given by

$$L^{-1}\left\{\widetilde{N}(x,s)\right\} = L^{-1}\left\{\widetilde{N}(x,s); t\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{e}^{st}\widetilde{N}(x,s)\mathrm{d}s. \tag{12.2}$$

The Fourier transform of a function $N(x,t)$ with respect to x is defined by

$$F\{N(x,t)\} = F^*(k,t) = \int_{-\infty}^{\infty} \mathrm{e}^{ikx} N(x,t)\mathrm{d}x. \tag{12.3}$$

The inverse Fourier transform with respect to k is given by the formula

$$F^{-1}\{F^*(k,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx} F^*(k,t)\mathrm{d}k. \tag{12.4}$$

From Mathai and Saxena [54] and Prudnikov et al. [55, page 355, (2.25.3)] it follows that the Laplace transform of the H-function is given by

$$L\left\{t^{\rho-1} H_{p,q}^{m,n}\left[zt^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = s^{-\rho} H_{p+1,q}^{m,n+1}\left[zs^{-\sigma}\Big|_{(b_q,B_q)}^{(1-\rho,\sigma),(a_p,\mathrm{A}_p)}\right], \tag{12.5}$$

where $\Re(s) > 0$, $\Re(\rho + \sigma \min_{1\le j\le m}(b_j/B_j) > 0$, $\sigma > 0$,

$$|\arg z| < \frac{1}{2}\pi\Omega, \quad \Omega > 0, \quad \Omega = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{12.6}$$

By virtue of the cancelation law for the H-function [54] it can be readily seen that

$$L^{-1}\left\{ s^{-\rho} H_{p,q}^{m,n}\left[zs^{\sigma} \middle|_{(b_q,B_q)}^{(a_p,A_p)} \right] \right\} = t^{\rho-1} H_{p+1,q}^{m,n}\left[zt^{-\sigma} \middle|_{(b_q,B_q)}^{(a_p,A_p),(\rho,\sigma)} \right], \tag{12.7}$$

where $\sigma > 0$, $\Re(s) > 0$, $\Re[\rho + \sigma \max_{1\le i\le n}((1-a_i)/A_i)] > 0$, $|\arg z| < (1/2)\pi\Omega_1$, $\Omega_1 > 0$, $\Omega_1 = \Omega - \rho$. In view of the results

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \tag{12.8}$$

the cosine transform of the H-function [54, page 49] is given by

$$\begin{aligned} &\int_0^{\infty} t^{\rho-1} \cos(kt) H_{p,q}^{m,n}\left[at^{\mu} \middle|_{(b_q,B_q)}^{(a_p,A_p)} \right] \mathrm{d}t \\ &\quad = \frac{\pi}{k^{\rho}} H_{q+1,p+2}^{m+1,n}\left[\frac{k^{\mu}}{a} \middle| \begin{matrix} (1-b_q,B_q),((1+\rho)/2,\mu/2) \\ (\rho,\mu),(a-a_p,A_p),((1+\rho)/2,\mu/2) \end{matrix} \right], \end{aligned} \tag{12.9}$$

where $\Re[\rho + \mu \min_{1\le j\le m}(b_j/B_j)] > 0$, $\Re[\rho + \mu \max_{1\le j\le n}((a_j-1)/A_j) < 0$, $|\arg a| < (1/2)\pi\Omega$, $\Omega > 0$; $\Omega = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j$, and $k > 0$.

The definitions of fractional integrals used in the analysis are defined below. The Riemann-Liouville fractional integral of order ν is defined by [78, page 45]

$${}_0D_t^{-\nu} f(x,t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(x,u)\mathrm{d}u, \tag{12.10}$$

where $\Re(\nu) > 0$. Following Samko et al. [16, page 37] we define the Riemann-Liouville fractional derivative for $\alpha > 0$ in the form

$${}_0D_t^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-u)^{n-\alpha-1} f(x,u)\mathrm{d}u, \quad n = [\alpha] + 1, \tag{12.11}$$

where $[\alpha]$ means the integral part of the number α. From Erdélyi et al. [79, page 182] we have

$$L\{{}_0D_t^{-\nu} f(x,t)\} = s^{-\nu} F(x,s), \tag{12.12}$$

where $F(x,s)$ is the Laplace transform of $f(x,t)$ with respect to t, $\Re(s) > 0, \Re(\nu) > 0$. The Laplace transform of the fractional derivative defined by (12.11) is given by Oldham and Spanier [80, page 134, (8.1.3)]

$$L\{{}_0D_t^{\alpha} f(x,t)\} = s^{\alpha}F(x,s) - \sum_{k=1}^{n} s^{k-1}\,{}_0D_t^{\alpha-k} f(x,t)\big|_{t=0}, \quad n-1 < \alpha \le n, \tag{12.13}$$

In certain boundary value problems arising in the theory of visco elasticity and in the hereditary solid mechanics the following fractional derivative of order $\alpha > 0$ is introduced by Caputo [81] in the form

$$\begin{aligned} D_t^{\alpha} f(x,t) &= \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(x,t)\mathrm{d}t, \quad m-1 < \alpha \le m,\ \Re(\alpha) > 0,\ m \in N \\ &= \frac{\partial^m}{\partial t^m} f(x,t), \quad \text{if } \alpha = m, \end{aligned} \tag{12.14}$$

where $(\partial^m/\partial t^m)f$ is the mth partial derivative of the function $f(x,t)$ with respect to t. The Laplace transform of this derivative is given by Podlubny [82] in the form

$$L\{D_t^{\alpha} f(t); s\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0_+), \quad m-1 < \alpha \le m. \tag{12.15}$$

The above formula is very useful in deriving the solution of differintegral equations of fractional order governing certain physical problems of reaction and diffusion. Making use of definitions (12.10) and (12.11) it readily follows that for $f(t) = t^{\rho}$ we obtain

$${}_0D_t^{-\nu} t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+\nu+1)} t^{\rho+\nu}, \quad \Re(\nu) > 0,\ \Re(\rho) > -1;\ t > 0, \tag{12.16}$$

$${}_0D_t^{\nu} t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\nu+1)} t^{\rho-\nu}, \quad \Re(\nu) < 0,\ \Re(\rho) > -1;\ t > 0. \tag{12.17}$$

On taking $\rho = 0$ in (12.17) we find that

$${}_0D_t^{\nu}[1] = \frac{1}{\Gamma(1-\nu)} t^{-\nu}, \quad t > 0,\ \Re(\nu) < 1. \tag{12.18}$$

From the above result, we infer that the Riemann-Liouville derivative of unity is not zero. We also need the Weyl fractional operator defined by

$${}_{-\infty}D_x^{\mu} = \frac{1}{\Gamma(n-\mu)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\infty}^{t} (t-u)^{n-\mu-1} f(u)\mathrm{d}u, \tag{12.19}$$

where $n = [\mu] + 1$ is the integer part of $\mu > 0$. Its Fourier transform is [83, page 59, A.11]

$$F\left\{{}_{-\infty}D_x^{\mu} f(x)\right\} = (ik)^{\mu} f^*(k), \tag{12.20}$$

where we define the Fourier transform as

$$h^*(q) = \int_{-\infty}^{\infty} h(x)\exp(iqx)\mathrm{d}x. \tag{12.21}$$

Following the convention initiated by Compte [84] we suppress the imaginary unit in Fourier space by adopting a slightly modified form of the above result in our investigations [83, page 59, A.12]

$$F\left\{{}_{-\infty}D_x^{\mu} f(x)\right\} = -|k|^{\mu} f^*(k), \tag{12.22}$$

instead of (12.20).

We now proceed to discuss the various applications of Mittag-Leffler functions in applied sciences. In order to discuss the application of Mittag-Leffler function in kinetic equations, we derive the solution of two kinetic equations in the next section.

13. Application in Kinetic Equations

Theorem 13.1. *If $\Re(\nu) > 0$, then the solution of the integral equation*

$$N(t) - N_0 = -c^{\nu}\,{}_0D_t^{-\nu} N(t) \tag{13.1}$$

is given by

$$N(t) = N_0 E_{\nu}(-c^{\nu}t^{\nu}), \tag{13.2}$$

where $E_{\nu}(t)$ is the Mittag-Leffler function defined in (1.1).

Proof. Applying Laplace transform to both sides of (13.1) and using (12.12) it gives

$$\widetilde{N}(s) = L\{N(t); s\} = N_0 s^{-1}\left[1 + \left(\frac{s}{c}\right)^{-\nu}\right]^{-1}. \tag{13.3}$$

By virtue of the relation

$$L^{-1}\left\{s^{-\rho}\right\} = \frac{t^{\rho-1}}{\Gamma(\rho)}, \quad \Re(\rho) > 0,\ \Re(s) > 0,\ s \in C, \tag{13.4}$$

it is seen that

$$L^{-1}\left[N_0 s^{-1}\left[1+\left(\frac{s}{c}\right)^{-\nu}\right]^{-1}\right] = N_0\sum_{k=0}^{\infty}(-1)^k c^{\nu k} L^{-1}\left\{s^{-\nu k-1}\right\} \tag{13.5}$$

$$= N_0\sum_{k=0}^{\infty}(-1)^k c^{\nu k}\frac{t^{\nu k}}{\Gamma(1+\nu k)} = N_0 E_\nu(-c^\nu t^\nu). \tag{13.6}$$

This completes the proof of Theorem 13.1. □

Remark 13.2. If we apply the operator ${}_0D_t^\nu$ from the left to (13.1) and make use of the formula

$${}_0D_t^\nu[1] = \frac{1}{\Gamma(1-\nu)}t^{-\nu}, \quad t>0,\ \Re(\nu)<1, \tag{13.7}$$

we obtain the fractional diffusion equation

$${}_0D_t^\nu N(t) - N_0\frac{t^{-\nu}}{\Gamma(1-\nu)} = -c^\nu N(t), \quad t>0,\ \Re(\nu)<1 \tag{13.8}$$

whose solution is also given by (13.6).

