Research Article

On the Neutrix Composition of the Delta and Inverse Hyperbolic Sine Functions

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Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. In the ordinary sense, the composition $\delta^{(s)}[(\sinh^{-1}x_+)^r]$ does not exist. In this study, it is proved that the neutrix composition $\delta^{(s)}[(\sinh^{-1}x_+)^r]$ exists and is given by $\delta^{(s)}[(\sinh^{-1}x_+)^r] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^k \binom{k}{i} ((-1)^k r c_{s,k,i} / 2^{k+1} k!) \delta^{(k)}(x)$, for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$, where $c_{s,k,i} = (-1)^s s! [(k-2i+1)^{rs-1} + (k-2i-1)^{rs+r-1}] / (2(rs+r-1)!)$. Further results are also proved.

1. Introduction

In the following, we let $\mathfrak{D}$ be the space of infinitely differentiable functions with compact support, let $\mathfrak{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$, and let $\mathfrak{D}'$ be the space of distributions defined on $\mathfrak{D}$.

Now, let $\rho(x)$ be a function in $\mathfrak{D}[-1, 1]$ having the following properties:

- (i) $\rho(x) \geq 0$,
- (ii) $\rho(x) = \rho(-x)$,
- (iii) $\int_{-1}^1 \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is

an arbitrary distribution in $\mathfrak{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(t) \rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

Since the theory of distributions is a linear theory, thus we can extend some of the operations which are valid for ordinary functions to the space of distributions and such operations are called regular operations such as: addition, multiplication by scalars; see [1]. Other operations can be defined only for a particular class of distributions or for certain restricted subclasses of distributions; these are called irregular operations such as: multiplication of distributions, convolution products, and composition of distributions; see [2–4]. Thus, there have been several attempts recently to define distributions of the form $F(f(x))$ in $\mathfrak{D}'$, where F and f are distributions in $\mathfrak{D}'$; see for example [5–8]. In the following, we are going to consider an alternative approach. As a starting point, we look at the following definition which is a generalization of Gel'fand and Shilov's definition of the composition involving the delta function [9], and was given in [6].

Definition 1.1. Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) , with $-\infty < a < b < \infty$, if

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle, \quad (1.1)$$

for all φ in $\mathfrak{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and N is the neutrix, see [10], having domain N' the positive and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \lambda > 0, \quad r = 1, 2, \dots \quad (1.2)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle, \quad (1.3)$$

for all φ in $\mathfrak{D}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$. The definition of the neutrix composition of distributions was originally given in [10] but was then simply called the composition of distributions.

The following three theorems were proved in [11], [8], and [12], respectively.

Theorem 1.2. *The neutrix composition $\delta^{(s)}(\operatorname{sgn} x |x|^\lambda)$ exists and*

$$\delta^{(s)}(\operatorname{sgn} x |x|^\lambda) = 0, \quad (1.4)$$

for $s = 0, 1, 2, \dots$ and $(s+1)\lambda = 1, 3, \dots$, and

$$\delta^{(s)}(\operatorname{sgn} x |x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!} \delta^{((s+1)\lambda-1)}(x), \quad (1.5)$$

for $s = 0, 1, 2, \dots$, and $(s+1)\lambda = 2, 4, \dots$

Theorem 1.3. The neutrix compositions $\delta^{(2s-1)}(\operatorname{sgn} x |x|^{1/s})$ and $\delta^{(s-1)}(|x|^{1/s})$ exist and

$$\begin{aligned} \delta^{(2s-1)}(\operatorname{sgn} x |x|^{1/s}) &= \frac{1}{2} (2s)! \delta'(x), \\ \delta^{(s-1)}(|x|^{1/s}) &= (-1)^{s-1} \delta(x), \end{aligned} \quad (1.6)$$

for $s = 1, 2, \dots$

Theorem 1.4. The neutrix composition $\delta^{(s)}(\sinh^{-1} x_+^{1/r})$ exists and

$$\delta^{(s)}\left[(\sinh^{-1} x_+)^{1/r}\right] = \sum_{k=0}^{(s+1)/r-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^k r c_{s,k,i}}{2^{k+1} k!} \delta^{(k)}(x), \quad (1.7)$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$, where

$$c_{r,s,k,i} = \frac{(-1)^s s! \left[(k-2i+1)^{rs+r-1} + (k-2i-1)^{rs+r-1} \right]}{2(rs+r-1)!}. \quad (1.8)$$

The next two theorems were proved in [13].

