Research Article

Sharp Estimates of m -Linear p -Adic Hardy and Hardy-Littlewood-Pólya Operators

Qingyan Wu and Zunwei Fu

Department of Mathematics, Linyi University, Linyi 276005, China

Correspondence should be addressed to Zunwei Fu, lyfzw@tom.com

Received 16 April 2011; Accepted 12 May 2011

Academic Editor: Mark A. Petersen

Copyright © 2011 Q. Wu and Z. Fu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The sharp estimates of the m -linear p -adic Hardy and Hardy-Littlewood-Pólya operators on Lebesgue spaces with power weights are obtained in this paper.

1. Introduction

In recent years, p -adic numbers are widely used in theoretical and mathematical physics (cf. [1–8]), such as string theory, statistical mechanics, turbulence theory, quantum mechanics, and so forth.

For a prime number p , let $\mathbb{Q}_p$ be the field of p -adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p -adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; If any nonzero rational number x is represented as $x = p^\gamma(m/n)$, where m and n are integers which are not divisible by p and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}. \quad (1.1)$$

From the standard p -adic analysis [6], we see that any nonzero p -adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \quad (1.2)$$

where a_j are integers, $0 \leq a_j \leq p-1$, $a_0 \neq 0$. The series (1.2) converges in the p -adic norm because $|a_j p^j|_p = p^{-j}$. Denote by $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

The space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, \dots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \dots, n$. The p -adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \quad (1.3)$$

Denote by

$$B_Y(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^Y\}, \quad (1.4)$$

the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^Y , and

$$S_Y(a) := \{x \in \mathbb{Q}_p^n : |x - a|_p = p^Y\} = B_Y(a) \setminus B_{Y-1}(a). \quad (1.5)$$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1, \quad (1.6)$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_Y(a)|_H = p^{Yn}, \quad |S_Y(a)|_H = p^{Yn}(1 - p^{-n}), \quad (1.7)$$

for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the p -adic field, see [6] or [9].

The space $\underbrace{\mathbb{Q}_p^n \times \mathbb{Q}_p^n \times \dots \times \mathbb{Q}_p^n}_m$ consists of points $(y_1, y_2, \dots, y_m)$, where $y_i = (y_{i1}, y_{i2}, \dots, y_{in}) \in \mathbb{Q}_p^n$, $i = 1, 2, \dots, m$. The p -adic norm of m -tuple $(y_1, y_2, \dots, y_m)$ is

$$|(y_1, y_2, \dots, y_m)|_p := \max_{1 \leq i \leq m} |y_i|_p. \quad (1.8)$$

Recently, p -adic analysis has received a lot of attention due to its application in mathematical physics. There are numerous papers on p -adic analysis, such as [10, 11] about Riesz potentials, [12–16] about p -adic pseudodifferential equations, and so forth. The harmonic analysis on p -adic field has been drawing more and more concern (cf. [17–21] and references therein).

The well-known Hardy's integral inequality [22] tells us that for $1 < q < \infty$,

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1} \|f\|_{L^q(\mathbb{R}^+)}, \quad (1.9)$$

where the classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t) dt, \quad (1.10)$$

for nonnegative integral function f on $\mathbb{R}^+$, and the constant $q/(q-1)$ is the best possible. Thus the norm of Hardy operator on $L^q(\mathbb{R}^+)$ is

$$\|H\|_{L^q(\mathbb{R}^+) \rightarrow L^q(\mathbb{R}^+)} = \frac{q}{q-1}. \quad (1.11)$$

Faris [23] introduced the following n -dimensional Hardy operator, for nonnegative function f on $\mathbb{R}^n$,

$$\mathcal{H}f(x) := \frac{1}{\Omega_n |x|^n} \int_{|y| < |x|} f(y) dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (1.12)$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. Christ and Grafakos [24] obtained that the norm of $\mathcal{H}$ on $L^q(\mathbb{R}^n)$ is

$$\|\mathcal{H}\|_{L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} = \frac{q}{q-1}, \quad (1.13)$$

which is the same as that of the 1-dimension Hardy operator. In [25], Fu et al. introduced the m -linear Hardy operator, which is defined by

$$\mathcal{H}^m(f_1, \dots, f_m)(x) = \frac{1}{\Omega_{mn} |x|^{mn}} \int_{|(y_1, \dots, y_m)| < |x|} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \quad (1.14)$$

where $x \in \mathbb{R}^n \setminus \{0\}$ and $f_1, \dots, f_m$ are nonnegative locally integrable functions on $\mathbb{R}^n$. And they obtained the precise norms of $\mathcal{H}^m$ on Lebesgue spaces with power weight. The authors of [26] also got the best constants of m -linear Hilbert, Hardy and Hardy-Littlewood-Pólya operators on Lebesgue spaces.

The study of multilinear averaging operators in Euclidean spaces is a byproduct of the recent interest in multilinear singular integral operator theory. This subject was established by Coifman and Meyer [27] in 1975. In this article, we consider the sharp estimates of m -linear p -adic Hardy and Hardy-Littlewood-Pólya operators. In contrast with [25], we use a new technique in calculations based on the feature of p -adic field, and Theorem 3.1 is also new. They cannot be obtained immediately by [25]. In [28], we defined the p -adic Hardy operator.

Definition 1.1. For a function f on $\mathbb{Q}_p^n$, we define the p -adic Hardy operator as follows

$$\mathcal{H}^p f(x) = \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(t) dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\}, \quad (1.15)$$

where $B(0, |x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$.

It is obvious that $|\mathcal{H}^p f| \leq \mathcal{M}^p f$, where $\mathcal{M}^p$ is the Hardy-Littlewood maximal operator [17] defined by

$$\mathcal{M}^p f(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_H} \int_{B_\gamma(x)} |f(y)| dy, \quad f \in L^1_{\text{loc}}(\mathbb{Q}_p^n). \quad (1.16)$$

The Hardy-Littlewood maximal operator plays an important role in harmonic analysis. The boundedness of $\mathcal{M}^p$ on $L^q(\mathbb{Q}_p^n)$ has been solved (see, e.g., [9]). But the best estimate of $\mathcal{M}^p$ on $L^q(\mathbb{Q}_p^n)$, $q > 1$, even that of Hardy-Littlewood maximal operator on Euclidean spaces $\mathbb{R}^n$ is very difficult to obtain. Instead, we have obtained the sharp estimates of $\mathcal{H}^p$ (and p -adic Hardy-Littlewood-Pólya operator) elsewhere.

