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Research Article

Maximum Norm Analysis of a Nonmatching Grids Method for Nonlinear Elliptic PDES

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We provide a maximum norm analysis of a finite element Schwarz alternating method for a nonlinear elliptic PDE on two overlapping subdomains with nonmatching grids. We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Under a Lipschitz asssumption on the nonlinearity, we establish, on each subdomain, an optimal L^∞ error estimate between the discrete Schwarz sequence and the exact solution of the PDE.

1. Introduction

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomains.

Extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems can be found in ([1–4] and the references therein). Also the effectiveness of Schwarz methods for these problems, especially those in fluid mechanics, has been demonstrated in many papers. See proceedings of the annual domain decomposition conference beginning with [5].

In this paper, we are interested in the error analysis in the maximum norm for a class of nonlinear elliptic problems in the context of overlapping nonmatching grids: we consider a domain which is the union of two overlapping subdomains where each subdomain has its own triangulation. This kind of discretizations are very interesting as they can be applied

to solve many practical problems which cannot be handled by global discretizations. They are earning particular attention of computational experts and engineers as they allow the choice of different mesh sizes and different orders of approximate polynomials in different subdomains according to the different properties of the solution and different requirements of the practical problems.

Quite a few works on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems are known in the literature (cf., e.g., [6–9]).

To prove the main result of this paper, we proceed as in [7]. More precisely, we develop an approach which combines a geometrical convergence result due to Lions [2] and a lemma which consists of estimating the error in the maximum norm between the continuous and discrete Schwarz iterates. The optimal convergence order is then derived making use of standard finite element L^∞-error estimate for linear elliptic equations.

In the present paper, the proof of this lemma stands on a Lipschitz continuous dependency with respect to both the boundary condition and the source term for linear elliptic equations (see Proposition 2.7) while in [7] the proof stands on a Lipschitz continuous dependency only with respect to the boundary condition for the elliptic obstacle problem.

To the best of our knowledge, this paper provides the first L^∞-error analysis for overlapping nonmatching grids for nonlinear elliptic PDEs. We also believe that our convergence result will have important implications in the computation of the solution of this type of problems on composite grids.

Now, we give an outline of the paper. In Section 2 we state a continuous alternating Schwarz sequences and define their respective finite element counterparts in the context of nonmatching overlapping grids. Section 3 is devoted to the L^∞-error analysis of the method.

2. Preliminaries

We begin by laying down some definitions and classical results related to linear elliptic equations.

2.1. Linear Elliptic Equations

Let Ω be a bounded polyhedral domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. We consider the bilinear form

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v)dx, \tag{2.1}$$

the linear form

$$(f,v) = \int_\Omega f(x) \cdot v(x)dx, \tag{2.2}$$

the right hand side

$$f, \text{ a regular function,} \tag{2.3}$$

the space

$$V^{(g)} = \left\{ v \in H^1(\Omega) \text{ such that } v = g \text{ on } \partial\Omega \right\}, \tag{2.4}$$

where g is a regular function defined on $\partial\Omega$.

We consider the linear elliptic equation: find $\xi \in V^{(g)}$ such that

$$a(\zeta, v) + c(\xi, v) = (f, v), \quad \forall v \in V^{(g)}, \tag{2.5}$$

where c is a positive constant such that

$$c \geq \beta > 0. \tag{2.6}$$

Let V_h be the space of finite elements consisting of continuous piecewise linear functions v vanishing on $\partial\Omega$ and φ_s, $s = 1, 2, \ldots, m(h)$ be the basis functions of V_h.

The discrete counterpart of (2.5) consists of finding $\zeta_h \in V_h^{(g)}$ such that

$$a(\zeta_h, v) + c(\zeta_h, v) = (f, v), \quad \forall v \in V_h^{(g)}, \tag{2.7}$$

where

$$V_h^{(g)} = \{ v \in V_h : v = \pi_h g \text{ on } \partial\Omega \} \tag{2.8}$$

and π_h is an interpolation operator on $\partial\Omega$.

