Research Article

Some New Fixed-Point Theorems for a (ψ, ϕ) -Pair Meir-Keeler-Type Set-Valued Contraction Map in Complete Metric Spaces

Chi-Ming Chen

Department of Applied Mathematics, National Hsinchu University of Education, Hsinchu, Taiwan

Correspondence should be addressed to Chi-Ming Chen, ming@mail.nhcue.edu.tw

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We obtain some new fixed point theorems for a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map in metric spaces. Our main results generalize and improve the results of Klim and Wardowski, (2007).

1. Introduction and Preliminaries

Let (X, d) be a metric space, Y a subset of X , and $f : Y \rightarrow X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that, for all $x, y \in Y$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \quad (1.1)$$

The well-known Banach's fixed-point theorem asserts that if $Y = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X . It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \rightarrow X$ is called a quasi-contraction if there exists $k < 1$ such that

$$d(fx, fy) \leq k\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \quad (1.2)$$

for any $x, y \in X$. In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed-point theorem.

Throughout this paper, by $\mathbb{R}$ we denote the set of all real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space. Let $C(X)$ denote a collection of all nonempty closed subsets of X and $CB(X)$ a collection of all nonempty closed and bounded subsets of X .

The existence of fixed points for various multivalued contractive mappings had been studied by many authors under different conditions. In 1969, Nadler Jr. [3] extended the famous Banach contraction principle from single-valued mapping to multivalued mapping and proved the below fixed-point theorem for multivalued contraction.

Theorem 1.1 (see [3]). *Let (X, d) be a complete metric space, and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \leq cd(x, y) \quad \forall x, y \in X, \quad (1.3)$$

where $\mathcal{H}$ denotes the Hausdorff metric on $CB(X)$ induced by d ; that is, $H(A, B) = \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\}$, for all $A, B \in CB(X)$ and $D(x, B) = \inf_{z \in B} d(x, z)$. Then T has a fixed point in X .

In 1989, Mizoguchi-Takahashi [4] proved the following fixed-point theorem.

Theorem 1.2 (see [4]). *Let (X, d) be a complete metric space, and let T be a map from X into $CB(X)$. Assume that*

$$\mathcal{H}(Tx, Ty) \leq \xi(d(x, y)) \cdot d(x, y), \quad (1.4)$$

for all $x, y \in X$, where $\xi : [0, \infty) \rightarrow [0, 1)$ satisfies $\limsup_{s \rightarrow t^+} \xi(s) < 1$ for all $t \in [0, \infty)$. Then T has a fixed point in X .

In 2006, Feng and Liu [5] gave the following theorem.

Theorem 1.3 (see [5]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow C(X)$ be a multivalued map. If there exist $b, c \in (0, 1)$, $c < b$ such that for any $x \in X$, there is $y \in T(x)$ satisfying the following two conditions:*

$$(i) \quad b \cdot d(x, y) \leq D(x, Tx),$$

$$(ii) \quad D(x, Ty) \leq c \cdot d(x, y).$$

Then T has a fixed point in X provided that the mapping $f : X \rightarrow \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous; that is, if for any $\{x_n\} \subset X$ and $x \in X$, $x_n \rightarrow x$, then $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$.

In 2007, Klim and Wardowski [6] proved the following fixed point theorem.

Theorem 1.4 (see [6]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow C(X)$ be a multivalued map. Assume that the following conditions hold:*

$$(i) \quad \text{the mapping } f : X \rightarrow \mathbb{R} \text{ defined by } f(x) = D(x, Tx), \quad x \in X, \text{ is lower semicontinuous;}$$

(ii) there exist $b \in (0, 1)$ and $\varphi : [0, \infty) \rightarrow [0, b)$ such that

$$\begin{aligned} & \forall_{t \in [0, \infty)} \quad \left\{ \lim_{r \rightarrow t^+} \sup \varphi(r) < b \right\}, \\ & \forall_{x \in X} \quad \exists_{y \in Tx} \quad \{bd(x, y) \leq D(x, Tx) \wedge D(y, Ty) \leq \varphi(d(x, y)) \cdot d(x, y)\}. \end{aligned} \quad (1.5)$$

Then T has a fixed point in X .