Definition 1.2. Let m be a positive integer and $f_1, \dots, f_m$ be nonnegative locally integrable functions on $\mathbb{Q}_p^n$. The m -linear p -adic Hardy operator is defined by

$$\mathcal{H}_m^p(f_1, \dots, f_m)(x) = \frac{1}{|x|_p^{mn}} \int_{|(y_1, \dots, y_m)|_p \leq |x|_p} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \quad (1.17)$$

where $x \in \mathbb{Q}_p^n \setminus \{0\}$.

The Hardy-Littlewood-Pólya's linear operator [26] is defined by

$$Tf(x) = \int_0^\infty \frac{f(y)}{\max(x, y)} dy. \quad (1.18)$$

In [26], the authors obtained that the norm of Hardy-Littlewood-Pólya's operator on $L^q(\mathbb{R}^+)$ (see also [22, page 254]), $1 < q < \infty$, is

$$\|T\|_{L^q(\mathbb{R}^+) \rightarrow L^q(\mathbb{R}^+)} = \frac{q^2}{q-1}. \quad (1.19)$$

We define the p -adic Hardy-Littlewood-Pólya operator as (see [28])

$$T^p(x) = \int_{\mathbb{Q}_p} \frac{f(y)}{\max(|x|_p, |y|_p)} dy, \quad x \in \mathbb{Q}_p^*. \quad (1.20)$$

Definition 1.3. Let m be a positive integer and $f_1, \dots, f_m$ be nonnegative locally integrable functions on $\mathbb{Q}_p^n$. The m -linear p -adic Hardy-Littlewood-Pólya operator is defined by

$$T_m^p(f_1, \dots, f_m)(x) = \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} \frac{f_1(y_1) \cdots f_m(y_m)}{[\max(|x|_p, |y_1|_p, \dots, |y_m|_p)]^m} dy_1 \cdots dy_m, \quad x \in \mathbb{Q}_p^*. \quad (1.21)$$

We obtain the sharp estimates of the m -linear p -adic Hardy operator on Lebesgue spaces with power weights in Section 2. In Section 3, we get the best estimate of m -linear p -adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights.

In the following sequel, for $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{Q}_p^n : |x|_p \leq p^k\}$ and $S_k = \{x \in \mathbb{Q}_p^n : |x|_p = p^k\}$.

2. Sharp Estimates of m -Linear p -Adic Hardy Operator

Theorem 2.1. Let $m \in \mathbb{Z}^+$, $f_i \in L^{q_i}(|x|_p^{\alpha_i q_i / q} dx)$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$, $1/q = \sum_{i=1}^m (1/q_i)$, $\alpha_i < qn(1 - (1/q_i))$ and $\alpha = \sum_{i=1}^m \alpha_i$. Then

$$\|\mathcal{H}^p(f_1, \dots, f_m)\|_{L^q(|x|_p^\alpha dx)} \leq C_{\mathcal{H}} \|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1 / q} dx)} \cdots \|f_m\|_{L^{q_m}(|x|_p^{\alpha_m q_m / q} dx)}, \quad (2.1)$$

where

$$C_{\mathcal{H}} = \frac{(1 - p^{-n})^m}{\prod_{i=1}^m (1 - p^{n/q_i + (\alpha_i/q) - n})} \quad (2.2)$$

is the best constant.

When $\alpha = 0$, we get the sharp estimates of the m -linear p -adic Hardy operator on Lebesgue spaces.

Corollary 2.2. Let $m \in \mathbb{Z}^+$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$ and $1/q = \sum_{i=1}^m 1/q_i$. Then

$$\left\| \mathcal{H}_m^p \right\|_{L^{q_1}(\mathbb{Q}_p^n) \times \dots \times L^{q_m}(\mathbb{Q}_p^n) \rightarrow L^q(\mathbb{Q}_p^n)} = \frac{(1 - p^{-n})^m}{\prod_{i=1}^m (1 - p^{(n/q_i) - n})}. \quad (2.3)$$

Proof of Theorem 2.1. Since the proof of the case when $m = 1$ is similar to and even simpler than that of the case when $m \geq 2$, for simplicity, we will only give the proof of case when $m \geq 2$. To make the proof clearer, we will discuss it in two parts.

(I) When $m = 2$

Firstly, we claim that the operator $\mathcal{H}^p$ and its restriction to the functions g satisfying $g(x) = g(|x|_p^{-1})$ have the same operator norm on $L^q(|x|_p^\alpha dx)$. In fact, set

$$g_i(x) = \frac{1}{1 - p^{-n}} \int_{|\xi_i|_p=1} f_i(|x|_p^{-1} \xi_i) d\xi_i, \quad x \in \mathbb{Q}_p^n, \quad i = 1, 2. \quad (2.4)$$

It's clear that $g_i(x) = g_i(|x|_p^{-1})$, $i = 1, 2$, and

$$\begin{aligned}
 \mathcal{L}_2^p(g_1, g_2)(x) &= \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} g_1(y_1) g_2(y_2) dy_1 dy_2 \\
 &= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} \prod_{i=1}^2 \left(\int_{|\xi_i|_p=1} f_i(|y_i|_p^{-1} \xi_i) d\xi_i \right) dy_1 dy_2 \\
 &= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} \prod_{i=1}^2 \left(\int_{|z_i|_p=|y_i|_p} f_i(z_i) |y_i|_p^{-n} dz_i \right) dy_1 dy_2 \\
 &= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(z_1, z_2)|_p \leq |x|_p} \prod_{i=1}^2 \left(\int_{|y_i|_p=|z_i|_p} |y_i|_p^{-n} dy_i \right) f_1(z_1) f_2(z_2) dz_1 dz_2 \\
 &= \frac{1}{|x|_p^{2n}} \int_{|(z_1, z_2)|_p \leq |x|_p} f_1(z_1) f_2(z_2) dz_1 dz_2 \\
 &= \mathcal{L}_2^p(f_1, f_2)(x).
 \end{aligned} \tag{2.5}$$