Remark 13.3. We note that Haubold and Mathai [46] have given the solution of (13.1) in terms of the series given by (13.5). The solution in terms of the Mittag-Leffler function is given in Saxena et al. [39].

Alternate Procedure

We now present an alternate method similar to that followed by Al-Saqabi and Tuan [85] for solving some differintegral equations, also, see Saxena and Kalla [44] for details.

Applying the operator $(-c^\nu)^m\,{}_0D_t^{-m\nu}$ to both sides of (13.1) we find that

$$(-c^\nu)^m\,{}_0D_t^{-m\nu}N(t) - (-c^\nu)^{m+1}\,{}_0D_t^{-\nu(m+1)}N(t) = N_0\,{}_0D_t^{-m\nu}[1], \quad m=0,1,2,\ldots. \tag{13.9}$$

Summing up the above expression with respect to m from 0 to ∞, it gives

$$\sum_{m=0}^{\infty}(-c^\nu)^m\,{}_0D_t^{-m\nu}N(t) - \sum_{m=0}^{\infty}(-c^\nu)^{m+1}\,{}_0D_t^{-\nu(m+1)}N(t) = N_0\sum_{m=0}^{\infty}(-c^\nu)^m\,{}_0D_t^{-m\nu}[1] \tag{13.10}$$

which can be written as

$$\sum_{m=0}^{\infty}(-c^\nu)^m\,{}_0D_t^{-m\nu}N(t) - \sum_{m=1}^{\infty}(-c^\nu)^m\,{}_0D_t^{-\nu m}N(t) = N_0\sum_{m=0}^{\infty}(-c^\nu)^m\,{}_0D_t^{-m\nu}[1]. \tag{13.11}$$

Simplifying the above equation by using the result

$$ {}_0D_t^{-\nu} t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)} t^{\mu+\nu-1}, \tag{13.12} $$

where $\min\{\Re(\nu), \Re(\mu)\} > 0$, we obtain

$$ N(t) = N_0 \sum_{m=0}^{\infty} (-c^{\nu})^m \frac{t^{m\nu}}{\Gamma(1+m\nu)} \tag{13.13} $$

for $m = 0, 1, 2, \ldots$. Rewriting the series on the right in terms of the Mittag-Leffler function, it yields the desired result (13.6). The next theorem can be proved in a similar manner.

Theorem 13.4. *If* $\min\{\Re(\nu), \Re(\mu)\} > 0$, *then the solution of the integral equation*

$$ N(t) - N_0 t^{\mu-1} = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{13.14} $$

is given by

$$ N(t) = N_0 \Gamma(\mu) t^{\mu-1} E_{\nu,\mu}(-c^{\nu} t^{\nu}), \tag{13.15} $$

where $E_{\nu,\mu}(t)$ *is the generalized Mittag-Leffler function defined in* (1.2).

Proof. Applying Laplace transform to both sides of (13.14) and using (1.11), it gives

$$ \widetilde{N}(s) = \{N(t); s\} = N_0 \Gamma(\mu) s^{-\mu} \left[1 + \left(\frac{s}{c}\right)^{-\nu}\right]^{-1}. \tag{13.16} $$

Using relation (13.4), it is seen that

$$ \begin{aligned} L^{-1}\left\{N_0 \Gamma(\mu) s^{-\mu} \left[1 + \left(\frac{s}{c}\right)^{-\nu}\right]^{-1}\right\} &= N_0 \sum_{k=0}^{\infty} (-1)^k c^{\nu k} L^{-1}\left\{s^{-\mu-\nu k}\right\} \\ &= N_0 \Gamma(\mu) t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k c^{\nu k} \frac{t^{\nu k}}{\Gamma(\mu+\nu k)} \\ &= N_0 \Gamma(\mu) t^{\mu-1} E_{\nu,\mu}(-c^{\nu} t^{\nu}). \end{aligned} \tag{13.17} $$

This completes the proof of Theorem 13.4. □

Alternate Procedure

We now give an alternate method similar to that followed by Al-Saqabi and Tuan [85] for solving the differintegral equations. Applying the operator $(-c^{\nu})^m\, {}_0D_t^{-m\nu}$ to both sides of (13.14), we find that

$$ (-c^{\nu})^m\, {}_0D_t^{-m\nu} N(t) - (-c^{\nu})^{m+1}\, {}_0D_t^{-\nu(m+1)} N(t) = N_0 (-c^{\nu})^m\, {}_0D_t^{-m\nu} t^{\mu-1} \tag{13.18} $$

for $m = 0, 1, 2, \ldots$. Summing up the above expression with respect to m from 0 to ∞, it gives

$$\sum_{m=0}^{\infty}(-c^{\nu})^{m}\,{}_{0}D_{t}^{-m\nu}N(t) - \sum_{m=0}^{\infty}(-c^{\nu})^{m+1}\,{}_{0}D_{t}^{-\nu(m+1)}N(t) = N_{0}\sum_{m=0}^{\infty}(-c^{\nu})^{m}\,{}_{0}D_{t}^{-m\nu}t^{\mu-1} \tag{13.19}$$

which can be written as

$$\sum_{m=0}^{\infty}(-c^{\nu})^{m}\,{}_{0}D_{t}^{-m\nu}N(t) - \sum_{m=1}^{\infty}(-c^{\nu})^{m}\,{}_{0}D_{t}^{-m\nu}N(t) = N_{0}\sum_{m=0}^{\infty}(-c^{\nu})^{m}\,{}_{0}D_{t}^{-m\nu}t^{\mu-1}. \tag{13.20}$$

Simplifying by using result (13.12) we obtain

$$N(t) = N_{0}\Gamma(\mu)\sum_{m=0}^{\infty}(-c^{\nu})^{m}\frac{t^{\mu+m\nu-1}}{\Gamma(m\nu+\mu)}; \quad m = 0, 1, 2, \ldots. \tag{13.21}$$

Rewriting the series on the right of (13.21) in terms of the generalized Mittag-Leffler function, it yields the desired result (13.5). Next we present a general theorem given by Saxena et al. [40].

Theorem 13.5. *If $c > 0$, $\Re(\nu) > 0$, then for the solution of the integral equation*

$$N(t) - N_{0}f(t) = -c^{\nu}\,{}_{0}D_{t}^{-\nu}N(t) \tag{13.22}$$

where $f(t)$ is any integrable function on the finite interval $[0, b]$, there exists the formula

$$N(t) = cN_{0}\int_{0}^{t}H_{1,2}^{1,1}\left[c^{\nu}(t-\tau)^{\nu}\Big|_{(-1/\nu,1),(-1,\nu)}^{(-1/\nu,1)}\right]f(\tau)\mathrm{d}\tau, \tag{13.23}$$

where $H_{1,2}^{1,1}(\cdot)$ is the H-function defined by (8.1).

The proof can be developed by identifying the Laplace transform of $N()+c^{\nu}\,{}_{0}D_{t}^{-\nu}N(t)$ as an H-function and then using the convolution property for the Laplace transform. In what follows, $E_{\beta,\gamma}^{\delta}(\cdot)$ will be employed to denote the generalized Mittag-Leffler function, defined by (11.1).

Note 4. For an alternate derivation of this theorem see Saxena and Kalla [44].

Next we will discuss time-fractional diffusion.