Recently, Pathak and Shahzad [7] introduced a new class of mapping $\Theta[0, A)$ and generalized the results of Klim and Wardowski [6]. Suppose that $A \in (0, \infty]$, $\Theta[0, A)$ denote the class of functions $\theta : [0, A) \rightarrow \mathbb{R}$ satisfying the following conditions:

- (1) θ is nondecreasing on $[0, A)$;
- (2) $\theta(t) > 0$ for all $t \in (0, A)$;
- (3) θ is subadditive in $(0, A)$; that is, $\theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2)$.

The following theorem was introduced in Pathak and Shahzad [7].

Theorem 1.5 (see [7]). *Let (X, d) be a complete metric space and suppose that $T : X \rightarrow C(X)$. Assume that the following conditions hold:*

- (i) the mapping $f : X \rightarrow \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous,
- (ii) there exists $\alpha : (0, \infty) \rightarrow (0, 1)$ such that

$$\forall_{t \in [0, \infty)} \quad \left\{ \lim_{r \rightarrow t^+} \sup \alpha(r) < 1 \right\}, \quad (1.6)$$

- (iii) there exists $\theta \in \Theta[0, A)$ satisfying the following condition:

$$\begin{aligned} & \forall_{x \in X} \quad \exists_{y \in Tx} \quad \{\theta(d(x, y)) \leq \theta(D(x, Tx))\}, \\ & \forall_{x \in X} \quad \exists_{y \in Tx} \quad \{\theta(D(y, Ty)) \leq \alpha(d(x, y)) \cdot \theta(d(x, y))\}. \end{aligned} \quad (1.7)$$

Then T has a fixed point in X .

Later, Kamran and Kiran [8] improved some results of Pathak and Shahzad [7] by allowing T to have values in closed subsets of X . They proved that the function $\theta \in \Theta[0, A)$ is positive homogenous in $[0, A)$, that is,

- (4) $\theta(at) \leq a\theta(t)$ for all $a > 0$, $t \in [0, A)$,

and denote by $\Theta_h[0, A)$ the class of functions $\theta \in \Theta[0, A)$ satisfying condition (4). They proved the following theorem.

Theorem 1.6 (see [8]). *Let (X, d) be a complete metric space and suppose that α is a function from $(0, \infty)$ to $[0, 1)$ such that*

$$\forall_{t \in [0, \infty)} \quad \left\{ \lim_{r \rightarrow t^+} \sup \alpha(r) < 1 \right\}. \quad (1.8)$$

Suppose that $T : X \rightarrow C(X)$. Assume that the following condition holds:

$$\theta(D(y, Ty)) \leq \alpha(d(x, y)) \cdot \theta(d(x, y)), \quad \text{for each } x \in X, y \in Tx, \quad (1.9)$$

where $\theta \in \Theta_n[0, A)$. Then

- (i) for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_{n \rightarrow \infty} x_n = \xi$;
- (ii) ξ is a fixed point of T if and only if the function $f(x) = D(x, Tx)$ is T -orbitally lower semicontinuous at ξ .

2. Main Results

In this section, we first recall the notion of the Meir-Keeler-type function (see [9]). A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is said to be a Meir-Keeler-type function, if, for each $\eta \in [0, \infty)$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$. We now define a new stronger Meir-Keeler-type function, as follows.

Definition 2.1. One calls $\psi : [0, \infty) \rightarrow [0, 1)$ the stronger Meir-Keeler-type function, if, for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \leq t < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(t) < \gamma_\eta$.

Remark 2.2. It is clear that, if the function $\xi : [0, \infty) \rightarrow [0, 1)$ satisfies

$$\limsup_{s \rightarrow t^+} \xi(s) < 1 \quad (2.1)$$

for all $t \in [0, \infty)$, then ξ is also a stronger Meir-Keeler-type function.

Example 2.3. (1) If $\psi : [0, \infty) \rightarrow [0, 1)$, $\psi(t) = k$ with $k \in (0, 1)$, then ψ is a stronger Meir-Keeler-type function.

(2) If $\psi : [0, \infty) \rightarrow [0, 1)$, $\psi(t) = t/(t+1)$, then ψ is a stronger Meir-Keeler-type function.