By Hölder's inequality, we get

$$\begin{aligned}
 \|g_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)} &= \left(\int_{\mathbb{Q}_p^n} \left| \frac{1}{1-p^{-n}} \int_{|\xi_i|_p=1} f_i(|x|_p^{-1} \xi_i) d\xi_i \right|^{q_i} |x|_p^{\alpha_i q_i/q} dx \right)^{1/q_i} \\
 &\leq \left\{ \int_{\mathbb{Q}_p^n} \frac{1}{(1-p^{-n})^{q_i}} \left(\int_{|\xi_i|_p=1} |f_i(|x|_p^{-1} \xi_i)|^{q_i} d\xi_i \right) \left(\int_{|\xi_i|_p=1} d\xi_i \right)^{q_i-1} |x|_p^{\alpha_i q_i/q} dx \right\}^{1/q_i} \\
 &= \left\{ \int_{\mathbb{Q}_p^n} \frac{1}{1-p^{-n}} \left(\int_{|\xi_i|_p=1} |f_i(|x|_p^{-1} \xi_i)|^{q_i} d\xi_i \right) |x|_p^{\alpha_i q_i/q} dx \right\}^{1/q_i} \\
 &= \frac{1}{(1-p^{-n})^{1/q_i}} \left\{ \int_{\mathbb{Q}_p^n} \left(\int_{|z_i|_p=|x|_p} |f_i(z_i)|^{q_i} dz_i \right) |x|_p^{(\alpha_i q_i/q)-n} dx \right\}^{1/q_i} \\
 &= \frac{1}{(1-p^{-n})^{1/q_i}} \left\{ \int_{\mathbb{Q}_p^n} \left(\int_{|x|_p=|z_i|_p} |x|_p^{(\alpha_i q_i/q)-n} dx \right) |f_i(z_i)|^{q_i} dz_i \right\}^{1/q_i} \\
 &= \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}, \quad i = 1, 2.
 \end{aligned} \tag{2.6}$$

Therefore,

$$\frac{\|\mathcal{L}_2^p(f_1, f_2)\|_{L^q(|x|_p^s dx)}}{\|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \|f_2\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}} \leq \frac{\|\mathcal{L}_2^p(g_1, g_2)\|_{L^q(|x|_p^s dx)}}{\|g_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \|g_2\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}}, \tag{2.7}$$

which implies the claim. In the following, without loss of generality, we may assume that $f_i \in L^{q_i}(|x|_p^{\alpha_i q_i / q} dx)$, $i = 1, 2$, which satisfy that $f_i(x) = f_i(|x|_p^{-1})$, $i = 1, 2$.

By changing of variables $y_i = |x|_p^{-1} z_i$, $i = 1, 2$, we have

$$\begin{aligned} \|\mathcal{H}_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p^n} \left| \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} f_1(y_1) f_2(y_2) dy_1 dy_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \\ &= \left(\int_{\mathbb{Q}_p^n} \left| \int_{|(z_1, z_2)|_p \leq 1} f_1(|x|_p^{-1} z_1) f_2(|x|_p^{-1} z_2) dz_1 dz_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \quad (2.8) \\ &= \left(\int_{\mathbb{Q}_p^n} \left| \int_{|(z_1, z_2)|_p \leq 1} f_1(|z_1|_p^{-1} x) f_2(|z_2|_p^{-1} x) dz_1 dz_2 \right|^q |x|_p^\alpha dx \right)^{1/q}. \end{aligned}$$

Then using Minkowski's integral inequality and Hölder's inequality $((q/q_1) + (q/q_2) = 1)$, we get

$$\begin{aligned} \|\mathcal{H}_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &\leq \int_{|(z_1, z_2)|_p \leq 1} \left(\int_{\mathbb{Q}_p^n} |f_1(|z_1|_p^{-1} x) f_2(|z_2|_p^{-1} x)|^q |x|_p^\alpha dx \right)^{1/q} dz_1 dz_2 \\ &\leq \int_{|(z_1, z_2)|_p \leq 1} \prod_{i=1}^2 \left(\int_{\mathbb{Q}_p^n} |f_i(|z_i|_p^{-1} x)|^{q_i} |x|_p^{\alpha_i q_i / q} dx \right)^{1/q_i} dz_1 dz_2 \quad (2.9) \\ &= \left(\int_{|(z_1, z_2)|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \right) \prod_{i=1}^2 \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i / q} dx)}. \end{aligned}$$

By calculation, we have

$$\begin{aligned} &\int_{|(z_1, z_2)|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \\ &= \int_{|z_1|_p \leq 1} \int_{|z_2|_p \leq |z_1|_p} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \\ &\quad + \int_{|z_2|_p \leq 1} \int_{|z_1|_p < |z_2|_p} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \\ &= \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} \left(\sum_{k=-\infty}^{\log_p |z_1|_p} \int_{S_k} |z_2|_p^{-(n/q_2) - (\alpha_2/q)} dz_2 \right) dz_1 \\ &\quad + \int_{|z_2|_p \leq 1} |z_2|_p^{-(n/q_2) - (\alpha_2/q)} \left(\sum_{k=-\infty}^{\log_p |z_2|_p - 1} \int_{S_k} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} dz_1 \right) dz_2 \end{aligned}$$

$$\begin{aligned}
&= \frac{(1-p^{-n})}{1-p^{(n/q_2)+(\alpha_2/q)-n}} \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q)-(\alpha/q)+n} dz_1 \\
&\quad + \frac{(1-p^{-n})p^{(n/q_1)+(\alpha_1/q)-n}}{1-p^{(n/q_1)+(\alpha_1/q)-n}} \int_{|z_2|_p \leq 1} |z_2|_p^{-(n/q)-(\alpha/q)+n} dz_2 \\
&= \frac{(1-p^{-n})^2}{\prod_{i=1}^2 (1-p^{(n/q_i)+(\alpha_i/q)-n})}.
\end{aligned} \tag{2.10}$$

Therefore,

$$\left\| \mathcal{L}_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \leq \frac{(1-p^{-n})^2}{\prod_{i=1}^2 (1-p^{(n/q_i)+(\alpha_i/q)-n})}. \tag{2.11}$$

Now let us prove that our estimate is sharp. For $0 < \epsilon < 1$ and $|e|_p > 1$, we take

$$f_i^\epsilon(x_i) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2. \tag{2.12}$$

Then by calculation, we have

$$\|f_1^\epsilon\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)}^{q_1} = \|f_2^\epsilon\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}^{q_2} = \frac{1-p^{-n}}{1-p^{-\epsilon q_2}}. \tag{2.13}$$

It is clear that when $|x|_p < 1$, $\mathcal{L}_2^p(f_1^\epsilon, f_2^\epsilon)(x) = 0$. But when $|x|_p \geq 1$,

$$\begin{aligned}
&\mathcal{L}_2^p(f_1^\epsilon, f_2^\epsilon)(x) \\
&= |x|_p^{-(n/q)-(\alpha/q)-(q_2\epsilon/q)} \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|x|_p, |y_2|_p \geq 1/|x|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2.
\end{aligned} \tag{2.14}$$