14. Application to Time-Fractional Diffusion

Theorem 14.1. *Consider the following time-fractional diffusion equation:*

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}N(x,t) = D\frac{\partial^{2}}{\partial x^{2}}N(x,t), \quad 0 < \alpha < 1, \tag{14.1}$$

where D is the diffusion constant and $N(x,t=0)=\delta(x)$ is the Dirac delta function and $\lim_{x\to\pm\infty}N(x,t)=0$. Then its fundamental solution is given by

$$N(x,t)=\frac{1}{|x|}H_{1,1}^{1,0}\left[\frac{|x|^2}{Dt^{\alpha}}\middle|\begin{matrix}(1,\alpha)\\(1,2)\end{matrix}\right]. \tag{14.2}$$

Proof. In order to find a closed form representation of the solution in terms of the H-function, we use the method of joint Laplace-Fourier transform, defined by

$$\widetilde{N}^{*}(k,s)=\int_0^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-st+ikx}N(x,t)\mathrm{d}x\,\mathrm{d}t, \tag{14.3}$$

where, according to the convention followed, ~ will denote the Laplace transform and * the Fourier transform. Applying the Laplace transform with respect to time variable t, Fourier transform with respect to space variable x, and using the given condition, we find that

$$s^{\alpha}\widetilde{N}^{*}(k,s)-s^{\alpha-1}=-Dk^2\widetilde{N}^{*}(k,s), \tag{14.4}$$

and then

$$\widetilde{N}^{*}(k,s)=\frac{s^{\alpha-1}}{s^{\alpha}+Dk^2}. \tag{14.5}$$

Inverting the Laplace transform, it yields

$$N^{*}(k,t)=L^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+Dk^2}\right\}=E_{\alpha}\left(-Dk^2t^{\alpha}\right), \tag{14.6}$$

where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function defined by (1.1). In order to invert the Fourier transform, we will make use of the integral

$$\int_0^{\infty}\cos(kt)E_{\alpha,\beta}\left(-at^2\right)\mathrm{d}t=\frac{\pi}{k}H_{1,1}^{1,0}\left[\frac{k^2}{a}\middle|\begin{matrix}(\beta,\alpha)\\(1,2)\end{matrix}\right], \tag{14.7}$$

where $\Re(\alpha)>0$, $\Re(\beta)>0$, $k>0$, $a>0$, and the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-ikx}f(k)\mathrm{d}k=\frac{1}{\pi}\int_0^{\infty}f(k)\cos(kx)\mathrm{d}k. \tag{14.8}$$

Then it yields the required solution as

$$N(x,t)=\frac{1}{|x|}H_{3,3}^{2,1}\left[\frac{|x|^2}{Dt^{\alpha}}\middle|\begin{matrix}(1,1),(1,\alpha),(1,1)\\(1,2),(1,1),(1,1)\end{matrix}\right]=\frac{1}{|x|}H_{1,1}^{1,0}\left[\frac{|x|^2}{Dt^{\alpha}}\middle|\begin{matrix}(1,\alpha)\\(1,2)\end{matrix}\right]. \tag{14.9}$$

□

Note 5. For $\alpha = 1$, (14.9) reduces to the Gaussian density

$$N(x,t) = \frac{1}{2(\pi D t)^{1/2}} \exp\left(-\frac{|x|^2}{4Dt}\right). \tag{14.10}$$

Fractional-space-diffusion will be discussed in the next section.

15. Application to Fractional-Space Diffusion

Theorem 15.1. *Consider the following fractional-space-diffusion equation:*

$$\frac{\partial}{\partial t} N(x,t) = D \frac{\partial^\alpha}{\partial x^\alpha} N(x,t), \quad 0 < \alpha < 1, \tag{15.1}$$

where D is the diffusion constant, $\partial^\alpha / \partial x^\alpha$ is the operator defined by (12.19), *and $N(x,t=0) = \delta(x)$ is the Dirac delta function and $\lim_{x\to\pm\infty} N(x,t) = 0$. Then its fundamental solution is given by*

$$N(x,t) = \frac{1}{a|x|} H_{2,2}^{1,1}\left[\frac{|x|}{(Dt)^{1/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right]. \tag{15.2}$$

The proof can be developed on similar lines to that of the theorem of the preceding section.

16. Application to Fractional Reaction-Diffusion Model

In the same way, we can establish the following theorem, which gives the fundamental solution of the reaction-diffusion model given below.

Theorem 16.1. *Consider the following reaction-diffusion model:*

$$\frac{\partial^\beta}{\partial t^\beta} N(x,t) = \eta\, {}_{-\infty}D_x^\alpha N(x,t), \quad 0 < \beta \le 1 \tag{16.1}$$

with the initial condition $N(x,t=0) = \delta(x)$, $\lim_{x\to\pm\infty} N(x,t) = 0$, where η is a diffusion constant and $\delta(x)$ is the Dirac delta function. Then for the solution of (16.1) *there holds the formula*

$$N(x,t) = \frac{1}{a|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^\beta)^{1/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(1,\beta/\alpha),(1,1/2)\\(1,1),(1,1/\alpha),(1,1/2)\end{matrix}\right]. \tag{16.2}$$

For details of the proof, the reader is referred to the original paper by Saxena et al. [43].

Corollary 16.2. *For the solution of the fraction reaction-diffusion equation*

$$\frac{\partial}{\partial t} N(x,t) = \eta\, {}_{-\infty}D_x^\alpha N(x,t), \tag{16.3}$$

with initial condition $N(x,t=0)=\delta(x)$ *there holds the formula*

$$N(x,t)=\frac{1}{\alpha|x|}H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{1/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right], \tag{16.4}$$

where $\alpha>0$.

Note 6. It may be noted that (16.4) is a closed form representation of a Lévy stable law. It is interesting to note that as $\alpha\to 2$ the classical Gaussian solution is recovered since

$$\begin{aligned}N(x,t)&=\frac{1}{2|x|}H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{1/\alpha}}\middle|\begin{matrix}(1,1/2),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right]\\&=\frac{1}{2\pi^{1/2}|x|}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[\frac{|x|}{2(\eta t)^{1/\alpha}}\right]^{2k+1}\\&=\left(4\pi(\eta t)^{2/\alpha}\right)^{-1/2}\exp\left[-\frac{|x|^2}{4(\eta t)^{2/\alpha}}\right].\end{aligned} \tag{16.5}$$

17. Application to Nonlinear Waves

It will be shown in this section that by the application of the inverse Laplace transforms of certain algebraic functions derived in Saxena et al. [43], we can establish the following theorem for nonlinear waves.

Theorem 17.1. *Consider the fractional reaction-diffusion equation:*

$${}_0D_t^{\alpha}N(x,t)+a_0D_t^{\beta}N(x,t)=\nu^2{}_{-\infty}D_x^{\gamma}N(x,t)+\zeta^2N(x,t)+\phi(x,t) \tag{17.1}$$

for $x\in R$, $t>0$, $0\le\alpha\le 1$, $0\le\beta\le 1$ *with initial conditions* $N(x,0)=f(x)$, $\lim_{x\to\pm\infty}N(x,t)=0$ *for* $x\in R$, *where* ν^2 *is a diffusion constant,* ζ *is a constant which describes the nonlinearity in the system, and* $\phi(x,t)$ *is nonlinear function which belongs to the area of reaction-diffusion, then there holds the following formula for the solution of* (17.1)

$$\begin{aligned}N(x,t)=&\sum_{r=0}^{\infty}\frac{(-a)^r}{2\pi}\int_{-\infty}^{\infty}t^{(\alpha-\beta)r}f^*(k)\exp(-kx)\\&\times\left[E_{\alpha,(\alpha-\beta)r+1}^{r+1}(-bt^{\alpha})+t^{\alpha-\beta}E_{\alpha,(\alpha-\beta)(r+1)+1}^{r+1}(-bt^{\alpha})\right]dk\\&+\sum_{r=0}^{\infty}\frac{(-a)^r}{2\pi}\int_0^t\zeta^{\alpha+(\alpha-\beta)r-1}\int_{-\infty}^{\infty}\phi(k,t-\zeta)\exp(-ikx)\\&\times E_{\alpha,\alpha+(\alpha-\beta)r}^{r+1}(-b\zeta^{\alpha})\mathrm{d}k\mathrm{d}\zeta,\end{aligned} \tag{17.2}$$

where $\alpha>\beta$ *and* $E_{\beta,\gamma}^{\delta}(\cdot)$ *is the generalized Mittag-Leffler function, defined by* (11.1), *and* $b=\nu^2|k|^{\gamma}-\zeta^2$.

Proof. Applying the Laplace transform with respect to the time variable t and using the boundary conditions, we find that

$$s^{\alpha}\widetilde{N}(x,s) - s^{\alpha-1}f(x) + as^{\beta}\widetilde{N}(x,s) - as^{\beta-1}f(x) = \nu^2\,{}_{-\infty}D_x^{\gamma}\widetilde{N}(x,s) + \zeta^2\widetilde{N}(x,s) + \widetilde{f}(x,s). \tag{17.3}$$

If we apply the Fourier transform with respect to the space variable x to (17.3) it yields

$$s^{\alpha}\widetilde{N}^*(k,s) - s^{\alpha-1}f^*(k) + as^{\beta}\widetilde{N}^*(k,s) - as^{\beta-1}f^*(k) = -\nu^2|k|^{\gamma}\widetilde{N}^*(k,s) + \zeta^2\widetilde{N}^*(k,s) + \widetilde{f}^*(k,s). \tag{17.4}$$