Theorem 2.1 (see (cf. [13])). *Under suitable regularity of the solution of problem* (2.5), *there exists a constant C independent of h such that*

$$\|\zeta - \zeta_h\| \leq Ch^2 |\log h|. \tag{2.9}$$

Lemma 2.2 (see (cf. [4])). *Let $w \in H_1(\Omega) \cap C(\overline{\Omega})$ satisfy $a(w, \phi) + c(w, \phi) \geq 0$ for all nonnegative $\phi \in H_0^1(\Omega)$, and $w \geq 0$ on $\partial\Omega$. Then $w \geq 0$ on $\overline{\Omega}$.*

The proposition below establishes a Lipschitz continuous dependency of the solution with respect to the data.

Notation 2.3. Let $(f; g)$; $(\tilde{f}, \tilde{g})$ be a pair of data and $\zeta = \partial(f, g)$; $\tilde{\zeta} = \partial(\tilde{f}, \tilde{g})$ the corresponding solutions to (2.5).

Proposition 2.4. *Under conditions of the preceding Lemma 2.2, we have*

$$\left\| \zeta - \tilde{\zeta} \right\|_{L^\infty(\Omega)} \leq \max\left\{ \left(\frac{1}{\beta} \right) \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)}, \| g - \tilde{g} \|_{L^\infty(\partial\Omega)} \right\}. \tag{2.10}$$

Proof. First, set

$$\Phi = \max\left\{\left(\frac{1}{\beta}\right)\left\|f-\tilde{f}\right\|_{L^\infty(\Omega)}, \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}\right\}. \tag{2.11}$$

Then

$$\begin{aligned}\tilde{f} &\le f + \left\|f-\tilde{f}\right\|_{L^\infty(\Omega)}\\ &\le f + \left(\frac{c}{\beta}\right)\left\|f-\tilde{f}\right\|_{L^\infty(\Omega)}\\ &\le f + c\max\left\{\left(\frac{1}{\beta}\right)\left\|f-\tilde{f}\right\|_{L^\infty(\Omega)}, \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}\right\}\\ &\le f + c\Phi.\end{aligned} \tag{2.12}$$

So

$$\begin{aligned}a\left(\tilde{\zeta},\phi\right) + c\left(\tilde{\zeta},\phi\right) &\le a(\zeta,\phi) + c(\zeta,\phi) + c(\Phi,\phi), \quad \forall\phi\ge 0,\ \phi\in H_0^1(\Omega)\\ &\le a(\zeta+\Phi,\phi) + c(\zeta+\Phi,\phi) = (f+c\Phi,\phi).\end{aligned} \tag{2.13}$$

On the other hand, we have

$$\zeta + \Phi - \tilde{\zeta} \ge 0 \quad \text{on } \partial\Omega. \tag{2.14}$$

So

$$\begin{aligned}a\left(\zeta+\Phi-\tilde{\zeta},\phi\right) + c\left(\zeta+\Phi-\tilde{\zeta},\phi\right) &\ge 0\\ \zeta+\Phi-\tilde{\zeta} \ge 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{2.15}$$

Thus, making use of Lemma 2.2, we get

$$\zeta + \Phi - \tilde{\zeta} \ge 0 \quad \text{on } \overline{\Omega}. \tag{2.16}$$

Similarly, interchanging the roles of the couples (f,g) and $(\tilde{f},\tilde{g})$, we obtain

$$\tilde{\zeta} + \Phi - \zeta \ge 0 \quad \text{on } \overline{\Omega}, \tag{2.17}$$

which completes the proof. □

Remark 2.5. Lemma 2.2 stays true in the discrete case.

Indeed, assume that the discrete maximum principle (d.m.p) holds; that is, the matrix resulting from the finite element discretization is an M-Matrix (cf. [10, 11]). Then we have the following

Lemma 2.6. *Let $w \in V_h$ satisfy $a(w,s) + c(w,s) \geq 0$, $s = 1,2,\ldots,m(h)$ and $w \geq 0$ on $\partial\Omega$. Then $w \geq 0$ on $\overline{\Omega}$.*

Proof. The proof is a direct consequence of the discrete maximum principle. □

Let $(f,g);(\tilde{f},\tilde{g})$ be a pair of data and $\zeta_h = \partial_h(f,g)$; $\tilde{\zeta}_h = \partial_h(\tilde{f},\tilde{g})$ the corresponding solutions to (2.7).