Since $|\epsilon|_p > 1$, we get

$$\begin{aligned}
& \left\| \mathcal{H}_2^p(f_1^\epsilon, f_2^\epsilon) \right\|_{L^q(|x|_p^\alpha dx)} \\
&= \left\{ \int_{|x|_p \geq 1} \left(|x|_p^{-(n/q) - (\alpha/q) - (q_2\epsilon/q)} \right. \right. \\
&\quad \times \left. \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|x|_p, |y_2|_p \geq 1/|x|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right)^q |x|_p^\alpha dx \Big\}^{1/q} \\
&\geq \left\{ \int_{|x|_p \geq |\epsilon|_p} \left(|x|_p^{-(n/q) - (\alpha/q) - (q_2\epsilon/q)} \right. \right. \\
&\quad \times \left. \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right)^q |x|_p^\alpha dx \Big\}^{1/q} \\
&= \left(\int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right) \left(\int_{|x|_p \geq |\epsilon|_p} |x|_p^{-n - \epsilon q_2} dx \right)^{1/q} \\
&= \left(\int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right) |\epsilon|_p^{-\epsilon q_2/q} \prod_{i=1}^2 \|f_i^\epsilon\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}. \tag{2.15}
\end{aligned}$$

By the same calculation as that in (2.10), we obtain that

$$\begin{aligned}
& \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \\
&= \frac{(1 - p^{-n})^2 \left[1 - (p|\epsilon|_p)^{(n/q) + (\alpha/q) + (q_2\epsilon/q) - 2n} \right]}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - n})}. \tag{2.16}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{|\epsilon|_p^{-\epsilon q_2/q} (1 - p^{-n})^2 \left[1 - (p|\epsilon|_p)^{(n/q) + (\alpha/q) + (q_2\epsilon/q) - 2n} \right]}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - n})} \\
&\leq \left\| \mathcal{H}_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)}. \tag{2.17}
\end{aligned}$$

Now take $\epsilon = p^{-k}$, $k = 1, 2, 3, \dots$. Then $|\epsilon|_p = p^k > 1$. Letting k approach to ∞ , then ϵ approaches to 0 and $|\epsilon|_p^{-\epsilon q_2/q}$ approaches to 1. Since $\alpha_i < qn(1 - (1/q_i))$, $i = 1, 2$, we have

$$\frac{(1 - p^{-n})^2}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) - n})} \leq \|\mathcal{L}_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)}. \quad (2.18)$$

Then (2.11) and (2.18) imply that

$$\|\mathcal{L}_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} = \frac{(1 - p^{-n})^2}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) - n})}. \quad (2.19)$$

(II) When $m \geq 3$

The proof of the upper bound in this case is similar to that of the previous case, and we can obtain that

$$\|\mathcal{L}_m^p(f_1, \dots, f_m)\|_{L^q(|x|_p^\alpha dx)} \leq C_{\mathcal{L}} \|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \cdots \|f_m\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)}, \quad (2.20)$$

where

$$C_{\mathcal{L}} = \int_{|(z_1, \dots, z_m)|_p \leq 1} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m. \quad (2.21)$$

Let

$$\begin{aligned} D_1 &= \{(z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |z_1|_p \leq 1, |z_k|_p \leq |z_1|_p, 1 < k \leq m\}, \\ D_i &= \{(z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |z_i|_p \leq 1, |z_j|_p < |z_i|_p, |z_k|_p \leq |z_i|_p, 1 \leq j < i < k \leq m\}, \\ D_m &= \{(z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |z_m|_p \leq 1, |z_j|_p < |z_m|_p, 1 \leq j < m\}. \end{aligned} \quad (2.22)$$

It is clear that

$$\bigcup_{j=1}^m D_j = \{(z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |(z_1, \dots, z_m)|_p \leq 1\}, \quad (2.23)$$

and $D_i \cap D_j = \emptyset$. Then

$$C_{\mathcal{L}} = \sum_{j=1}^m \int_{D_j} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m := \sum_{j=1}^m I_j. \quad (2.24)$$

Now let us calculate I_j , $j = 1, 2, \dots, m$, respectively,

$$\begin{aligned}
 I_1 &= \int_{D_1} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m \\
 &= \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} \left(\prod_{k=2}^m \int_{|z_k|_p \leq |z_1|_p} |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_k \right) dz_1 \\
 &= \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} \left(\prod_{k=2}^m \left(\sum_{j=-\infty}^{\log_p |z_1|_p} \int_{S_j} |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_k \right) \right) dz_1 \quad (2.25) \\
 &= \frac{(1 - p^{-n})^{m-1}}{\prod_{k=2}^m (1 - p^{(n/q_k) + (\alpha_k/q) - n})} \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q) - (\alpha/q) + (m-1)n} dz_1 \\
 &= \frac{(1 - p^{-n})^m}{(1 + p^{(n/q) + (\alpha/q) - mn}) \prod_{k=2}^m (1 - p^{(n/q_k) + (\alpha_k/q) - n})}.
 \end{aligned}$$

Similarly, for $i = 2, \dots, m-1$, we have

$$\begin{aligned}
 I_i &= \int_{D_i} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m \\
 &= \int_{|z_i|_p \leq 1} |z_i|_p^{-(n/q_i) - (\alpha_i/q)} \left(\prod_{j=1}^{i-1} \int_{|z_j|_p \leq |z_i|_p} |z_j|_p^{-(n/q_j) - (\alpha_j/q)} dz_j \right) \\
 &\quad \times \left(\prod_{k=i+1}^m \int_{|z_k|_p \leq |z_i|_p} |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_k \right) dz_i \\
 &= \frac{(1 - p^{-n})^{m-1} \prod_{j=1}^{i-1} p^{(n/q_j) + (\alpha_j/q) - n}}{\prod_{1 \leq k \leq m, k \neq i} (1 - p^{(n/q_k) + (\alpha_k/q) - n})} \int_{|z_i|_p \leq 1} |z_i|_p^{-(n/q) - (\alpha/q) + (m-1)n} dz_i \\
 &= \frac{(1 - p^{-n})^m \prod_{j=1}^{i-1} p^{(n/q_j) + (\alpha_j/q) - n}}{(1 + p^{(n/q) + (\alpha/q) - mn}) \prod_{1 \leq k \leq m, k \neq i} (1 - p^{(n/q_k) + (\alpha_k/q) - n})}, \\
 I_m &= \int_{|z_m|_p \leq 1} |z_m|_p^{-(n/q_m) - (\alpha_m/q)} \left(\prod_{j=1}^{m-1} \int_{|z_j|_p \leq |z_m|_p} |z_j|_p^{-(n/q_j) - (\alpha_j/q)} dz_j \right) dz_m \\
 &= \frac{(1 - p^{-n})^m \prod_{j=1}^{m-1} p^{(n/q_j) + (\alpha_j/q) - n}}{(1 + p^{(n/q) + (\alpha/q) - mn}) \prod_{j=1}^{m-1} (1 - p^{(n/q_j) + (\alpha_j/q) - n})}. \quad (2.26)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
C_{\mathcal{L}} &= \frac{(1-p^{-n})^m}{(1+p^{(n/q)+(\alpha/q)-mn}) \prod_{k=2}^m (1-p^{(n/q_k)+(\alpha_k/q)-n})} \\
&+ \sum_{i=2}^{m-1} \frac{(1-p^{-n})^m \prod_{j=1}^{m-1} p^{(n/q_j)+(\alpha_j/q)-n}}{(1+p^{(n/q)+(\alpha/q)-mn}) \prod_{k=1}^{m-1} (1-p^{(n/q_k)+(\alpha_k/q)-n})} \\
&+ \frac{(1-p^{-n})^m \prod_{j=1}^{m-1} p^{(n/q_j)+(\alpha_j/q)-n}}{(1+p^{(n/q)+(\alpha/q)-mn}) \prod_{j=1}^{m-1} (1-p^{(n/q_j)+(\alpha_j/q)-n})} \\
&= \frac{(1-p^{-n})^m}{\prod_{i=1}^m (1-p^{(n/q_i)+(\alpha_i/q)-n})}.
\end{aligned} \tag{2.27}$$