Definition 2.4. Let $\psi : [0, \infty) \rightarrow [0, 1)$, $\phi : [0, \infty) \rightarrow [b, 1)$ be two functions where $0 < b < 1$. Then the mappings ψ, ϕ are called a (ψ, ϕ) -pair Meir-Keeler-type function, if, for each $\eta > 0$, there exists $\delta > 0$ such that, for $t \in [0, \infty)$ with $\eta \leq t < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(t)/\phi(t) < \gamma_\eta$.

Remark 2.5. It is clear that if the functions $\psi : [0, \infty) \rightarrow [0, 1)$, $\phi : [0, \infty) \rightarrow [b, 1)$ satisfy

$$\limsup_{s \rightarrow t^+} \frac{\psi(s)}{\phi(s)} < 1, \quad (2.2)$$

for all $t \in [0, \infty)$, then ψ, ϕ are also a (ψ, ϕ) -pair Meir-Keeler-type function.

Example 2.6. If $\psi : [0, \infty) \rightarrow [0, 1)$, $\psi(t) = t/(4t + 1)$ and $\phi : [0, \infty) \rightarrow [0, 1)$, $\phi(t) = t/(3t + 1)$, then ψ, ϕ are a (ψ, ϕ) -pair Meir-Keeler-type function.

Definition 2.7. Let (X, d) be a metric space, let $\psi : [0, \infty) \rightarrow [0, 1)$, $\phi : [0, \infty) \rightarrow [b, 1)$ be two functions where $0 < b < 1$, and let $T : X \rightarrow 2^X$ be a set-valued map. Then T is called a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map, if the following conditions hold:

(C1) for each $\eta > 0$, there exists $\delta > 0$ such that for $x \in X$ with $\eta \leq D(x, Tx) < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that

$$\frac{\psi(D(x, Tx))}{\phi(D(x, Tx))} < \gamma_\eta, \quad (2.3)$$

(C2) for all $x \in X$, there exists $y \in Tx$ such that

$$\begin{aligned} \phi(D(x, Tx)) \cdot d(x, y) &\leq D(x, Tx), \\ D(y, Ty) &\leq \psi(D(x, Tx)) \cdot d(x, y). \end{aligned} \quad (2.4)$$

In this paper, we obtain some new fixed-point theorems for a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map in metric spaces. Our main results generalize and improve the results of Klim and Wardowski [6]. We now state our main theorem as follows.

Theorem 2.8. *Let (X, d) be a complete metric space, and let $T : X \rightarrow C(X)$ be a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map. Then T has a fixed point in X provided the mapping $f : X \rightarrow \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous.*

Proof. Given $x_0 \in X$ and by (C2), there exists $x_1 \in X$ such that $x_1 \in Tx_0$. Since T is a (ψ, ϕ) -pair Meir-Keeler type set-valued contraction map, there exists $x_1 \in Tx_0$ such that

$$\begin{aligned} \phi(D(x_0, Tx_0)) \cdot d(x_0, x_1) &\leq D(x_0, Tx_0), \\ D(x_1, Tx_1) &\leq \psi(D(x_0, Tx_0)) \cdot d(x_0, x_1). \end{aligned} \quad (2.5)$$

Continuing this process, we can choose a sequence $\{x_n\} \subset X$ with $x_{n+1} \in Tx_n$ such that, for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \phi(D(x_n, Tx_n)) \cdot d(x_n, x_{n+1}) &\leq D(x_n, Tx_n), \\ D(x_{n+1}, Tx_{n+1}) &\leq \psi(D(x_n, Tx_n)) \cdot d(x_n, x_{n+1}). \end{aligned} \quad (2.6)$$

Therefore, we can deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} D(x_{n+1}, Tx_{n+1}) &\leq \frac{\psi(D(x_n, Tx_n))}{\phi(D(x_n, Tx_n))} \cdot D(x_n, Tx_n) \\ &< D(x_n, Tx_n). \end{aligned} \quad (2.7)$$

Thus, the sequence $\{D(x_n, Tx_n)\}_{n=0}^\infty$ is decreasing and bounded below. Then there exists $\eta \geq 0$ such that

$$\lim_{n \rightarrow \infty} D(x_n, Tx_n) = \eta. \quad (2.8)$$

Hence, there exists $\kappa_0 \in \mathbb{N}$ and $\delta > 0$ such that, for all $n \geq \kappa_0$,

$$\eta \leq D(x_n, Tx_n) < \eta + \delta. \quad (2.9)$$

By the condition (C1), we have that there exists $\gamma_\eta \in [0, 1)$ such that

$$\frac{\psi(D(x_n, Tx_n))}{\phi(D(x_n, Tx_n))} < \gamma_\eta, \quad \forall n \geq \kappa_0. \quad (2.10)$$