Proposition 2.7. *Let the d.m.p hold. Then, under conditions of Lemma 2.6, we have*

$$\left\|\zeta_h - \tilde{\zeta}_h\right\|_{L^\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta}\right)\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}\right\}. \tag{2.18}$$

Proof. The proof is similar to that of the continuous case. Indeed, as the basis functions of the space V_h are positive, it suffices to use the discrete maximum principle. □

2.2. Schwarz Alternating Methods for Nonlinear PDEs

Consider the nonlinear PDE

$$\begin{aligned} -\Delta u + cu &= f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.19}$$

or in its weak form

$$a(u,v) + c(u,v) = (f(u),v), \quad \forall v \in H_0^1(\Omega), \tag{2.20}$$

where $f(\cdot)$ is a nondecreasing nonlinearity. Thanks to [12], problem (2.19) has a unique solution.

Let us also assume that $f(\cdot)$ is a Lipschitz continuous on $\mathbb{R}$; that is,

$$|f(x) - f(y)| \leq k|x - y|, \quad \forall x, y \in \mathbb{R} \tag{2.21}$$

such that

$$\frac{1}{2} < \frac{k}{\beta} < 1, \tag{2.22}$$

where β is the constant defined in (2.6).

We decompose Ω into two overlapping smooth subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2. \tag{2.23}$$

We denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$ and assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$, $i \neq j$ is empty. Let

$$V_i^{(w_j)} = \left\{ v \in H^1(\Omega_i) \text{ such that } v = w_j \text{ on } \Gamma_i \right\}. \tag{2.24}$$

We associate with problem (2.20) the following system: find $(u_1, u_2) \in V_1^{(u_2)} \times V_2^{(u_1)}$ solution to

$$\begin{aligned} a_1(u_1, v) + c(u_1, v) &= \left(f(u_1), v\right), \quad \forall v \in H_0^1(\Omega_1), \\ a_2(u_2, v) + c(u_2, v) &= \left(f(u_2), v\right), \quad \forall v \in H_0^1(\Omega_2), \end{aligned} \tag{2.25}$$

where

$$\begin{gathered} a_i(u, v) = \int_{\Omega_i} (\nabla u \cdot \nabla v) dx, \\ u_i = u/\Omega_i, \quad i = 1, 2. \end{gathered} \tag{2.26}$$

2.3. The Continuous Schwarz Sequences

Let u_0 be an initialization in $C_0(\overline{\Omega})$ (i.e., continuous functions vanishing on $\partial\Omega$) such that

$$a(u_0, v) + c(u_0, v) = \left(f, v\right), \quad \forall v \in H_0^1(\Omega). \tag{2.27}$$

Starting from $u_2^0 = u^0/\Omega_2$, we respectively define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in V^{(u_2^n)}$ solves

$$a_1\left(u_1^{n+1}, v\right) + c\left(u_1^{n+1}, v\right) = \left(f\left(u_1^{n+1}\right), v\right), \quad \forall v \in H_0^1(\Omega_1);\ n \geq 0, \tag{2.28}$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in V^{(u_1^{n+1})}$ solves

$$a_2\left(u_2^{n+1}, v\right) + c\left(u_2^{n+1}, v\right) = \left(f\left(u_2^{n+1}\right), v\right), \quad \forall v \in H_0^1(\Omega_2);\ n \geq 0. \tag{2.29}$$

Theorem 2.8 (see (cf. [2, pages 51–63])). *The sequences (u_1^{n+1}); (u_2^{n+1}); $n \geq 0$ produced by the Schwarz alternating method converge geometrically to the solution (u_1, u_2) of the system* (2.25). *More*

precisely, there exist two constants $k_1, k_2 \in (0;1)$ *which depend on* (Ω_1, Γ_2) *and* (Ω_2, Γ_1)*, respectively, such that for all* $n \geq 0$*;*

$$\begin{aligned} \left\| u_1 - u_1^{n+1} \right\|_{L^\infty(\Omega_1)} &\leq k_1^n k_2^n \left\| u^0 - u \right\|_{L^\infty(\Gamma_1)}, \\ \left\| u_2 - u_2^{n+1} \right\|_{L^\infty(\Omega_2)} &\leq k_1^{n+1} k_2^n \left\| u^0 - u \right\|_{L^\infty(\Gamma_2)}. \end{aligned} \tag{2.30}$$