To show that $C_{\mathcal{L}}$ is the best constant, we should prove that it is also the lower bound of the norm of $\mathcal{L}_m^p$ from $L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times \dots \times L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)$ to $L^q(|x|_p^\alpha dx)$. For $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we take

$$f_i^\epsilon(x_i) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_m \epsilon/q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2, \dots, m. \tag{2.28}$$

By simple calculation, we have

$$\|f_1^\epsilon\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)}^{q_1} = \dots = \|f_m^\epsilon\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)}^{q_m} = \frac{1-p^{-n}}{1-p^{-\epsilon q_m}}. \tag{2.29}$$

And when $|x|_p < 1$, $\mathcal{L}_m^p(f_1^\epsilon, \dots, f_m^\epsilon)(x) = 0$. But when $|x|_p \geq 1$,

$$\begin{aligned}
&\mathcal{L}_m^p(f_1^\epsilon, \dots, f_m^\epsilon)(x) \\
&= |x|_p^{-(n/q)-(\alpha/q)-(q_m \epsilon/q)} \int_{|(y_1, \dots, y_m)|_p \leq 1, |y_1|_p \geq 1/|x|_p, \dots, |y_m|_p \geq 1/|x|_p} \prod_{i=1}^m |y_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2 \epsilon/q_i)} dy_1 \dots dy_m.
\end{aligned} \tag{2.30}$$

Then by the similar discussion to that in previous case, we can obtain that

$$\|\mathcal{L}_m^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times \dots \times L^{q_m}(|x|_p^{\alpha_m q_m/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \geq C_{\mathcal{L}}. \tag{2.31}$$

Theorem 2.1 is established by (2.20), (2.27), and (2.31). $\square$

3. Sharp Estimate of m -Linear p -Adic Hardy-Littlewood-Pólya Operator

We get the following best estimate of m -linear p -adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights.

Theorem 3.1. Let $m \in \mathbb{Z}^+$, $f_i \in L_i^q(|x|_p^{\alpha_i q_i/q} dx)$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$, $1/q = \sum_{i=1}^m 1/q_i$, $\alpha_i < q(1 - (1/q_i))$ and $\alpha = \sum_{i=1}^m \alpha_i$. Then

$$\|T_m^p(f_1, \dots, f_m)\|_{L^q(|x|_p^\alpha dx)} = C_T \|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \cdots \|f_m\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)}, \quad (3.1)$$

where

$$C_T = \frac{(1-p^{-1})^m (1-q^{-m})}{(1-p^{-(1/q)-(\alpha/q)}) \prod_{i=1}^m (1-p^{(1/q_i)+(\alpha_i/q)-1})}, \quad (3.2)$$

is the best constant.

In particular, when $\alpha = 0$, we obtain the norm of the m -linear p -adic Hardy-Littlewood-Pólya operator on Lebesgue spaces.

Corollary 3.2. Let $m \in \mathbb{Z}^+$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$ and $1/q = \sum_{i=1}^m 1/q_i$. Then

$$\|T_m^p\|_{L^{q_1}(\mathbb{Q}_p) \times \dots \times L^{q_m}(\mathbb{Q}_p) \rightarrow L^q(\mathbb{Q}_p)} = \frac{(1-p^{-1})^m (1-q^{-m})}{(1-p^{-1/q}) \prod_{i=1}^m (1-p^{(1/q_i)-1})}. \quad (3.3)$$

Proof of Theorem 3.1. Just as the proof of Theorem 2.1, we will only give the proof of case when $m \geq 2$.

(I) Case $m = 2$

By definition and the change of variables $y_i = xz_i$, $i = 1, 2$, we have

$$\begin{aligned} \|T_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p} \left| \iint_{\mathbb{Q}_p} \frac{f_1(y_1) f_2(y_2)}{[\max(|x|_p, |y_1|_p, |y_2|_p)]^2} dy_1 dy_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \\ &\leq \left(\int_{\mathbb{Q}_p} \left(\iint_{\mathbb{Q}_p} \frac{|f_1(y_1) f_2(y_2)|}{[\max(|x|_p, |y_1|_p, |y_2|_p)]^2} dy_1 dy_2 \right)^q |x|_p^\alpha dx \right)^{1/q} \\ &= \left(\int_{\mathbb{Q}_p} \left(\iint_{\mathbb{Q}_p} \frac{|f_1(xz_1) f_2(xz_2)|}{[\max(1, |z_1|_p, |z_2|_p)]^2} dz_1 dz_2 \right)^q |x|_p^\alpha dx \right)^{1/q}. \end{aligned} \quad (3.4)$$