Theorem 1.5. The neutrix composition $\delta^{(s)}[\ln^r(1+|x|)]$ exists and

$$\delta^{(s)}[\ln^r(1+|x|)] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{s-i} \left[1 + (-1)^k \right] s! (i+1)^{rs+r-1}}{2r(rs+r-1)! k!} \delta^{(k)}(x). \quad (1.9)$$

for $s = 0, 1, 2, \dots$, and $r = 1, 2, \dots$

In particular, the composition $\delta[\ln(1+|x|)]$ exists and

$$\delta[\ln(1+|x|)] = \delta(x). \quad (1.10)$$

Theorem 1.6. The neutrix composition $\delta^{(s)}[\ln(1+|x|^{1/r})]$ exists and

$$\delta^{(s)}\left[\ln\left(1+|x|^{1/r}\right)\right] = \sum_{k=0}^{m-1} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{r+s+i-1} \left[1 + (-1)^k \right] r(i+1)^s}{2k!} \delta^{(k)}(x), \quad (1.11)$$

for $s=0, 1, 2, \dots$ and $r=2, 3, \dots$, where m is the smallest non-negative integer greater than $(s-r+1)r^{-1}$.

In particular, the composition $\delta^{(s)}[\ln(1 + |x^{1/r}|)]$ exists and

$$\delta^{(s)}\left[\ln\left(1 + \left|x^{1/r}\right|\right)\right] = 0, \quad (1.12)$$

for $s = 0, 1, 2, \dots, r-2$ and $r = 2, 3, \dots$ and

$$\delta^{(r-1)}\left[\ln\left(1 + \left|x^{1/r}\right|\right)\right] = (-1)^{r-1} r! \delta(x), \quad (1.13)$$

for $r = 2, 3, \dots$

2. Main Results

We now prove the following theorem.

Theorem 2.1. The neutrix composition $\delta^{(s)}[(\sinh^{-1} x_+)^r]$ exists and

$$\delta^{(s)}\left[(\sinh^{-1} x_+)^r\right] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^k r c_{s,k,i}}{2^{k+1} k!} \delta^{(k)}(x), \quad (2.1)$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$, where

$$c_{r,s,k,i} = \frac{(-1)^s s! \left[(k-2i+1)^{rs+r-1} + (k-2i-1)^{rs+r-1} \right]}{2(rs+r-1)!}. \quad (2.2)$$

In particular, the neutrix composition $\delta(\sinh^{-1} x_+)$ exists and

$$\delta(\sinh^{-1} x_+) = \frac{1}{2} \delta(x). \quad (2.3)$$

Proof. To prove (2.1), we first of all evaluate

$$\int_{-1}^1 \delta_n^{(s)}\left[(\sinh^{-1} x_+)^r\right] x^k dx. \quad (2.4)$$

We have

$$\begin{aligned} \int_{-1}^1 \delta_n^{(s)}\left[(\sinh^{-1} x_+)^r\right] x^k dx &= n^{s+1} \int_{-1}^1 \rho^{(s)}\left[(n \sinh^{-1} x_+)^r\right] x^k dx \\ &= n^{s+1} \int_0^1 \rho^{(s)}\left[n(\sinh^{-1} x)^r\right] x^k dx \\ &\quad + n^{s+1} \int_{-1}^0 \rho^{(s)}(0) x^k dx \\ &= I_1 + I_2. \end{aligned} \quad (2.5)$$

It is obvious that

$$N - \lim_{n \rightarrow \infty} I_2 = N - \lim_{n \rightarrow \infty} \int_{-1}^0 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^k dx = 0, \quad (2.6)$$

for $k = 0, 1, 2, \dots$

Making the substitution $t = n(\sinh^{-1} x)^r$, we have for large enough n

$$\begin{aligned} I_1 &= \frac{n^{s-r+1}}{r} \int_0^1 t^{1/(r-1)} \sinh^k \left(\frac{t}{n} \right)^{1/r} \cosh \left(\frac{t}{n} \right)^{1/r} \rho^{(s)}(t) dt \\ &\quad \times \int_0^1 t^{1/(r-1)} \left\{ \exp \left[(k-2i+1) \left(\frac{t}{n} \right)^{1/r} \right] + \exp \left[(k-2i-1) \left(\frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) dt, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} &n^{(s-1)/(r+1)} \int_0^1 t^{1/(r-1)} \left\{ \exp \left[(k-2i+1) \left(\frac{t}{n} \right)^{1/r} \right] + \exp \left[(k-2i-1) \left(\frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) dt \\ &= \sum_{p=0}^{\infty} \int_0^1 \frac{[(k-2i+1)^p + (k-2i-1)^p] t^{(p/r)+(1/r)-1}}{p! n^{(p/r)+(1/r)-s-1}} \rho^{(s)}(t) dt. \end{aligned} \quad (2.8)$$