So for each $n \in \mathbb{N}$ with $n \geq \kappa_0$, by (2.6), we can deduce that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\psi(D(x_n, Tx_n))}{\phi(D(x_n, Tx_n))} \\ &\leq \frac{\psi(D(x_{n-1}, Tx_{n-1}))}{\phi(D(x_n, Tx_n))} \cdot d(x_{n-1}, x_n) \\ &\leq \frac{\psi(D(x_{n-1}, Tx_{n-1}))}{\phi(D(x_n, Tx_n))} \cdot \frac{D(x_{n-1}, Tx_{n-1})}{\phi(D(x_{n-1}, Tx_{n-1}))} \\ &\leq \frac{1}{b} \cdot \frac{\psi(D(x_{n-1}, Tx_{n-1}))}{\phi(D(x_{n-1}, Tx_{n-1}))} \cdot D(x_{n-1}, Tx_{n-1}) \\ &\leq \frac{1}{b} \cdot \gamma_\eta \cdot D(x_{n-1}, Tx_{n-1}) \\ &\leq \frac{1}{b} \cdot \gamma_\eta^2 \cdot D(x_{n-2}, Tx_{n-2}) \\ &\vdots \\ &\leq \frac{1}{b} \cdot \gamma_\eta^{n-\kappa_0} \cdot D(x_{\kappa_0}, Tx_{\kappa_0}). \end{aligned} \quad (2.11)$$

Take $m, n \in \mathbb{N}$ with $m > n > \kappa_0$. Then we get

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \frac{1}{b} \cdot \frac{\gamma_\eta^{n-\kappa_0} \cdot D(x_{\kappa_0}, Tx_{\kappa_0})}{1 - \gamma_\eta}, \quad (2.12)$$

and so we conclude that

$$d(x_n, x_m) \longrightarrow 0, \quad \text{as } m, n \longrightarrow \infty, \quad (2.13)$$

since $0 \leq \gamma_\eta < 1$. Thus, $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in X . Since X is complete, there exists $\mu \in X$ such that $x_n \rightarrow \mu$ as $n \rightarrow \infty$.

Since $f : X \rightarrow \mathbb{R}$, $f(x) = d(x, Tx)$, $x \in X$, is lower semicontinuous, we have

$$0 \leq d(\mu, T\mu) = f(\mu) \leq \liminf_{\infty} d(x_n, Tx_n) = 0. \quad (2.14)$$

The closeness of $T\mu$ implies $\mu \in T\mu$. □

The following is a simple example for Theorem 2.8, and it generalize the result of Klim and Wardowski [6].

Example 2.9. Let $X = [0, 1]$ be a metric space with the standard metric d . Let $T : X \rightarrow C(X)$ be defined by

$$T(x) = \left\{ \frac{1}{3}x^2 \right\}, \quad \forall x \in X. \quad (2.15)$$

Let $\psi : [0, \infty) \rightarrow [0, 1)$, $\phi : [0, \infty) \rightarrow [2/3, 1)$ be defined by

$$\psi(t) = \frac{4}{9} + \frac{1}{9(t+1)}, \quad \phi(t) = \frac{2}{3} + \frac{1}{9(t+1)}, \quad \forall t \in [0, \infty). \quad (2.16)$$

Then T is a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map, and $0 \in X$ is a fixed point of T .

In particular, if we let $\phi(t) = 2/3$, then this example satisfies all of the conditions of Theorem 1.4 (that was introduced in Klim and Wardowski [6]).

Using Example 3.1 in [6] and Example 1 in [10], we get the following another example for Theorem 2.8.