By Minkowski's integral inequality and Hölder's inequality $((q/q_1) + (q/q_2) = 1)$, we get

$$\begin{aligned}
 \|T_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &\leq \iint_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p} |f_1(xz_1)f_2(xz_2)|^q |x|_p^\alpha dx \right)^{1/q} \frac{1}{\left[\max(1, |z_1|_p, |z_2|_p) \right]^2} dz_1 dz_2 \\
 &\leq \iint_{\mathbb{Q}_p} \prod_{i=1}^2 \left(\int_{\mathbb{Q}_p} |f_i(xz_i)|^{q_i} |x|_p^{\alpha_i q_i / q} dx \right)^{1/q_i} \frac{1}{\left[\max(1, |z_1|_p, |z_2|_p) \right]^2} dz_1 dz_2 \\
 &\leq \left(\iint_{\mathbb{Q}_p} \frac{\prod_{i=1}^2 |z_i|_p^{-(1/q_i) - (\alpha_i/q)}}{\left[\max(1, |z_1|_p, |z_2|_p) \right]^2} dz_1 dz_2 \right) \prod_{i=1}^2 \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i / q} dx)}.
 \end{aligned} \tag{3.5}$$

By calculation, we have

$$\begin{aligned}
 &\iint_{\mathbb{Q}_p} \frac{\prod_{i=1}^2 |z_i|_p^{-(1/q_i) - (\alpha_i/q)}}{\left[\max(1, |z_1|_p, |z_2|_p) \right]^2} dz_1 dz_2 \\
 &= \int_{|z_1|_p \leq 1} \int_{|z_2|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(1/q_i) - (\alpha_i/q)} dz_1 dz_2 \\
 &\quad + \int_{|z_1|_p > 1} \int_{|z_2|_p \leq |z_1|_p} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - 2} |z_2|_p^{-(1/q_2) - (\alpha_2/q)} dz_1 dz_2 \\
 &\quad + \int_{|z_2|_p > 1} \int_{|z_1|_p < |z_2|_p} |z_1|_p^{-(1/q_1) - (\alpha_1/q)} |z_2|_p^{-(1/q_2) - (\alpha_2/q) - 2} dz_1 dz_2 \\
 &:= L_0 + L_1 + L_2.
 \end{aligned} \tag{3.6}$$

By definition,

$$\begin{aligned}
 L_0 &= \int_{|z_1|_p \leq 1} \int_{|z_2|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(1/q_i) - (\alpha_i/q)} dz_1 dz_2 \\
 &= \frac{(1 - p^{-1})^2}{(1 - p^{(1/q_1) + (\alpha_1/q) - 1})(1 - p^{(1/q_2) + (\alpha_2/q) - 1})}, \\
 L_1 &= \int_{|z_1|_p > 1} \int_{|z_2|_p \leq |z_1|_p} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - 2} |z_2|_p^{-(1/q_2) - \alpha_2/q} dz_1 dz_2 \\
 &= \frac{(1 - p^{-1})}{1 - p^{(1/q_2) + (\alpha_2/q) - 1}} \int_{|z_1|_p > 1} |z_1|_p^{-(1/q) - (\alpha/q) - 1} dz_1 \\
 &= \frac{(1 - p^{-1})^2 p^{-(1/q) - (\alpha/q)}}{(1 - p^{(1/q_2) + (\alpha_2/q) - 1})(1 - p^{-(1/q) - (\alpha/q)})}.
 \end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned} L_2 &= \int_{|z_2|_p > 1} \int_{|z_1|_p < |z_2|_p} |z_1|_p^{-(1/q_1) - (\alpha_1/q)} |z_2|_p^{-(1/q_2) - (\alpha_2/q) - 2} dz_1 dz_2 \\ &= \frac{(1 - p^{-1})^2 p^{(1/q_1) + (\alpha_1/q) - 1} p^{-(1/q) - (\alpha/q)}}{(1 - p^{(1/q_1) + (\alpha_1/q) - 1}) (1 - p^{-(1/q) - (\alpha/q)})}. \end{aligned} \quad (3.8)$$

Substituting (3.7) and (3.8) into (3.6), we get

$$\iint_{\mathbb{Q}_p} \frac{\prod_{i=1}^2 |z_i|_p^{-(1/q_i) - (\alpha_i/q)}}{\left[\max(1, |z_1|_p, |z_2|_p) \right]^2} dz_1 dz_2 = \frac{(1 - p^{-1})^2 (1 - q^{-2})}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) - 1})}. \quad (3.9)$$

Then (3.5) and (3.9) imply that

$$\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \leq \frac{(1 - p^{-1})^2 (1 - p^{-2})}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) - 1})}. \quad (3.10)$$

On the other hand, for $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we take

$$f_i^\epsilon(x_i) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2 \epsilon / q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2. \quad (3.11)$$

Then

$$\|f_1^\epsilon\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)}^{q_1} = \|f_2^\epsilon\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}^{q_2} = \frac{1 - p^{-1}}{1 - p^{-\epsilon q_2}}. \quad (3.12)$$

Since $|\epsilon|_p > 1$, we have

$$\begin{aligned} &\|T_2^p(f_1^\epsilon, f_2^\epsilon)\|_{L^q(|x|_p^\alpha dx)} \\ &= \left(\int_{\mathbb{Q}_p} \left| \iint_{\mathbb{Q}_p} \frac{f_1^\epsilon(x_1) f_2^\epsilon(x_2)}{\left[\max(|x|_p, |x_1|_p, |x_2|_p) \right]^2} dx_1 dx_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \\ &\geq \left(\int_{|x|_p \geq 1} \left(\int_{|x_1|_p \geq 1} \int_{|x_2|_p \geq 1} \frac{\prod_{i=1}^2 |x_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2 \epsilon / q_i)}}{\left[\max(|x|_p, |x_1|_p, |x_2|_p) \right]^2} dx_1 dx_2 \right)^q |x|_p^\alpha dx \right)^{1/q} \\ &= \left(\int_{|x|_p \geq 1} \left(\int_{|y_1|_p \geq 1/|x|_p} \int_{|y_2|_p \geq 1/|x|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2 \epsilon / q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2 \right)^q |x|_p^{-1 - q_2 \epsilon} dx \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
&\geq \left(\int_{|x|_p \geq |\epsilon|_p} \left(\int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2 \right)^q |x|_p^{-1-q_2\epsilon} dx \right)^{1/q} \\
&= \prod_{i=1}^2 \|f_i^\epsilon\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)} |\epsilon|_p^{-q_2\epsilon/q} \int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2.
\end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned}
&\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \\
&\geq |\epsilon|_p^{-q_2\epsilon/q} \int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2.
\end{aligned} \tag{3.14}$$