Solving for $\widetilde{N}^*$ it gives

$$\widetilde{N}^*(k,s) = \frac{\left(s^{\alpha-1} + as^{\beta-1}\right)f^*(k) + \widetilde{f}^*(k,s)}{s^{\alpha} + as^{\beta} + b}, \tag{17.5}$$

where $b = \nu^2|k|^{\gamma} - \zeta^2$. For inverting (17.5) it is convenient to first invert the Laplace transform and then the Fourier transform. Inverting the Laplace transform with the help of the result Saxena et al. [41, equation (28)]

$$L^{-1}\left\{\frac{s^{\rho-1}}{s^{\alpha} + as^{\beta} + b}; t\right\} = t^{\alpha-\rho}\sum_{r=0}^{\infty}(-a)^r t^{(\alpha-\beta)r}E^{r+1}_{\alpha,\alpha+(\alpha-\beta)r-\rho+1}(-bt^{\alpha}), \tag{17.6}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\rho) > 0, |as^{\beta}/(s^{\alpha} + b)| < 1$ and provided that the series in (17.6) is convergent, it yields

$$\begin{aligned} N^*(k,t) = {} & \sum_{r=0}^{\infty}(-a)^r t^{(\alpha-\beta)r}f^*(k) \\ & \times \left[E^{r+1}_{\alpha,(\alpha-\beta)r+1}(-bt^{\alpha}) + t^{\alpha-\beta}E^{r+1}_{\alpha,(\alpha-\beta)(r+1)+1}(-bt^{\alpha})\right] \\ & + \sum_{r=0}^{\infty}(-a)^r\int_0^t \phi^*(k,t-\zeta)\zeta^{\alpha+(\alpha-\beta)r-1}E^{r+1}_{\alpha,(\alpha-\beta)r+\alpha}(-b\zeta^{\alpha})\mathrm{d}\zeta. \end{aligned} \tag{17.7}$$

Finally, the inverse Fourier transform gives the desired solution (17.3). □

18. Generalized Mittag-Leffler Type Functions

The multi-index (m-tuple) Mittag-Leffler function is defined in Kiryakova [86] by means of the power series

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty}\phi_k z^k = \sum_{k=0}^{\infty}\frac{z^k}{\prod_{j=1}^{m}\Gamma\left(\mu_j + k/\rho_j\right)}. \tag{18.1}$$

Here $m > 1$ is an integer, $\rho_1, \ldots, \rho_m$ and $\mu_1, \ldots, \mu_m$ are arbitrary real parameters.

The following theorem is proved by Kiryakova [86, page 244] which shows that the multiindex Mittag-Leffler function is an entire function and also gives its asymptotic estimate, order, and type.

Theorem 18.1. *For arbitrary sets of indices $\rho_i > 0$, $-\infty < \mu_i < \infty$, $i = 1, \ldots, m$ the multiindex Mittag-Leffler function defined by* (18.1) *is an entire function of order*

$$\rho = \left[\sum_{i=1}^{m} \frac{1}{\rho_i}\right]^{-1}, \quad \text{that is,} \quad \frac{1}{\rho} = \frac{1}{\rho_1} + \cdots + \frac{1}{\rho_m} \tag{18.2}$$

and type

$$\sigma = \left(\frac{\rho_1}{\rho}\right)^{\rho/\rho_1} \cdots \left(\frac{\rho_m}{\rho}\right)^{\rho/\rho_m}. \tag{18.3}$$

Furthermore, for every positive ϵ, the asymptotic estimate

$$\left|E_{(1/\rho_i),(\mu_i)}(z)\right| < \exp\left((\sigma + \epsilon)|z|^{\rho}\right), \tag{18.4}$$

holds for $|z| \geq r_0(\epsilon)$, $r_0(\epsilon)$ sufficiently large.

It is interesting to note that for $m = 2$, (18.2) reduces to the generalized Mittag-Leffler function considered by Dzherbashyan [87] denoted by $\phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2)$ and defined in the following form [62, Appendix]:

$$E_{(1/\rho_1,1/\rho_2);(\mu_1,\mu_2)}(z) = \phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\Gamma(\mu_2 + k/\rho_2)}, \tag{18.5}$$

and shown to be an entire function of order

$$\rho = \frac{\rho_1\rho_2}{\rho_1 + \rho_2} \quad \text{and type } \sigma = \left(\frac{\rho_1}{\rho_2}\right)^{\rho_2/\rho_1} \left(\frac{\rho_2}{\rho_1}\right)^{\rho_1/\rho_2}. \tag{18.6}$$

Relations between multiindex Mittag-Leffler function with H-function, generalized Wright function and other special functions are given by Kiryakova; for details, see the original papers Kiryakova [86, 88]. Saxena et al. [38] investigated the relations between the multiindex Mittag-Leffler function and the Riemann-Liouville fractional integrals and derivatives. The results derived are of general nature and give rise to a number of known as well as unknown results in the theory of generalized Mittag-Leffler functions, which serve as a backbone for the fractional calculus. Two interesting theorems established by Saxena et al. [38] are described below.

Theorem 18.2. *Let $\alpha > 0$, $\rho_i > 0$, $\mu_i > 0$, $i = 1, \ldots, m$, and further, let $I_{0_+}^{\alpha}$ be the left-sided Riemann-Liouville fractional integral. Then there holds the relation*

$$\left(I_{0_+}^{\alpha}\left[t^{\rho_i - 1} E_{(1/\rho_i),(\mu_i)}\left(at^{1/\rho_i}\right)\right]\right)(x) = x^{\alpha+\rho_i-1} E_{(1/\rho_i),(\mu_1+\alpha,\mu_2,\ldots,\mu_m)}\left(at^{1/\rho_i}\right). \tag{18.7}$$

Theorem 18.3. *Let $\alpha > 0$, $\rho_i > 0$, $\mu_i > 0$, $i = 1, \ldots, m$, and further, let I_-^α be the right-sided Riemann-Liouville fractional integral. Then there holds the relation*

$$\left(I_-^\alpha\left[t^{-\rho_i-\alpha}E_{(1/\rho_i),(\mu_i)}\left(at^{-1/\rho_i}\right)\right]\right)(x) = x^{-\mu_i}E_{(1/\rho_i),(\mu_1+\alpha,\mu_2,\ldots,\mu_m)}\left(at^{-1/\rho_i}\right). \tag{18.8}$$

Another generalization of the Mittag-Leffler function is recently given by Sharma [89] in terms of the M-series defined by

$$\begin{aligned} {}_pM_q^\alpha &= {}_pM_q^\alpha(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha;z) \\ &= \sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r}{(b_1)_r\cdots(b_q)_r\Gamma(\alpha r+1)}z^r, \quad p \le q+1. \end{aligned} \tag{18.9}$$

Remark 18.4. According to Saxena [90], the M-series discussed by Sharma [89] is not a new special function. It is, in disguise, a special case of the generalized Wright function ${}_p\psi_q(z)$, which was introduced by Wright [57], as shown below

$$\begin{aligned} \kappa\,{}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1,1),\ldots,(a_p,1),(1,1)\\(b_1,1),\ldots,(b_q,1),(1,\alpha)\end{matrix};\ z\right] &= \sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r(1)_r}{(b_1)_r\cdots(b_q)_r\Gamma(\alpha r+1)}\frac{z^r}{r!} \\ &= \sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r}{(b_1)_r\cdots(b_q)_r\Gamma(\alpha r+1)}z^r \\ &= {}_pM_q^\alpha(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha;z), \end{aligned} \tag{18.10}$$

where

$$\kappa = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}. \tag{18.11}$$

Fractional integration and fractional differentiation of the M-series are discussed by Sharma [89]. The two results proved in Sharma [89] for the function defined by (18.9) are reproduced below. For $\Re(\nu) > 0$

$$\begin{aligned} &\left(I_{0_+}^\nu\left[{}_pM_q^\alpha(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha;z)\right]\right)(x) \\ &\quad = \frac{z^\nu}{\Gamma(1+\nu)}\,{}_{p+1}M_{q+1}^\alpha(a_1,\ldots,a_p,1;b_1,\ldots,b_q,1+\nu;\alpha;z), \end{aligned} \tag{18.12}$$

and for $\Re(\nu) < 0$

$$\begin{aligned} &\left(D_{0_+}^\nu\left[{}_pM_q^\alpha(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha;z)\right]\right)(x) \\ &\quad = \frac{z^\nu}{\Gamma(1-\nu)}\,{}_{p+1}M_{q+1}^\alpha(a_1,\ldots,a_p,1;b_1,\ldots,b_q,1-\nu;\alpha;z). \end{aligned} \tag{18.13}$$

19. Mittag-Leffler Distributions and Processes

19.1. Mittag-Leffler Statistical Distribution and Its Properties

A statistical distribution in terms of the Mittag-Leffler function $E_\alpha(y)$ was defined by Pillai [91] in terms of the distribution function or cumulative density function as follows:

$$G_y(y) = 1 - E_\alpha(-y^\alpha) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} y^{\alpha k}}{\Gamma(1+\alpha k)}, \quad 0 < \alpha \le 1,\ y > 0 \tag{19.1}$$

and $G_y(y) = 0$ for $y \le 0$. Differentiating on both sides with respect to y we obtain the density function $f(y)$ as follows:

$$\begin{aligned} f(y) &= \frac{\mathrm{d}}{\mathrm{d}y} G_y(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1} y^{\alpha k}}{\Gamma(1+\alpha k)}\right] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \alpha k y^{\alpha k-1}}{\Gamma(1+\alpha k)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} y^{\alpha k}}{\Gamma(1+\alpha k)} = \sum_{k=0}^{\infty} \frac{(-1)^{k} y^{\alpha+\alpha k-1}}{\Gamma(\alpha+\alpha k)} \end{aligned} \tag{19.2}$$

by replacing k by $k+1$

$$= y^{\alpha-1} E_{\alpha,\alpha}(-y^\alpha), \quad 0 < \alpha \le 1,\ y > 0, \tag{19.3}$$

where $E_{\alpha,\beta}(x)$ is the generalized Mittag-Leffler function.