2.4. The Discretization

For $i = 1, 2$, let τ^{h_i} be a standard regular and quasiuniform finite element triangulation in Ω_i; h_i, being the meshsize. The two meshes being mutually independent $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other. We consider the following discrete spaces:

$$V_{h_i} = \left\{ v \in C(\Omega_i) \cap H_0^1(\Omega_i) \text{ such that } v/K \in P_1,\ \forall K \in \tau^{h_i} \right\} \tag{2.31}$$

and for every $w \in C(\Omega_i)$, we set

$$V_{h_i}^{(w)} = \{ v \in V_{h_i} : v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega;\ v = \pi_{h_i}(w) \text{ on } \Gamma_i \}, \tag{2.32}$$

where π_{h_i} denote an interpolation operator on Γ_i.

The Discrete Maximum Principle (see [10, 11])

We assume that the respective matrices resulting from the discretizations of problems (2.28) and (2.29) are M-matrices.

Note that as the two meshes h_1 and h_2 are independent over the overlapping subdomains, it is impossible to formulate a global approximate problem which would be the direct discrete counterpart of problem (2.20).

2.5. The Discrete Schwarz Sequences

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (2.28) and (2.29).

Indeed, let u_{0h} be the discrete analog of u_0, defined in (2.27); we, respectively, define by $u_{1h}^{n+1} \in V_{h_1}^{(u_{2h}^n)}$ such that

$$a_1\left(u_{1h}^{n+1}, v\right) + c\left(u_{1h}^{n+1}, v\right) = \left(f\left(u_{1h}^{n+1}\right), v\right) \quad \forall v \in V_{h_1};\ n \geq 0 \tag{2.33}$$

and $u_{2h}^{n+1} \in V_{h_2}^{(u_{1h}^{n+1})}$ such that

$$a_2\left(u_{2h}^{n+1}, v\right) + c\left(u_{2h}^{n+1}, v\right) = \left(f\left(u_{2h}^{n+1}\right), v\right) \quad \forall v \in V_{h_2};\ n \geq 0. \tag{2.34}$$

3. L^∞-Error Analysis

This section is devoted to the proof of the main result of the present paper. To that end we begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

3.1. Two Auxiliary Schwarz Sequences

For $w_{2h}^0 = u_{2h}^0$, we define the sequences (w_{1h}^{n+1}) and (w_{2h}^{n+1}) such that $w_{1h}^{n+1} \in V_{h_1}^{(u_2^n)}$ solves

$$a_1\left(w_{1h}^{n+1}, v\right) + c\left(w_{1h}^{n+1}, v\right) = \left(f\left(u_1^{n+1}\right), v\right), \quad \forall v \in V_{h_1};\ n \geq 0 \tag{3.1}$$

and $w_{2h}^{n+1} \in V_{h_2}^{(u_1^{n+1})}$ solves

$$a_2\left(w_{2h}^{n+1}, v\right) + c\left(w_{2h}^{n+1}, v\right) = \left(f\left(u_2^{n+1}\right), v\right), \quad \forall v \in V_{h_2};\ n \geq 0, \tag{3.2}$$

respectively.

It is then clear that w_{1h}^{n+1} and w_{2h}^{n+1} are the finite element approximation of u_1^{n+1} and u_2^{n+1} defined in (2.28), (2.29), respectively. Then, as $f(\cdot)$ is continuous, $\|f(u_i^n)\|_\infty \leq C$ (independent of n), and, therefore, making use of standard maximum norm estimates for linear elliptic problems, we have

$$\left\|u_i^n - w_{ih}^n\right\|_{L^\infty(\Omega_i)} \leq Ch^2\left|\log h\right|, \tag{3.3}$$

where C is a constant independent of both h and n.