As the calculation of (3.6)–(3.8), we obtain that

$$\begin{aligned}
&\int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2 \\
&= \frac{(1-p^{-1})^2 \prod_{i=1}^2 \left[1 - (p|\epsilon|_p)^{(1/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - 1} \right]}{\prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - 1})} \\
&\quad + \frac{(1-p^{-1})^2 p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)}}{(1 - p^{(1/q_2) + (\alpha_2/q) + \epsilon - 1}) (1 - p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)})} \\
&\quad + \frac{(1-p^{-1})^2 p^{(1/q_1) + (\alpha_1/q) + (q_2\epsilon/q_1) - 1} p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)}}{(1 - p^{(1/q_1) + (\alpha_1/q) + (q_2\epsilon/q_1) - 1}) (1 - p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)})}.
\end{aligned} \tag{3.15}$$

Now take $\epsilon = p^{-k}$, $k \in \mathbb{Z}^+$ and let k approach to ∞ , then by (3.9), (3.14), (3.15), and the fact that $\alpha_i < qn(1 - (1/q_i))$, $i = 1, 2$, we have

$$\begin{aligned}
&\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \\
&\geq \frac{(1-p^{-1})^2 (1-q^{-2})}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) - 1})}.
\end{aligned} \tag{3.16}$$

Then by (3.10) and (3.16), we get

$$\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} = \frac{(1-p^{-1})^2 (1-q^{-2})}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) - 1})}. \tag{3.17}$$

(II) Case $m \geq 3$

The upper bound estimate for the norm can be obtained by the same way as that when $m = 2$, and we can obtain that

$$\left\| T_m^p(f_1, \dots, f_m) \right\|_{L^q(|x|_p^\alpha dx)} \leq C_T \prod_{i=1}^m \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}, \quad (3.18)$$

where

$$C_T = \int_{|(z_1, \dots, z_m)|_p \leq 1} \frac{\prod_{j=1}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)}}{\left[\max(1, |z_1|_p, \dots, |z_m|_p) \right]^m} dz_1 \cdots dz_m. \quad (3.19)$$

Let

$$\begin{aligned} E_0 &= \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p, \\ E_1 &= \left\{ (z_1, \dots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_1|_p > 1, |z_k|_p \leq |z_1|_p, 1 < k \leq m \right\}, \\ E_i &= \left\{ (z_1, \dots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_i|_p > 1, |z_j|_p < |z_i|_p, |z_k|_p \leq |z_i|_p, 1 \leq j < i < k \leq m \right\}, \\ E_m &= \left\{ (z_1, \dots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_m|_p > 1, |z_j|_p < |z_m|_p, 1 \leq j < m \right\}. \end{aligned} \quad (3.20)$$

Obviously,

$$\bigcup_{k=1}^m E_k = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p, \quad E_i \cap E_j = \emptyset, \quad i \neq j, \quad 1 \leq i, j \leq m. \quad (3.21)$$

Then

$$C_T = \sum_{k=0}^m \int_{E_k} \frac{\prod_{j=1}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)}}{\left[\max(1, |z_1|_p, \dots, |z_m|_p) \right]^m} dz_1 \cdots dz_m := \sum_{k=0}^m J_k. \quad (3.22)$$

Now let us calculate $J_k, k = 0, 1, \dots, m$, respectively,

$$\begin{aligned} J_0 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \prod_{j=1}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_1 \cdots dz_m \\ &= \prod_{j=1}^m \int_{\mathbb{Z}_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j = \prod_{j=1}^m \left(\sum_{k=-\infty}^0 \int_{S_k} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) \\ &= \frac{(1 - p^{-1})^m}{\prod_{j=1}^m (1 - p^{(1/q_j) + (\alpha_j/q) - 1})}, \end{aligned}$$

$$\begin{aligned}
J_1 &= \int_{|z_1|_p > 1} \int_{|z_2|_p \leq |z_1|_p} \cdots \int_{|z_m|_p \leq |z_1|_p} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - m} \prod_{j=2}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_1 \cdots dz_m \\
&= \int_{|z_1|_p > 1} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - m} \left(\prod_{j=2}^m \int_{|z_j|_p \leq |z_1|_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) dz_1 \\
&= \int_{|z_1|_p > 1} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - m} \prod_{j=2}^m \left(\frac{(1 - p^{-1}) |z_1|_p^{-(1/q_j) - (\alpha_j/q) + 1}}{1 - p^{(1/q_j) + (\alpha_j/q) - 1}} \right) dz_1 \\
&= \frac{(1 - p^{-1})^{m-1}}{\prod_{j=2}^m (1 - p^{(1/q_j) + (\alpha_j/q) - 1})} \int_{|z_1|_p > 1} |z_1|_p^{-(1/q) - (\alpha/q) - 1} dz_1 \\
&= \frac{(1 - p^{-1})^{m-1} p^{-(1/q) - (\alpha/q)}}{\prod_{j=2}^m (1 - p^{(1/q_j) + (\alpha_j/q) - 1}) (1 - p^{-(1/q) - (\alpha/q)})}.
\end{aligned} \tag{3.23}$$

Similar to J_1 , for $1 < i < m$, it is true that

$$\begin{aligned}
J_i &= \int_{|z_i|_p > 1} |z_i|_p^{-(1/q_i) - (\alpha_i/q) - m} \left(\prod_{j=1}^{i-1} \int_{|z_j|_p < |z_i|_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) \\
&\quad \times \left(\prod_{k=i+1}^m \int_{|z_k|_p \leq |z_i|_p} |z_k|_p^{-(1/q_k) - (\alpha_k/q)} dz_k \right) dz_i \\
&= \frac{(1 - p^{-1})^{m-1} \prod_{j=1}^{i-1} p^{(1/q_j) + (\alpha_j/q) - 1}}{\prod_{1 \leq k \leq m, k \neq i} (1 - p^{(1/q_k) + (\alpha_k/q) - 1})} \int_{|z_i|_p > 1} |z_i|_p^{-(1/q) - (\alpha/q) - 1} dz_i \\
&= \frac{(1 - p^{-1})^m \left(\prod_{j=1}^{i-1} p^{(1/q_j) + (\alpha_j/q) - 1} \right) p^{-(1/q) - (\alpha/q)}}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{1 \leq k \leq m, k \neq i} (1 - p^{(1/q_k) + (\alpha_k/q) - 1})}, \tag{3.24} \\
J_m &= \int_{|z_m|_p > 1} |z_m|_p^{-(1/q_m) - (\alpha_m/q) - m} \left(\prod_{j=1}^{m-1} \int_{|z_j|_p < |z_m|_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) dz_m \\
&= \frac{(1 - p^{-1})^{m-1} \prod_{j=1}^{m-1} p^{(1/q_j) + (\alpha_j/q) - 1}}{\prod_{j=1}^{m-1} (1 - p^{(1/q_j) + (\alpha_j/q) - 1})} \int_{|z_m|_p > 1} |z_m|_p^{-(1/q) - (\alpha/q) - 1} dz_m \\
&= \frac{(1 - p^{-1})^m \left(\prod_{j=1}^{m-1} p^{(1/q_j) + (\alpha_j/q) - 1} \right) p^{-(1/q) - (\alpha/q)}}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{j=1}^{m-1} (1 - p^{(1/q_j) + (\alpha_j/q) - 1})}.
\end{aligned}$$