It is straightforward to observe that for the density in (19.3) the distribution function is that in (19.1). The Laplace transform of the density in (19.3) is the following:

$$L_f(t) = \int_0^\infty \mathrm{e}^{-tx} f(x)\mathrm{d}x = \int_0^\infty \mathrm{e}^{-tx} x^{\alpha-1} E_{\alpha,\alpha}(-x^\alpha)\mathrm{d}x = (1+t^\alpha)^{-1}, \quad |t^\alpha| < 1. \tag{19.4}$$

Note that (19.4) is a special case of the general class of Laplace transforms discussed in [92, Section 2.3.7]. From (19.4) one can also note that there is a structural representation in terms of positive Lévy distribution. A positive Lévy random variable $u > 0$, with parameter α is such that the Laplace transform of the density of $u > 0$ is given by e^{-t^α}. That is,

$$E\left[\mathrm{e}^{-tu}\right] = \mathrm{e}^{-t^\alpha}, \tag{19.5}$$

where $E(\cdot)$ denotes the expected value of $(\cdot)$ or the statistical expectation of $(\cdot)$. When $\alpha = 1$ the random variable is degenerate with the density function

$$f_1(x) = \begin{cases} 1, & \text{for } x = 1, \\ 0, & \text{elsewhere.} \end{cases} \tag{19.6}$$

Consider an exponential random variable with density function

$$f_1(x) = \begin{cases} e^{-x}, & \text{for } 0 \le x < \infty \\ 0, & \text{elsewhere} \end{cases} \qquad L_{f_1}(t) = (1+t)^{-1} \tag{19.7}$$

and with the Laplace transform $L_{f_1}(t)$.

Theorem 19.1. *Let $y > 0$ be a Lévy random variable with Laplace transform as in* (19.5) *and let x and y be independently distributed. Then $u = yx^{1/\alpha}$ is distributed as a Mittag-Leffler random variable with Laplace transform as in* (19.4).

Proof. For establishing this result we will make use of a basic result on conditional expectations, which will be stated as a lemma. □

Lemma 19.2. *For two random variables x and y having a joint distribution,*

$$E(x) = E\left[E(x \mid y)\right] \tag{19.8}$$

whenever all the expected values exist, where the inside expectation is taken in the conditional space of x given y and the outside expectation is taken in the marginal space of y.

Now by applying (19.8) we have the following: let the density of u be denoted by $g(u)$. Then the Laplace transform of g is given by

$$E\left[e^{-(tx^{1/\alpha})y} \mid x\right] = e^{-t^{\alpha}x.} \tag{19.9}$$

But the right side of (19.9) is in the form of a Laplace transform of the density of x with parameter t^{α}. Hence the expected value of the right side is

$$L_g(t) = (1+t^{\alpha})^{-1} \tag{19.10}$$

which establishes the result. From (19.9) one property is obvious. Suppose that we consider an arbitrary random variable y with the Laplace transform of the form

$$L_g(t) = e^{-[\phi(t)]} \tag{19.11}$$

whenever the expected value exists, where $\phi(t)$ be such that

$$\phi\left(tx^{1/\alpha}\right) = x\phi(t), \qquad \lim_{t\to 0}\phi(t) = 0. \tag{19.12}$$

Then from (19.9) we have

$$E\left[e^{-(tx^{1/\alpha})y} \mid x\right] = e^{-x[\phi(t)]}. \tag{19.13}$$

Now, let x be an arbitrary positive random variable having Laplace transform, denoted by $L_x(t)$ where $L_x(t) = \psi(t)$. Then from (19.10) we have

$$L_g(t) = \psi\left[\phi(t)\right]. \tag{19.14}$$

For example, if y is a random variable whose density has the Laplace transform, denoted by $L_y(t) = \phi(t)$, with $\phi(tx^{1/\alpha}) = x\phi(t)$, and if x is a real random variable having the gamma density,

$$f_x(x) = \frac{x^{\beta-1}e^{-x/\delta}}{\delta^{\beta}\Gamma(\beta)}, \quad x \geq 0,\ \beta > 0,\ \delta > 0 \tag{19.15}$$

and $f_x(x) = 0$ elsewhere, and if x and y are statistically independently distributed and if $u = yx^{1/\alpha}$ then the Laplace transform of the density of u, denoted by $L_u(t)$ is given by

$$L_u(t) = \left[1 + \delta\left\{\phi(t)\right\}\right]^{-\beta}. \tag{19.16}$$

Note 7. Since we did not put any restriction on the nature of the random variables, except that the expected values exist, the result in (19.14) holds whether the variables are continuous, discrete or mixed.

Note 8. Observe that for the result in (19.14) to hold we need only the conditional Laplace transform of y given x be of the form in (19.13) and the marginal Laplace transform of x be $\psi(t)$. Then the result in (19.14) will hold. Thus statistical independence of x and y is not a basic requirement for the result in (19.14) to hold.

Thus from (19.15) we may write a particular case as

$$z = yx^{1/\alpha}, \tag{19.17}$$

where x is distributed as in (19.7), and y as in (19.5) then z will be distributed as in (19.4) or (19.10) when x and y are assumed to be independently distributed.

Note 9. The representation of the Mittag-Leffler variable as well as the properties described in Jayakumar [93, page 1432] and in Jayakumar and Suresh [94, page 53] are to be rewritten and corrected because the exponential variable and Lévy variable seem to be interchanged there.

By taking the natural logarithms on both sides of (19.17) we have

$$\frac{1}{\alpha}\ln x + \ln y = \ln z. \tag{19.18}$$

Then the first moment of $\ln z$ is available from (19.18) by computing $E[\ln x]$ and $E[\ln y]$. But $E[\ln x]$ is available from the following procedure:

$$E\left[e^{-t\ln x}\right] = E\left[e^{\ln x^{-t}}\right] = E\left[x^{-t}\right] = \int_0^{\infty} x^{-t}e^{-x}dx = \Gamma(1-t) \quad \text{for } \Re(1-t) > 0 \tag{19.19}$$

which will be $\Gamma(\beta - t)/\Gamma(\beta)$ for the density in (19.15). Hence

$$E[\ln x] = -\frac{\mathrm{d}}{\mathrm{d}t} E\left[\mathrm{e}^{-t\ln x}\right]\Big|_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(1-t)|_{t=0}. \tag{19.20}$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(1-t) = -\Gamma(1-t)\frac{\mathrm{d}}{\mathrm{d}t}\ln\Gamma(1-t) = -\Gamma(1-t)\psi(1-t), \tag{19.21}$$

where $\psi(\cdot)$ is the psi function of $(\cdot)$, see Mathai [95] for details. Hence by taking the limits $t \to 0$

$$E[\ln x] = -\Gamma(1-t)\psi(1-t)\big|_{t=0} = -\psi(1) = \gamma, \tag{19.22}$$

where γ is Euler's constant, see Mathai [95] for details.

19.2. Mellin-Barnes Representation of the Mittag-Leffler Density

Consider the density function in (19.3). After writing in series form and then looking at the corresponding Mellin-Barnes representation we have the following:

$$g(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha}) = \sum_{k=0}^{\infty}\Gamma(1+k)\frac{(-1)^k}{k!}\frac{x^{\alpha-1+\alpha k}}{\Gamma(\alpha+\alpha k)} \tag{19.23}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\alpha}\frac{\Gamma(1/\alpha - s/\alpha)\Gamma(1-1/\alpha+s/\alpha)}{\Gamma(1-s)}x^{-s}\mathrm{d}s, \quad 1-\alpha < c < 1 \tag{19.24}$$

(by expanding as the sum of residues at the poles of $\Gamma(1-1/\alpha+s/\alpha)$)

$$= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha s)}x^{\alpha s-1}\mathrm{d}s = g(x), \quad 0 < c_1 < 1,\ 0 < \alpha \le 1 \tag{19.25}$$

by putting $1/\alpha - s/\alpha = s_1$. Here the point $s = 0$ is removable. By taking the Laplace transform of $g(x)$ from (19.23) we have

$$\begin{aligned} L_g(t) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha+\alpha k)}\int_0^{\infty}x^{\alpha+\alpha k-1}\mathrm{e}^{-tx}\mathrm{d}x \\ &= \sum_{k=0}^{\infty}(-1)^k t^{-\alpha-\alpha k} = t^{-\alpha}\left(1+t^{-\alpha}\right)^{-1} = (1+t^{\alpha})^{-1}, \quad |t^{\alpha}| < 1. \end{aligned} \tag{19.26}$$

19.2.1. Generalized Mittag-Leffler Density

Consider the generalized Mittag-Leffler function

$$g_1(x) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\gamma+k)}{k!\Gamma(\alpha k + \alpha\gamma)} x^{\alpha\gamma-1+\alpha k} = x^{\alpha\gamma-1} E_{\alpha\gamma,\alpha}^{\gamma}(-x^{\alpha}), \quad \alpha > 0,\ \gamma > 0. \tag{19.27}$$

Laplace transform of $g_1(x)$ is the following:

$$L_{g_1}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\gamma)_k}{\Gamma(\alpha\gamma+\alpha k)} \int_0^{\infty} x^{\alpha\gamma+\alpha k-1} e^{-tx} dx = \sum_{k=0}^{\infty} (-1)^k \frac{(\gamma)_k}{k!} t^{-\alpha\gamma-\alpha k} = (1+t^{\alpha})^{-\gamma}, \quad |t^{\alpha}| < 1. \tag{19.28}$$

In fact, this is a special case of the general class of Laplace transforms connected with Mittag-Leffler function considered in Mathai et al. [45].