Notation 3.1. From now on, we shall adopt the following notations:

$$\begin{gathered} |\cdot|_1 = \|\cdot\|_{L^\infty(\Gamma_1)}; \qquad |\cdot|_2 = \|\cdot\|_{L^\infty(\Gamma_2)}, \\ \|\cdot\|_1 = \|\cdot\|_{L^\infty(\Omega_1)}; \qquad \|\cdot\|_2 = \|\cdot\|_{L^\infty(\Omega_2)}, \\ \pi_{h_1} = \pi_{h_2} = \pi_h. \end{gathered} \tag{3.4}$$

3.2. The Main Results

The following lemma will play a key role in proving the main result of this paper.

Lemma 3.2. *Let $\rho = k/\beta$. Then, under assumption* (2.22)*, there exists a constant C independent of both h and n such that*

$$\left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_i \leq \frac{Ch^2\left|\log h\right|}{1-\rho}; \quad i = 1, 2. \tag{3.5}$$

Proof. We know from standard L^∞-error estimate for linear problem (see [13]) that there exists a constant C independent of h such that

$$\left\|u^0 - u_h^0\right\|_{L^\infty(\Omega)} \leq Ch^2|\log h|. \tag{3.6}$$

Now, since $1/2 < \rho < 1$, then $1 < \rho/(1-\rho)$, and therefore

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq Ch^2|\log h| \leq \frac{\rho Ch^2|\log h|}{1-\rho}. \tag{3.7}$$

Let us now prove (3.5) by induction. Indeed for $n = 1$, using the Proposition 2.7, we have in domain 1

$$\begin{aligned}
\left\|u_1^1 - u_{1h}^1\right\|_1 &\leq \left\|u_1^1 - w_{1h}^1\right\|_1 + \left\|w_1^1 - u_{1h}^1\right\|_1 \\
&\leq Ch^2|\log h| + \left\|w_1^1 - u_{1h}^1\right\|_1 \\
&\leq Ch^2|\log h| + \max\left\{\left(\frac{1}{\beta}\right)\left\|f\left(u_1^1\right) - f\left(u_{1h}^1\right)\right\|_1, \left|u_2^0 - u_{2h}^0\right|_1\right\} \\
&\leq Ch^2|\log h| + \max\left\{\left(\frac{1}{\beta}\right)\left\|f\left(u_1^1\right) - f\left(u_{1h}^1\right)\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\right\} \\
&\leq Ch^2|\log h| + \max\left\{\rho\left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\right\}.
\end{aligned} \tag{3.8}$$

We then have to distinguish between two cases:

(1)

$$\max\left\{\rho\left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\right\} = \rho\left\|u_1^1 - u_{1h}^1\right\|_1 \tag{3.9}$$

or

(2)

$$\max\left\{\rho\left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\right\} = \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.10}$$

Case (1) implies

$$\begin{gathered}
\left\|u_1^1 - u_{1h}^1\right\|_1 \leq Ch^2|\log h| + \rho\left\|u_1^1 - u_{1h}^1\right\|_1, \\
\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho\left\|u_1^1 - u_{1h}^1\right\|_1.
\end{gathered} \tag{3.11}$$

Then

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{Ch^2|\log h|}{1-\rho},$$
$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{\rho Ch^2|\log h|}{1-\rho}. \tag{3.12}$$

Case (2) implies

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq Ch^2|\log h| + \left\|u_2^0 - u_{2h}^0\right\|_2, \tag{3.13}$$

$$\rho\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.14}$$

So, by multiplying (3.13) by ρ we get

$$\rho\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho Ch^2|\log h| + \rho\left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.15}$$

So, $\rho\|u_1^1 - u_{1h}^1\|_1$ is bounded by both $\rho Ch^2|\log h| + \rho\|u_2^0 - u_{2h}^0\|_2$ and $\|u_2^0 - u_{2h}^0\|_2$. This implies that

(a)

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho Ch^2|\log h| + \rho\left\|u_2^0 - u_{2h}^0\right\|_2 \tag{3.16}$$

or

(b)

$$\rho Ch^2|\log h| + \rho\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.17}$$