Example 2.10. Let $X = [0, 1]$ be a metric space with the standard metric d . Let $T : X \rightarrow C(X)$ be defined as in Example 3.1 of Klim and Wardowski [6]:

$$T(x) = \begin{cases} \left\{ \frac{1}{2}x^2 \right\}, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{ \frac{17}{96}, \frac{1}{4} \right\}, & \text{if } x = \frac{15}{32}. \end{cases} \quad (2.17)$$

Let $\psi : [0, \infty) \rightarrow [0, 1)$ be defined as in Example 1 of Ćirić [10]:

$$\psi(t) = \begin{cases} \max \left\{ \frac{1}{12}, \frac{23}{12}t \right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \left\{ \frac{23}{24} \right\}, & \text{if } t \in \left(\frac{1}{2}, \infty\right), \end{cases} \quad (2.18)$$

and let $\phi : [0, \infty) \rightarrow [1/12, 1)$ be defined by

$$\phi(t) = \sqrt{\psi(t)} = \begin{cases} \max\left\{\sqrt{\frac{1}{12}}, \sqrt{\frac{23t}{12}}\right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \sqrt{\frac{23}{24}}, & \text{if } t \in \left(\frac{1}{2}, \infty\right). \end{cases} \quad (2.19)$$

Clearly, a function

$$f(x) = D(x, Tx) = \begin{cases} x - \frac{1}{2}x^2, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \frac{7}{32}, & \text{if } x = \frac{15}{32}, \end{cases} \quad (2.20)$$

is lower semicontinuous. Then ϕ, ψ are a (ψ, ϕ) -pair Meir-Keeler-type function, and T is a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map. Moreover, by Theorem 2.8, we have that $0 \in X$ is a fixed point of T .

If we let $T : X \rightarrow C(X)$ be closed, then we also have the following fixed result.

Theorem 2.11. *Let (X, d) be a complete metric space, and let $T : X \rightarrow C(X)$ be a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map and closed. Then T has a fixed point in X .*

Proof. Following the proof of Theorem 2.8, we get that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X . Since X is complete, there exists $\mu \in X$ such that $x_n \rightarrow \mu$ as $n \rightarrow \infty$. Since T is closed and $x_{n+1} \in Tx_n$, we have that $\mu \in T\mu$. $\square$

The following is a simple example for Theorem 2.11.

Example 2.12. Let $X = [0, 1]$ be a metric space with the metric $d(x, y) := x$ for all $(x, y) \in X \times X$. Let $T : X \rightarrow C(X)$ be defined by

$$T(x) = \left\{ \frac{1}{4}x^2 \right\}, \quad \forall x \in X. \quad (2.21)$$

Let $\psi : [0, \infty) \rightarrow [0, 1)$, $\phi : [0, \infty) \rightarrow [1/4, 1)$ be defined by

$$\psi(t) = \frac{1}{4}, \quad \phi(t) = \frac{1}{2}, \quad \forall t \in [0, \infty). \quad (2.22)$$

Then T is a (ψ, ϕ) -pair Meir-Keeler-type set-valued contraction map and closed, and $0 \in X$ is a fixed point of T .

Applying Theorem 2.8 and Remark 2.5, we are easy to get the following result.

Theorem 2.13. Let (X, d) be a complete metric space, let $\psi : [0, \infty) \rightarrow [0, 1)$, $\phi : [0, \infty) \rightarrow [b, 1)$ be two functions where $0 < b < 1$, and let $T : X \rightarrow C(X)$ be a set-valued contraction map. Suppose the following conditions hold:

(1) for each $t \in [0, \infty)$,

$$\limsup_{s \rightarrow t^+} \frac{\psi(s)}{\phi(s)} < 1, \quad (2.23)$$

(2) for all $x \in X$, there exists $y \in Tx$ such that

$$\begin{aligned} \phi(D(x, Tx)) \cdot d(x, y) &\leq D(x, Tx), \\ D(y, Ty) &\leq \psi(D(x, Tx)) \cdot d(x, y). \end{aligned} \quad (2.24)$$

Then T has a fixed point in X provided the mapping $f : X \rightarrow \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous.

The following is a simple example for Theorem 2.13.

Example 2.14. Let $X = [0, 1]$ be a metric space with the metric d , $d(x, y) := x$ for all $(x, y) \in X \times Y$. Let $T : X \rightarrow C(X)$ be defined as in Example 3.1 of Klim and Wardowski [6]:

$$T(x) = \begin{cases} \left\{ \frac{1}{2}x^2 \right\}, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{ \frac{17}{96}, \frac{1}{4} \right\}, & \text{if } x = \frac{15}{32}. \end{cases} \quad (2.25)$$