Consequently, we have

$$C_T = \sum_{k=0}^m J_k = \frac{(1-p^{-1})^m (1-q^{-m})}{(1-p^{-(1/q)-(\alpha/q)}) \prod_{i=1}^m (1-p^{(1/q_i)+(\alpha_i/q)-1})}. \quad (3.25)$$

To obtain that C_T is also the lower bound, for $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we define

$$f_i^\epsilon = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(1/q_i)-(\alpha_i/q)-(\epsilon/q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2, \dots, m. \quad (3.26)$$

By the similar discussion to that in Case $m = 2$, we can also get that

$$\left\| T_m^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times \dots \times L^{q_m}(|x|_p^{\alpha_m q_m/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \geq C_T. \quad (3.27)$$

Combining (3.18) with (3.27), we complete the proof of Theorem 3.1. $\square$

Acknowledgments

This paper was partially supported by NSF of China (Grant nos. 10871024 and 10901076) and NSF of Shandong Province (Grant nos. Q2008A01 and ZR2010AL006).

References

- [1] S. Albeverio and W. Karwowski, "A random walk on p -adics—the generator and its spectrum," *Stochastic Processes and Their Applications*, vol. 53, no. 1, pp. 1–22, 1994.
- [2] V. A. Avetisov, A. H. Bikulov, S. V. Kozyrev, and V. A. Osipov, " p -adic models of ultrametric diffusion constrained by hierarchical energy landscapes," *Journal of Physics A*, vol. 35, no. 2, pp. 177–189, 2002.
- [3] A. Khrennikov, *p -Adic Valued Distributions in Mathematical Physics*, vol. 309 of *Mathematics and Its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 1994.
- [4] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
- [5] V. S. Varadarajan, "Path integrals for a class of p -adic Schrödinger equations," *Letters in Mathematical Physics*, vol. 39, no. 2, pp. 97–106, 1997.
- [6] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p -Adic Analysis and Mathematical Physics*, vol. 1 of *Series on Soviet and East European Mathematics*, World Scientific, River Edge, NJ, USA, 1994.
- [7] V. S. Vladimirov and I. V. Volovich, " p -adic quantum mechanics," *Communications in Mathematical Physics*, vol. 123, no. 4, pp. 659–676, 1989.
- [8] I. V. Volovich, "Harmonic analysis and p -adic strings," *Letters in Mathematical Physics*, vol. 16, no. 1, pp. 61–67, 1988.
- [9] M. H. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, NJ, USA, 1975.
- [10] S. Haran, "Riesz potentials and explicit sums in arithmetic," *Inventiones Mathematicae*, vol. 101, no. 3, pp. 697–703, 1990.
- [11] S. Haran, "Analytic potential theory over the p -adics," *Annales de l'Institut Fourier*, vol. 43, no. 4, pp. 905–944, 1993.
- [12] S. Albeverio, A. Yu. Khrennikov, and V. M. Shelkovich, "Harmonic analysis in the p -adic Lizorkin spaces: fractional operators, pseudo-differential equations, p -adic wavelets, Tauberian theorems," *The Journal of Fourier Analysis and Applications*, vol. 12, no. 4, pp. 393–425, 2006.

- [13] N. M. Chuong and N. V. Co, "The Cauchy problem for a class of pseudodifferential equations over p -adic field," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 629–645, 2008.
- [14] N. M. Chuong, Yu. V. Egorov, A. Khrennikov, Y. Meyer, and D. Mumford, Eds., *Harmonic, Wavelet and p -Adic Analysis*, World Scientific, Singapore, 2007.
- [15] A. N. Kochubei, "A non-Archimedean wave equation," *Pacific Journal of Mathematics*, vol. 235, no. 2, pp. 245–261, 2008.
- [16] W. A. Zuniga-Galindo, "Pseudo-differential equations connected with p -adic forms and local zeta functions," *Bulletin of the Australian Mathematical Society*, vol. 70, no. 1, pp. 73–86, 2004.
- [17] Y.-C. Kim, "Carleson measures and the BMO space on the p -adic vector space," *Mathematische Nachrichten*, vol. 282, no. 9, pp. 1278–1304, 2009.
- [18] Y.-C. Kim, "Weak type estimates of square functions associated with quasiradial Bochner-Riesz means on certain Hardy spaces," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 266–280, 2008.
- [19] K. S. Rim and J. Lee, "Estimates of weighted Hardy-Littlewood averages on the p -adic vector space," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1470–1477, 2006.
- [20] K. M. Rogers, "A van der Corput lemma for the p -adic numbers," *Proceedings of the American Mathematical Society*, vol. 133, no. 12, pp. 3525–3534, 2005.
- [21] K. M. Rogers, "Maximal averages along curves over the p -adic numbers," *Bulletin of the Australian Mathematical Society*, vol. 70, no. 3, pp. 357–375, 2004.
- [22] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1952.
- [23] W. G. Faris, "Weak Lebesgue spaces and quantum mechanical binding," *Duke Mathematical Journal*, vol. 43, no. 2, pp. 365–373, 1976.
- [24] M. Christ and L. Grafakos, "Best constants for two nonconvolution inequalities," *Proceedings of the American Mathematical Society*, vol. 123, no. 6, pp. 1687–1693, 1995.
- [25] Z. W. Fu, L. Grafakos, S. Z. Lu, and F. Y. Zhao, "Sharp bounds for m -linear Hardy and Hilbert operators," *Houston Journal of Mathematics*. In press.
- [26] A. Bényi and T. Oh, "Best constants for certain multilinear integral operators," *Journal of Inequalities and Applications*, vol. 2006, Article ID 28582, 12 pages, 2006.
- [27] R. R. Coifman and Y. Meyer, "On commutators of singular integrals and bilinear singular integrals," *Transactions of the American Mathematical Society*, vol. 212, pp. 315–331, 1975.
- [28] Z. W. Fu, Q. Y. Wu, and S. Z. Lu, "Sharp estimates for p -adic Hardy, Hardy-Littlewood-Pólya operators and commutators," <http://arxiv.org/abs/1105.2888>.