That is,

(a)

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho Ch^2|\log h|}{1-\rho} \tag{3.18}$$

or

(b)

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \geq \frac{\rho Ch^2|\log h|}{1-\rho}. \tag{3.19}$$

It follows that only the case (a) is true, that is,

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 |\log h|}{1-\rho}. \tag{3.20}$$

Thus

$$\begin{aligned}\left\|u_1^1 - u_{1h}^1\right\|_1 &\leq Ch^2|\log h| + \left\|u_2^0 - u_{2h}^0\right\|_2 \\ &\leq Ch^2|\log h| + \frac{\rho C h^2|\log h|}{1-\rho} \\ &\leq \frac{Ch^2|\log h|}{1-\rho}.\end{aligned} \tag{3.21}$$

So, in both cases (1) and (2), we have

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{Ch^2|\log h|}{1-\rho}. \tag{3.22}$$

Similarly, we have in domain 2

$$\begin{aligned}\left\|u_2^1 - u_{2h}^1\right\|_2 &\leq Ch^2|\log h| + \left\|w_2^1 - u_{2h}^1\right\|_2 \\ &\leq Ch^2|\log h| + \max\left\{\left(\frac{1}{\beta}\right)\left\|f\left(u_2^1\right) - f\left(u_{2h}^1\right)\right\|_2, \left|u_1^1 - u_{1h}^1\right|_2\right\} \\ &\leq Ch^2|\log h| + \max\left\{\left(\frac{1}{\beta}\right)\left\|f\left(u_2^1\right) - f\left(u_{2h}^1\right)\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\right\} \\ &\leq Ch^2|\log h| + \max\left\{\rho\left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\right\}.\end{aligned} \tag{3.23}$$

So

(1)

$$\max\left\{\rho\left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\right\} = \rho\left\|u_2^1 - u_{2h}^1\right\|_2 \tag{3.24}$$

or

(2)

$$\max\left\{\rho\left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\right\} = \left\|u_1^1 - u_{1h}^1\right\|_1. \tag{3.25}$$

Case (1) implies

$$\begin{aligned} \left\|u_2^1 - u_{2h}^1\right\|_2 &\le Ch^2|\log h| + \rho\left\|u_2^1 - u_{2h}^1\right\|_2, \\ \left\|u_1^1 - u_{1h}^1\right\|_1 &\le \rho\left\|u_2^1 - u_{2h}^1\right\|_2. \end{aligned} \tag{3.26}$$

So

$$\begin{aligned} \left\|u_2^1 - u_{2h}^1\right\|_2 &\le \frac{Ch^2|\log h|}{1-\rho}, \\ \left\|u_1^1 - u_{1h}^1\right\|_1 &\le \rho\left\|u_2^1 - u_{2h}^1\right\|_2 \\ &\le \frac{\rho Ch^2|\log h|}{1-\rho} \\ &\le \frac{Ch^2|\log h|}{1-\rho}, \end{aligned} \tag{3.27}$$

while case (2) implies

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \le Ch^2|\log h| + \left\|u_1^1 - u_{1h}^1\right\|_1, \tag{3.28}$$

$$\rho\left\|u_2^1 - u_{2h}^1\right\|_1 \le \left\|u_1^1 - u_{1h}^1\right\|_1. \tag{3.29}$$

So, by multiplying (3.28) by ρ we get

$$\rho\left\|u_2^1 - u_{2h}^1\right\|_2 \le \rho Ch^2|\log h| + \rho\left\|u_1^1 - u_{1h}^1\right\|_1, \tag{3.30}$$

and hence $\rho\|u_2^1 - u_{2h}^1\|_2$ is bounded by both $\rho Ch^2|\log h| + \rho\|u_1^1 - u_{1h}^1\|_1$ and $\|u_1^1 - u_{1h}^1\|_1$. Then

(a)

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \le \rho Ch^2|\log h| + \rho\left\|u_1^1 - u_{1h}^1\right\|_1 \tag{3.31}$$

or

(b)

$$Ch^2|\log h| + \rho\left\|u_1^1 - u_{1h}^1\right\|_1 \le \left\|u_1^1 - u_{1h}^1\right\|_1, \tag{3.32}$$

which implies

(a)