19.3. Mittag-Leffler Density as an H-Function

$g_1(x)$ of (19.27) can be written as a Mellin-Barnes integral and then as an H-function

$$g_1(x) = \frac{1}{\Gamma(\eta)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\eta-s)}{\Gamma(\alpha\eta-\alpha s)} x^{\alpha\eta-1} (x^{\alpha})^{-s} ds, \quad \Re(\eta) > 0,\ 0 < c < R(\eta) = \frac{x^{\alpha\eta-1}}{\Gamma(\eta)} H_{1,2}^{1,1}\left[x^{\alpha}\Big|_{(0,1),(1-\alpha\eta,\alpha)}^{(1-\eta,1)}\right] \tag{19.29}$$

$$= \frac{1}{\alpha\Gamma(\eta)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(\eta-1/\alpha+s/\alpha)\Gamma(1/\alpha-s/\alpha)}{\Gamma(1-s)} x^{-s} ds \tag{19.30}$$

(by taking $\alpha\eta - 1 - \alpha s = -s_1$)

$$= \frac{1}{\alpha\Gamma(\eta)} H_{1,2}^{1,1}\left[x\Big|_{(\eta-1/\alpha,1/\alpha),(0,1)}^{(1-1/\alpha,1/\alpha)}\right]. \tag{19.31}$$

Since g and g_1 are represented as inverse Mellin transforms, in the Mellin-Barnes representation, one can obtain the $(s-1)$th moments of g and g_1 from (19.30). That is,

$$M_{g_1}(s) = E\left(x^{s-1}\right) \quad \text{in } g_1 = \frac{1}{\Gamma(\gamma)} \frac{\Gamma(\eta-1/\alpha+s/\alpha)\Gamma(1/\alpha-s/\alpha)}{\alpha\Gamma(1-s)}, \tag{19.32}$$

for $1-\alpha < \Re(s) < 1$, $0 < \alpha \leq 1$, $\eta > 0$

$$\begin{aligned} M_g(t) &= E\left(x^{s-1}\right) \quad \text{in } g \\ &= \frac{\Gamma(1-1/\alpha+s/\alpha)\Gamma(1/\alpha-s/\alpha)}{\alpha\Gamma(1-s)} \end{aligned} \tag{19.33}$$

for $1-\alpha < \Re(s) < 1$, $0 < \alpha \leq 1$, obtained by putting $\eta = 1$ in (19.32) also. Since

$$\lim_{\alpha\to 1} \frac{\Gamma(1/\alpha-s/\alpha)}{\Gamma(1-s)} = 1 \tag{19.34}$$

for $\alpha \to 1$, (19.32) reduces to

$$M_{g_1}(t) = \frac{1}{\Gamma(\eta)}\Gamma(\eta-1+s) \quad \text{for } \alpha \longrightarrow 1. \tag{19.35}$$

Its inverse Mellin transform is then

$$g_1 = \frac{1}{\Gamma(\eta)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\eta-1+s)x^{-s}\mathrm{d}s = \frac{1}{\Gamma(\eta)}x^{\eta-1}\mathrm{e}^{-x}, \quad x \geq 0,\ \eta > 0 \tag{19.36}$$

which is the one-parameter gamma density and for $\eta = 1$ it reduces to the exponential density. Hence the generalized Mittag-Leffler density g_1 can be taken as an extension of a gamma density such as the one in (19.36) and the Mittag-Leffler density g as an extension of the exponential density for $\eta = 1$. Is there a structural representation for the random variable giving rise to the Laplace transform in (19.26) corresponding to (19.11)? The answer is in the affirmative and it is illustrated in (19.16).

Note 10. Pillai [91, Theorem 2.2], Lin [96, Lemma 3], and others list the ρth moment of the Mittag-Leffler density g in (19.3) as follows:

$$E(x^\rho) = \frac{\Gamma(1-\rho/\alpha)\Gamma(1+\rho/\alpha)}{\Gamma(1-\rho)}, \quad -\alpha < \Re(\rho) < \alpha < 1. \tag{19.37}$$

Therefore,

$$\begin{aligned} E\left(x^{s-1}\right) &= \frac{\Gamma(1+1/\alpha-s/\alpha)\Gamma(1-1/\alpha+s/\alpha)}{\Gamma(2-s)} \\ &= \frac{(1/\alpha-s/\alpha)\Gamma(1/\alpha-s/\alpha)\Gamma(1-1/\alpha+s/\alpha)}{(1-s)\Gamma(1-s)} = \frac{1}{\alpha}\frac{\Gamma(1/\alpha-s/\alpha)\Gamma(1-1/\alpha+s/\alpha)}{\Gamma(1-s)} \end{aligned} \tag{19.38}$$

which is the expression in (19.33). Hence the two expressions are one and the same.

Note 11. If $y = ax, a > 0$ and if x has a Mittag-Leffler distribution then the density of y can also be represented as a Mittag-Leffler function with the Laplace transform

$$L_x(t) = (1 + t^\alpha)^{-1} \Longrightarrow L_y(t) = \left(1 + (at)^\alpha\right)^{-1}, \quad a > 0, \ \left|(at)^\alpha\right| < 1. \tag{19.39}$$

Note 12. From the representation that

$$E\left(x^h\right) = \frac{\Gamma(1 - h/\alpha)\Gamma(1 + h/\alpha)}{\Gamma(1 - h)}, \quad -\alpha < \Re(h) < \alpha < 1 \tag{19.40}$$

we have

$$E\left(x^0\right) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(1)} = 1. \tag{19.41}$$

Further, $g(x)$ in (19.23) is a nonnegative function for all x, with

$$E\left(x^h\right) = \int_0^\infty x^h g(x)\mathrm{d}x = 1 \quad \text{for } h = 0. \tag{19.42}$$

Hence $g(x)$ is a density function for a positive random variable x. Note that from the series form for the Mittag-Leffler function it is not possible to show that $\int_0^\infty g(x)\mathrm{d}x = 1$ directly.

19.4. Structural Representation of the Generalized Mittag-Leffler Variable

Let u be the random variable corresponding to the Laplace transform (19.28) with t^α replaced by δt^α and γ by η. Let u be a positive Lévy variable with the Laplace transform $\mathrm{e}^{-t^\alpha}, 0 < \alpha \leq 1$ and let v be a gamma random variable with parameters η and δ or with he Laplace transform $(1 + \delta t)^{-\eta}, \eta > 0, \delta > 0$. Let u and v be statistically independently distributed.

Lemma 19.3. *Let u, v as defined above. Then*

$$w \sim uv^{1/\alpha}, \tag{19.43}$$

where w is a generalized Mittag-Leffler variable with Laplace transform $(1 + \delta t^\alpha)^{-\beta}$, $|\delta t^\alpha| < 1$, where $\sim$ means "distributed as" or both sides have the same distribution.

Proof. Denoting the Laplace transform of the density of w by $L_w(t)$ and treating it as an expected value

$$\begin{aligned} L_w(t) &= E\left[\mathrm{e}^{-tv^{1/\alpha}u}\right] = E\left[E\left[\mathrm{e}^{-(tv^{1/\alpha}u)} \mid v\right]\right] \\ &= E\left[\mathrm{e}^{-(tv^{1/\alpha})^\alpha}\right] = E\left[\mathrm{e}^{-t^\alpha v}\right] = (1 + \delta t^\alpha)^{-\eta} \end{aligned} \tag{19.44}$$

from the Laplace transform of a gamma variable. This establishes the result.

From the structural representation in (19.43), taking the Mellin transforms and writing as expected values, we have

$$E\left(w^{s-1}\right) = E\left(u^{s-1}\right)E\left(v^{1/\alpha}\right)^{s-1} \tag{19.45}$$

due to statistical independence of u and v. The left side is available from (19.32) as

$$E\left(w^{s-1}\right) = \frac{\Gamma(\eta - 1/\alpha + s/\alpha)\Gamma(1/\alpha - s/\alpha)\delta^{(s-1)/\alpha}}{\alpha\Gamma(\eta)\Gamma(1-s)}. \tag{19.46}$$

Let us compute $E[v^{1/\alpha}]^{s-1}$ from the gamma density. That is,

$$E\left[v^{1/\alpha}\right]^{s-1} = \frac{1}{\delta^{\eta}\Gamma(\eta)}\int_0^{\infty}\left(v^{1/\alpha}\right)^{s-1}v^{\eta-1}\mathrm{e}^{-v/\delta}\mathrm{d}v = \frac{\Gamma(\eta - 1/\alpha + s/\alpha)\delta^{(s-1)/\alpha}}{\Gamma(\eta)} \tag{19.47}$$

for $\Re(s) > 1 - \alpha\eta, 0 < \alpha \leq 1, \eta > 0$. Comparing (19.46) and (19.47) we have the (s − 1)th moment of a Lévy variable

$$E\left[u^{s-1}\right] = \frac{\Gamma(1/\alpha - s/\alpha)}{\alpha\Gamma(1-s)} = \frac{\Gamma(1 + 1/\alpha - s/\alpha)}{\Gamma(2-s)}, \quad \Re(s) < 1,\ 0 < \alpha \leq 1. \tag{19.48}$$

□

Note 13. Lin [96] gives the ρth moment of a Lévy variable with parameter α as

$$E[u^{\rho}] = \frac{\Gamma(1-\rho/\alpha)}{\Gamma(1-\rho)}. \tag{19.49}$$

Hence for $\rho = s - 1$ we have

$$\begin{aligned} E\left[u^{s-1}\right] &= \frac{\Gamma(1 + 1/\alpha - s/\alpha)}{\Gamma(2-s)} = \frac{(1/\alpha - s/\alpha)\Gamma(1/\alpha - s/\alpha)}{(1-s)\Gamma(1-s)} \\ &= \frac{\Gamma(1/\alpha - s/\alpha)}{\alpha\Gamma(1-s)} \quad \text{for } \Re(s) < 1. \end{aligned} \tag{19.50}$$

This is (19.48), and hence both the representations are one and the same.