Let $\psi : [0, \infty) \rightarrow [0, 1)$ be defined as in Example 1 of Ćirić [10]:

$$\psi(t) = \begin{cases} \max \left\{ \frac{1}{12}, \frac{23}{12}t \right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \left\{ \frac{23}{24} \right\}, & \text{if } t \in \left(\frac{1}{2}, \infty\right), \end{cases} \quad (2.26)$$

and let $\phi : [0, \infty) \rightarrow [1/12, 1)$ be defined by

$$\phi(t) = \sqrt{\psi(t)}. \quad (2.27)$$

Clearly, a function

$$f(x) = D(x, Tx) = x \quad (2.28)$$

is lower semicontinuous. Clearly, $\limsup_{s \rightarrow t^+} (\psi(s)/\phi(s)) < 1$. We also conclude the following.

Case 1. If $x \in [0, 15/32) \cup (15/32, 1]$, then $y = T(x) = (1/2)x^2$, and ψ, ϕ satisfy the condition (2) of Theorem 2.13.

Case 2. If $x = 15/32$, then $y = T(x) = 1/4$ (resp., $y = T(x) = 17/96$), and ψ, ϕ also satisfy the condition (2) of Theorem 2.13.

Thus, by Theorem 2.13, we have that $0 \in X$ is a fixed point of T .

Using Example 2.10, we also get the following example for Theorem 2.13.

Example 2.15. Let $X = [0, 1]$ be a metric space with the standard metric d . Let $T : X \rightarrow C(X)$ be defined as

$$T(x) = \begin{cases} \left\{ \frac{1}{2}x^2 \right\}, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{ \frac{17}{96}, \frac{1}{4} \right\}, & \text{if } x = \frac{15}{32}. \end{cases} \quad (2.29)$$

Let $\psi : [0, \infty) \rightarrow [0, 1)$ be defined as

$$\psi(t) = \begin{cases} \max\left\{ \frac{1}{12}, \frac{23}{12}t \right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \left\{ \frac{23}{24} \right\}, & \text{if } t \in \left(\frac{1}{2}, \infty\right), \end{cases} \quad (2.30)$$

and $\phi : [0, \infty) \rightarrow [1/12, 1)$ be defined by

$$\phi(t) = \sqrt{\psi(t)}. \quad (2.31)$$

Clearly, $\limsup_{s \rightarrow t^+} (\psi(s)/\phi(s)) < 1$, and ϕ, ψ satisfies all of the conditions of Theorem 2.13. So, we have that $0 \in X$ is a fixed point of T .

If we let the function $\phi : [0, \infty) \rightarrow [b, 1)$ be $\phi(t) = b$ for all $t \in [0, \infty)$ and let the function $\psi : [0, \infty) \rightarrow [0, b)$, $b \in (0, 1)$, be a stronger Meir-Keeler-type function; that is for if, for each $\eta > 0$, there exists $\delta > 0$ such that, for $t \in [0, \infty)$ with $\eta \leq t < \delta + \eta$, there exists $\gamma_\eta \in [0, b)$ such that $\psi(t) < \gamma_\eta$, then, by Theorem 2.8, it is easy to get the following theorem.

Theorem 2.16. Let (X, d) be a complete metric space, let $\psi : [0, \infty) \rightarrow [0, b)$, $b \in (0, 1)$ be a stronger Meir-Keeler-type function, and let $T : X \rightarrow C(X)$ be a set-valued contraction map. Suppose that, for all $x \in X$, there exists $y \in Tx$ such that

$$\begin{aligned} b \cdot d(x, y) &\leq D(x, Tx), \\ D(y, Ty) &\leq \psi(D(x, Tx)) \cdot d(x, y). \end{aligned} \quad (2.32)$$

Then T has a fixed point in X provided that the mapping $f : X \rightarrow \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous.

The following is a simple example for Theorem 2.16.

Example 2.17. Let $X = [0, 1]$ be a metric space with the standard metric d . Let $T : X \rightarrow C(X)$ be defined by

$$T(x) = \left\{ \frac{1}{3}x^2 \right\}, \quad \forall x \in X. \quad (2.33)$$

Let $\psi : [0, \infty) \rightarrow [0, 2/3)$ be defined by

$$\psi(t) = \frac{4}{9} + \frac{1}{9(t+1)}, \quad \forall t \in [0, \infty). \quad (2.34)$$

Then ψ a stronger Meir-Keeler-type function, and $0 \in X$ is a fixed point of T .

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