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{\rho C h^2 |\log h|}{1-\rho} < \frac{C h^2 |\log h|}{1-\rho} \tag{3.33}$$

or

(b)

$$\frac{\rho C h^2 |\log h|}{1-\rho} \leq \left\|u_1^1 - u_{1h}^1\right\|_1 < \frac{C h^2 |\log h|}{1-\rho}. \tag{3.34}$$

Hence, (a) and (b) are true because they both coincide with (3.22). So, there is either a contradiction and thus case (2) is impossible or case (2) is possible only if

$$\left\|u_1^1 - u_{1h}^1\right\|_1 = \rho C h^2 |\log h| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1, \tag{3.35}$$

that is,

$$\left\|u_1^1 - u_{1h}^1\right\|_1 = \frac{\rho C h^2 |\log h|}{1-\rho}. \tag{3.36}$$

Thus

$$\begin{aligned} \left\|u_2^1 - u_{2h}^1\right\|_2 &\leq C h^2 |\log h| + \left\|u_1^1 - u_{1h}^1\right\|_1 \\ &\leq C h^2 |\log h| + \frac{\rho C h^2 |\log h|}{1-\rho} \\ &\leq \frac{C h^2 |\log h|}{1-\rho}. \end{aligned} \tag{3.37}$$

That is, both cases (1) and (2) imply

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \frac{C h^2 |\log h|}{1-\rho}. \tag{3.38}$$

Now, let us assume that

$$\left\|u_2^n - u_{2h}^n\right\|_2 \leq \frac{C h^2 |\log h|}{1-\rho} \tag{3.39}$$

and prove that

$$\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1 \le \frac{1}{1-\rho}Ch^2|\log h|,$$
$$\left\|u_2^{n+1}-u_{2h}^{n+1}\right\|_2 \le \frac{Ch^2|\log h|}{1-\rho}. \tag{3.40}$$

Indeed, we have in domain 1

$$\begin{aligned}\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1 &\le Ch^2|\log h| + \left\|w_1^{n+1}-u_{1h}^{n+1}\right\|_1\\ &\le Ch^2|\log h| + \max\left\{\left(\frac{1}{\beta}\right)\left\|f\left(u_1^{n+1}\right)-f\left(u_{1h}^{n+1}\right)\right\|_1, \left|u_2^n-u_{2h}^n\right|_1\right\}\\ &\le Ch^2|\log h| + \max\left\{\left(\frac{1}{\beta}\right)\left\|f\left(u_1^{n+1}\right)-f\left(u_{1h}^{n+1}\right)\right\|_1, \left\|u_2^n-u_{2h}^n\right\|_2\right\}\\ &\le Ch^2|\log h| + \max\left\{\rho\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1, \left\|u_2^n-u_{2h}^n\right\|_2\right\}.\end{aligned} \tag{3.41}$$

We have again to distinguish between two cases

(1)

$$\max\left\{\rho\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1, \left\|u_2^n-u_{2h}^n\right\|_2\right\} = \rho\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1 \tag{3.42}$$

or

(2)

$$\max\left\{\rho\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1, \ \left\|u_2^n-u_{2h}^n\right\|_2\right\} = \left\|u_2^n-u_{2h}^n\right\|_2. \tag{3.43}$$

Case (1) implies

$$\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1 \le Ch^2|\log h| + \rho\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1,$$
$$\left\|u_2^n-u_{2h}^n\right\|_2 \le \rho\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1. \tag{3.44}$$

Then

$$\left\|u_1^{n+1}-u_{1h}^{n+1}\right\|_1 \le \frac{Ch^2|\log h|}{1-\rho},$$
$$\left\|u_2^n-u_{2h}^n\right\|_2 \le \frac{Ch^2|\log h|}{1-\rho}. \tag{3.45}$$

Case (2) implies

$$\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq Ch^2|\log h| + \|u_2^n - u_{2h}^n\|_2, \tag{3.46}$$

$$\rho\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq \|u_2^n - u_{2h}^n\|_2. \tag{3.47}$$