Hence the Lévy density, denoted by $g_2(u)$, can be written as

$$g_2(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(1/\alpha - s/\alpha)}{\alpha\Gamma(1-s)}u^{-s}\mathrm{d}s. \tag{19.51}$$

Its Laplace transform is then

$$\begin{aligned} L_{g_2}(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1/\alpha - s/\alpha)}{\alpha\Gamma(1-s)}\left[\int_0^\infty u^{1-s-1}\mathrm{e}^{-tu}\mathrm{d}u\right]\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)t^{-1+s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{1}{\alpha}\Gamma\left(\frac{s}{\alpha}\right)t^{-s}\mathrm{d}s \end{aligned} \tag{19.52}$$

by making the substitution $-1+s=-s_1$. Then evaluating as the sum of the residues at $s/\alpha = -\nu$, $\nu = 0,1,2,\ldots$ we have

$$L_{g_2}(t) = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}t^{\alpha\nu} = \mathrm{e}^{-t^\alpha}. \tag{19.53}$$

This verifies the result about the Laplace transform of the positive Lévy variable with parameter α. Note that when $\alpha = 1$, (19.53) gives the Laplace transform of a degenerate random variable taking the value 1 with probability 1.

Mellin convolution of certain Lévy variables can be seen to be again a Lévy variable.

Lemma 19.4. *Let x_j be a positive Lévy variable with parameter α_j, $0<\alpha_j\le 1$ and let $x_1,\ldots,x_p$ be statistically independently distributed. Then*

$$u = x_1 x_2^{1/\alpha_1}\cdots x_p^{1/\alpha_1\alpha_2\cdots\alpha_{p-1}} \tag{19.54}$$

is distributed as a Lévy variable with parameter $\alpha_1\alpha_2\cdots\alpha_p$.

Proof. From (19.48)

$$\begin{aligned} E\left[\mathrm{e}^{-tx_j}\right] &= \mathrm{e}^{-t^{\alpha_j}}, \quad 0<\alpha_j\le 1,\ j=1,\ldots,p \\ E\left[\mathrm{e}^{-tu}\right] &= E\left[\mathrm{e}^{-tx_1x_2^{1/\alpha_1}\cdots x_p^{1/\alpha_1\cdots\alpha_{p-1}}}\right] = E\left[E\left[\mathrm{e}^{-tx_1\cdots x_p^{1/\alpha_1\cdots\alpha_{p-1}}}\Big|_{x_2,\ldots,x_p}\right]\right] \\ &= E\left[\mathrm{e}^{-t^{\alpha_1}x_2x_3^{1/\alpha_2}\cdots x_p^{1/\alpha_2\cdots\alpha_{p-1}}}\right]. \end{aligned} \tag{19.55}$$

Repeated application of the conditional argument gives the final result as

$$E\left[\mathrm{e}^{-tu}\right] = \mathrm{e}^{-t^\alpha}, \quad \alpha = \alpha_1\alpha_2\cdots\alpha_p, \quad 0<\alpha_1\cdots\alpha_p\le 1 \tag{19.56}$$

which means that u is distributed as a Lévy with parameter $\alpha_1\cdots\alpha_p$.

From the representation in (19.43) we can compute the moments of the natural logarithms of Mittag-Leffler, Lévy and gamma variables

$$w = uv^{1/\alpha} \Longrightarrow \ln w = \ln u + \frac{1}{\alpha}\ln v. \tag{19.57}$$

But from (19.46) and (19.49) we have the hth moments of u and v given by

$$\begin{aligned} E\left[u^h\right] &= \frac{\Gamma(1-h/\alpha)}{\Gamma(1-h)}, \quad \Re(h) < \alpha \le 1, \\ E\left[v^{1/\alpha}\right]^h &= \frac{\Gamma(\eta + h/\alpha)\delta^{h/\alpha}}{\Gamma(\eta)}, \quad \Re\left(\eta + \frac{h}{\alpha}\right) > 0. \end{aligned} \tag{19.58}$$

From (19.46)

$$E\left[w^h\right] = \frac{\Gamma(\eta + h/\alpha)\Gamma(1-h/\alpha)\delta^{h/\alpha}}{\Gamma(\eta)\Gamma(1-h)}. \tag{19.59}$$

But for a positive random variable z

$$E\left[z^h\right] = E\left[\mathrm{e}^{\ln z^h}\right]. \tag{19.60}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}h} E\left[z^h\right]\Big|_{h=0} = E\left[\ln z \mathrm{e}^{h\ln z}\right]_{h=0} = E[\ln z]. \tag{19.61}$$

Therefore from (19.57) to (19.59) we have the following:

$$\begin{aligned} E[\ln w] &= \frac{\mathrm{d}}{\mathrm{d}h}\left\{\frac{\delta^{h/\alpha}\Gamma(\eta + h/\alpha)\Gamma(1-h/\alpha)}{\Gamma(\eta)\Gamma(1-h)}\right\}\Bigg|_{h=0} \\ &= \frac{1}{\alpha}\psi(\eta) - \frac{1}{\alpha}\psi(1) + \psi(1) + \frac{1}{\alpha}\ln\delta \end{aligned} \tag{19.62}$$

by taking the logarithmic derivative, where ψ is a psi function, see, for example, Mathai [95]

$$E\left[\ln v^{1/\alpha}\right] = \frac{\mathrm{d}}{\mathrm{d}h}\left\{\frac{\delta^{h/\alpha}\Gamma(\eta + h/\alpha)}{\Gamma(\eta)}\right\}\Bigg|_{h=0} = \frac{1}{\alpha}\psi(\eta) + \frac{1}{\alpha}\ln\delta \tag{19.63}$$

or

$$\begin{aligned} E[\ln v] &= \psi(\eta) + \ln\delta, \\ E[\ln u] &= \frac{\mathrm{d}}{\mathrm{d}h}\left\{\frac{\Gamma(1-h/\alpha)}{\Gamma(1-h)}\right\}\Bigg|_{h=0} = -\frac{1}{\alpha}\psi(1) + \psi(1), \end{aligned} \tag{19.64}$$

where $\psi(1) = -\gamma$ where γ is the Euler's constant. □

Note 14. The relations on the expected values of the logarithms of Mittag-Leffler variable, positive Lévy variable and exponential variable, given in Jayakumar [93, page 1432], where $\eta = 1$, are not correct.

19.5. A Pathway from Mittag-Leffler Distribution to Positive Lévy Distribution

Consider the function

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (\eta)_k}{k!\Gamma(\alpha\eta + \alpha k)} \frac{x^{\alpha\eta-1+\alpha k}}{(a^{1/\alpha})^{\alpha\eta+\alpha k}}, \quad \eta > 0,\ a > 0,\ 0 < \alpha \le 1 \\ &= \frac{x^{\alpha\eta-1}}{a^{\eta}} E_{\alpha,\alpha\eta}^{\eta}\left(-\frac{x^{\alpha}}{a}\right). \end{aligned} \tag{19.65}$$

Thus $x = a^{1/\alpha} y$ where y is a generalized Mittag-Leffler variable. The Laplace transform of f is given by the following:

$$\begin{aligned} L_f(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k (\eta)_k}{k! a^{\eta+k}} \int_0^{\infty} \frac{x^{\alpha\eta+\alpha k-1} \mathrm{e}^{-tx}}{\Gamma(\alpha\eta + \alpha k)} \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\eta)_k}{k! a^{\eta+k}} t^{-\alpha\eta-\alpha k} = [1 + a t^{\alpha}]^{-\eta}, \quad |a t^{\alpha}| < 1. \end{aligned} \tag{19.66}$$

If η is replaced by $\eta/(q-1)$ and a by $a(q-1)$ with $q > 1$, then we have a Laplace transform

$$L_f(t) = \left[1 + a(q-1)t^{\alpha}\right]^{-\eta/(q-1)}, \quad q > 1. \tag{19.67}$$