So, by multiplying (3.46) by ρ, we get

$$\begin{gathered}\rho\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq \rho Ch^2|\log h| + \rho\|u_2^n - u_{2h}^n\|_2, \\ \rho\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq \|u_2^n - u_{2h}^n\|_2.\end{gathered} \tag{3.48}$$

Hence, we can see that $\rho\|u_1^{n+1} - u_{1h}^{n+1}\|_1$ is bounded by both $\rho Ch^2|\log h| + \rho\|u_2^n - u_{2h}^n\|_2$ and $\|u_2^n - u_{2h}^n\|_2$. So, we have

$$\|u_2^n - u_{2h}^n\|_2 \leq \rho Ch^2|\log h| + \rho\|u_2^n - u_{2h}^n\|_2 \tag{3.49}$$

or

$$\rho Ch^2|\log h| + \rho\|u_2^n - u_{2h}^n\|_2 \leq \|u_2^n - u_{2h}^n\|_2, \tag{3.50}$$

which implies

(a)

$$\|u_2^n - u_{2h}^n\|_2 \leq \frac{\rho Ch^2|\log h|}{1-\rho} < \frac{Ch^2|\log h|}{1-\rho} \tag{3.51}$$

or

(b)

$$\frac{\rho Ch^2|\log h|}{1-\rho} \leq \|u_2^n - u_{2h}^n\|_2 < \frac{Ch^2|\log h|}{1-\rho}. \tag{3.52}$$

So, (a) and (b) are true because they both coincide with (3.39). This means that there is either contradiction and then case (2) is impossible, or case (2) is possible and then we must have

$$\rho Ch^2|\log h| + \rho\|u_2^n - u_{2h}^n\|_2 = \|u_2^n - u_{2h}^n\|_2, \tag{3.53}$$

that is

$$\left\|u_2^n - u_{2h}^n\right\|_2 = \frac{\rho C h^2 |\log h|}{1-\rho}. \tag{3.54}$$

So

$$\begin{aligned}\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 &\le Ch^2|\log h| + \left\|u_2^n - u_{2h}^n\right\|_2 \\ &\le Ch^2|\log h| + \frac{\rho C h^2|\log h|}{1-\rho} \\ &= \frac{Ch^2|\log h|}{1-\rho}.\end{aligned} \tag{3.55}$$

Thus, both cases (1) and (2) imply

$$\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \le \frac{Ch^2|\log h|}{1-\rho}. \tag{3.56}$$

Estimate in domain 2 can be proved similarly using estimate (3.56). □

Theorem 3.3. *Let $h = \max(h_1, h_2)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\|u_i - u_{ih}^{n+1}\right\|_1 \le Ch^2|\log h|, \quad \forall i = 1, 2. \tag{3.57}$$

Proof. Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted.

Indeed, Let $k = k_1 k_2$. Then making use of Theorem 2.8 and Lemma 3.2, we get

$$\begin{aligned}\left\|u_1 - u_{1h}^{n+1}\right\|_1 &\le \left\|u_1 - u_1^{n+1}\right\|_1 + \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \\ &\le k_1^n k_2^n \left|u^0 - u\right|_1 + \frac{Ch^2|\log h|}{1-\rho} \\ &\le k^{2n}\left|u^0 - u\right|_1 + \frac{Ch^2|\log h|}{1-\rho}.\end{aligned} \tag{3.58}$$

So, for n large enough, we have

$$k^{2n} \leq h^2 \tag{3.59}$$

and thus

$$\begin{aligned} \left\| u_1 - u_{1h}^{n+1} \right\|_1 &\leq Ch^2 + Ch^2 |\log h| \\ &\leq Ch^2 |\log h|, \end{aligned} \tag{3.60}$$

which is the desired result. □

Conclusion

We have established an error estimate for the finite element Schwarz alternating method for a nonlinear elliptic PDE on two subdomains with nonmatching grids combining a geometrical convergence result due to Lions and a standard finite element L^∞-error analysis for linear elliptic equations.The same approach may be extended to other types of problems such as linear parabolic PDEs (see [2]) and singularly perturbed advection-diffusion equations (see [14]).

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