If $q \to 1_+$ then

$$L_f(t) \longrightarrow \mathrm{e}^{-a\eta t^{\alpha}} = L_{f_1}(t) \tag{19.68}$$

which is the Laplace transform of a constant multiple of a positive Lévy variable with parameter α. Thus q here creates a pathway of going from the general Mittag-Leffler density f to a positive Lévy density f_1 with parameter α, the multiplying constant being $(a\eta)^{1/\alpha}$. For a discussion of a general rectangular matrix-variate pathway model see Mathai [97]. The result in (19.68) can be put in a more general setting. Consider an arbitrary real random variable y with the Laplace transform, denoted by $L_y(t)$, and given by

$$L_y(t) = \mathrm{e}^{-\phi(t)}, \tag{19.69}$$

where $\phi(t)$ is a function such that $\phi(tx^{\gamma}) = x\phi(t)$, $\phi(t) \ge 0$, $\lim_{t\to 0}\phi(t) = 0$ for some real positive γ. Let

$$u = y x^{\gamma}, \tag{19.70}$$

where x and y are independently distributed with y having the Laplace transform in (19.69) and x having a two-parameter gamma density with shape parameter β and scale parameter δ or with the Laplace transform

$$L_x(t) = (1+\delta t)^{-\beta}. \tag{19.71}$$

Now consider the Laplace transform of u in (19.70), denoted by $L_u(t)$. Then,

$$\begin{aligned} L_u(t) &= E\left[e^{-tu}\right] = E\left[e^{-tyx^\gamma}\right] = E\left[E\left[e^{tyx^\gamma} \mid x\right]\right] \\ &= E\left[e^{-\phi(tx^\gamma)}\right] = E\left[e^{-x\phi(t)}\right] \end{aligned} \tag{19.72}$$

from the assumed property of $\phi(t)$

$$= \left[1+\delta\phi(t)\right]^{-\beta}. \tag{19.73}$$

If δ is replaced by $\delta(q-1)$ and β by $\beta/(q-1)$ with $q>1$ then we get a path through q. That is, when $q \to 1_+$,

$$L_u(t) = \left[1+\delta(q-1)\phi(t)\right]^{-\beta/(q-1)} \longrightarrow e^{-\delta\beta\phi(t)} = e^{-\phi(\delta^\gamma\beta^\gamma t)}. \tag{19.74}$$

If $\phi(t) = t^\alpha$, $0<\alpha\le 1$, then

$$L_u(t) = e^{-(\delta\beta)^{\gamma\alpha}t^\alpha} \tag{19.75}$$

which means that u goes to a constant multiple of a positive Lévy variable with parameter α, the constant being $(\delta\beta)^\gamma$.

19.6. Linnik or α-Laplace Distribution

A Linnik random variable is defined as that real scalar random variable whose characteristic function is given by

$$\phi(t) = \frac{1}{1+|t|^\alpha}, \quad 0<\alpha\le 2,\ -\infty<t<\infty. \tag{19.76}$$

For $\alpha = 2$, (19.76) corresponds to the characteristic function of a Laplace random variable and hence Pillai and Jayakuma [98] called the distribution corresponding to (19.76) as the α-Laplace distribution. For positive variable, (19.76) reduces to the characteristic function of a Mittag-Leffler variable. Infinite divisibility, characterizations, other properties and related materials may be seen from the review paper Jayakumar and Suresh [94] and the many references therein, Pakes [99] and Mainardi and Pagnini [100]. Multivariate generalization of Mittag-Leffler and Linnik distributions may be seen from Lim and Teo [101]. Since the steps for deriving results on Linnik distribution are parallel to those of the Mittag-Leffler variable, further discussion of Linnik distribution is omitted.

19.7. Multivariable Generalization of Mittag-Leffler, Linnik, and Lévy Distributions

A multivariate Linnik distribution can be defined in terms of a multivariate Lévy vector. Let $T' = (t_1, \ldots, t_p)$, $X' = (x_1, \ldots, x_p)$, prime denoting the transpose. A vector variable having positive Lévy distribution is given by the characteristic function

$$E\left[e^{iT'X}\right] = e^{-(T'\Sigma T)^{\alpha/2}}, \quad 0 < \alpha \leq 2, \tag{19.77}$$

where $\Sigma = \Sigma' > 0$ is a real positive definite $p \times p$ matrix. Consider the representation

$$u = y^{1/\alpha} X, \tag{19.78}$$

where the $p \times 1$ vector X, having a multivariable Lévy distribution with parameter α, and y a real scalar gamma random variable with the parameters δ and β, are independently distributed. Then the characteristic function of the random vector variable u is given by the following:

$$\begin{aligned} E\left[e^{iy^{1/\alpha}T'X}\right] &= E\left[E\left[e^{iy^{1/\alpha}T'X}\right]\Big|_y\right] \\ &= E\left[e^{-y[|T'\Sigma T|]^{\alpha/2}}\right] = \left[1 + \delta\left|T'\Sigma T\right|^{\alpha/2}\right]^{-\beta}. \end{aligned} \tag{19.79}$$

Then the distribution of u, with the characteristic function in (19.79) is called a vector-variable Linnik distribution. Some properties of this distribution are given in Lim and Teo [101].

20. Mittag-Leffler Stochastic Processes

The stochastic process $\{x(t), t > 0\}$ having stationary independent increment with $x(0) = 0$ and $x(1)$ having the Laplace transform

$$L_{x(1)}(\lambda) = (1 + \lambda^\alpha)^{-1}, \quad 0 < \alpha \leq 1, \ \lambda > 0, \tag{20.1}$$

which is the Laplace transform of a Mittag-Leffler random variable, is called the Mittag-Leffler stochastic process. Then the Laplace transform of $x(t)$, denoted by $L_{x(t)}(\lambda)$, is given by

$$L_{x(t)}(\lambda) = \left[(1 + \lambda^\alpha)^{-1}\right]^t = [1 + \lambda^\alpha]^{-t}. \tag{20.2}$$

The density corresponding to the Laplace transform (20.2) or the density of $x(t)$ is then available as the following:

$$\begin{aligned} f_{x(t)}(x) &= \sum_{k=0}^{\infty}(-1)^k \frac{(t)_k}{k!} \frac{x^{\alpha k+\alpha t-1}}{\Gamma(\alpha k+\alpha t)} \\ &= x^{\alpha t-1} E_{\alpha,\alpha t}^{t}(-x^{\alpha}), \quad 0<\alpha\le 1,\ x\ge 0,\ t>0. \end{aligned} \tag{20.3}$$

The distribution function of $x(t)$ is given by

$$\begin{aligned} F_{x(t)}(x) &= \int_0^x f_{x(t)}(y)\,\mathrm{d}y = \sum_{k=0}^{\infty}(-1)^k \frac{(t)_k}{k!} \frac{x^{\alpha k+\alpha t}}{(\alpha k+\alpha t)\Gamma(\alpha k+\alpha t)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(t+k)}{\Gamma(t)} \frac{x^{\alpha k+\alpha t}}{\Gamma(1+\alpha k+\alpha t)}, \quad 0<\alpha\le 1,\ t>0. \end{aligned} \tag{20.4}$$

This form is given by Pillai [91] and by his students.

20.1. Linear First-Order Autoregressive Processes

Consider the stochastic process

$$x_n = \begin{cases} e_n, & \text{with probability } p,\ 0\le p\le 1, \\ e_n + a x_{n-1} & \text{with probability } 1-p,\ 0<a\le 1. \end{cases} \tag{20.5}$$

Let the sequence $\{e_n\}$ be independently and identically distributed with Laplace transform $L_e(\lambda)$, and let $\{x_n\}$ be identically distributed with Laplace transform $L_x(\lambda)$. From the representation in (20.5)

$$L_{x_n}(\lambda) = pL_e(\lambda) + (1-p)L_e(\lambda)L_{x_{n-1}}(a\lambda). \tag{20.6}$$

Therefore,

$$L_e(\lambda) = \frac{L_{x_n}(\lambda)}{p+(1-p)L_{x_{n-1}}(a\lambda)} = \frac{L_x(\lambda)}{p+(1-p)L_x(a\lambda)} \tag{20.7}$$

assuming stationarity. When $p=0$,

$$L_e(\lambda) = \frac{L_x(\lambda)}{L_x(a\lambda)}, \quad 0<a\le 1 \tag{20.8}$$

which defines class L distributions, for all a, $0<a<1$. When $p=0$, (20.8) implies that the innovation sequence $\{e_n\}$ belongs to class L distributions. Then (20.8) can lead to two autoregressive situations, the first-order exponential autoregressive process EAR(1) and the first-order Mittag-Leffler autoregressive process MLAR(1).

Concluding Remarks

The various Mittag-Leffler functions discussed in this paper will be useful for investigators in various disciplines of applied sciences and engineering. The importance of Mittag-Leffler function in physics is steadily increasing. It is simply said that deviations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Leffler functions (power-law). Currently more and more such phenomena are discovered and studied.

It is particularly important for the disciplines of stochastic systems, dynamical systems theory, and disordered systems. Eventually, it is believed that all these new research results will lead to the discovery of truly nonequilibrium statistical mechanics. This is statistical mechanics beyond Boltzmann and Gibbs. This nonequilibrium statistical mechanics will focus on entropy production, reaction, diffusion, reaction-diffusion, and so forth, and may be governed by fractional calculus.

Right now, fractional calculus and H-function (Mittag-Leffler function) are very important in research in physics.

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