It follows that

$$\begin{aligned} &N - \lim_{n \rightarrow \infty} n^{s-1/r+1} \int_0^1 t^{1/(r-1)} \left\{ \exp \left[(k-2i+1) \left(\frac{t}{n} \right)^{1/r} \right] + \exp \left[(k-2i-1) \left(\frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) dt \\ &= \frac{(-1)^s s! [(k-2i+1)^{rs+r-1} + (k-2i-1)^{rs+r-1}]}{2(rs+r-1)!} \\ &= c_{r,s,k,i}, \end{aligned} \quad (2.9)$$

and by applying the neutrix limit we obtain

$$N - \lim_{n \rightarrow \infty} I_1 = N - \lim_{n \rightarrow \infty} \int_0^1 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^k dx = \frac{1}{2^{k+1} r} \sum_{i=0}^k \binom{k}{i} (-1)^i c_{r,s,k,i} \quad (2.10)$$

for $k = 0, 1, 2, \dots$

When $k = sr + r$, we have

$$\begin{aligned}
 |I_1| &= \int_0^1 \left| \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^{sr+r} \right| dx \\
 &= n^{s+1} \int_0^1 \left| \rho_n^{(s)} \left[n \left(\sinh^{-1} x \right)^r \right] x^{sr+r} \right| dx \\
 &\leq \frac{n^{(s-1)/(r+1)}}{2^{sr+r}} \exp(sr+r+1) \int_0^1 \left| \left[1 - \exp \left[-2 \left(\frac{t}{n} \right)^{1/r} \right] \right]^{sr+r} \rho^{(s)}(t) \right| dt \\
 &= \frac{n^{(s-1)/(r+1)}}{2^{sr+r}} \exp(sr+r+1) \int_0^1 \left[2 \left(\frac{t}{n} \right)^{1/r} + O(n^{-2/r}) \right]^{sr+r} \left| \rho^{(s)}(t) \right| dt \\
 &\leq n^{-1/r} \exp(sr+r+1) \int_0^1 \left[1 + O(n^{-2/r}) \right] \left| \rho^{(s)}(t) \right| dt \\
 &= O(n^{-1/r}).
 \end{aligned} \tag{2.11}$$

Thus, if ψ is an arbitrary continuous function, then

$$\lim_{n \rightarrow \infty} \int_0^1 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^{sr+r} \psi(x) dx = 0. \tag{2.12}$$

We also have

$$\int_{-1}^0 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] \psi(x) dx = n^{s+1} \int_{-1}^0 \rho^{(s)}(0) \psi(x) dx, \tag{2.13}$$

and it follows that

$$N - \lim_{n \rightarrow \infty} \int_{-1}^0 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] \psi(x) dx = 0. \tag{2.14}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1, 1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{rs+r}}{(rs+r)!} \varphi^{(rs+r)}(\xi x), \tag{2.15}$$

where $0 < \xi < 1$, and so

$$\begin{aligned}
& N - \lim_{n \rightarrow \infty} \left\langle \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^{1/r} \right], \varphi(x) \right\rangle \\
&= N - \lim_{n \rightarrow \infty} \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} \int_0^1 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^k dx \\
&\quad + N - \lim_{n \rightarrow \infty} \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^0 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^k dx \\
&\quad + \lim_{n \rightarrow \infty} \frac{1}{(sr+r)!} \int_0^1 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^{sr+r} \varphi^{(sr+r)}(\xi x) dx \\
&\quad + \lim_{n \rightarrow \infty} \frac{1}{(sr+r)!} \int_{-1}^0 \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] x^{sr+r} \varphi^{(sr+r)}(\xi x) dx \\
&= \sum_{k=0}^{sr+r-1} \sum_{i=0}^k \binom{k}{i} \frac{r c_{r,s,k,i} \varphi^{(k)}(0)}{2^{k+1} k!} + 0 \\
&= \sum_{k=0}^{sr+r-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^k r c_{r,s,k,i}}{2^{k+1} k!} \left\langle \delta^{(k)}(x), \varphi(x) \right\rangle,
\end{aligned} \tag{2.16}$$

on using (2.3) to (2.14). This proves (2.1) on the interval $(-1, 1)$.

It is clear that $\delta^{(s)}[(\sinh^{-1} x_+)^r] = 0$ for $x > 0$ and so (2.1) holds for $x > -1$.

Now, suppose that φ is an arbitrary function in $\mathfrak{D}[a, b]$, where $a < b < 0$. Then,

$$\int_a^b \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] \varphi(x) dx = n^{s+1} \int_a^b \rho^{(s)}(0) \varphi(x) dx \tag{2.17}$$

and so

$$N - \lim_{n \rightarrow \infty} \int_a^b \delta_n^{(s)} \left[\left(\sinh^{-1} x_+ \right)^r \right] \varphi(x) dx = 0. \tag{2.18}$$

It follows that $\delta^{(s)}[(\sinh^{-1} x_+)^r] = 0$ on the interval (a, b) . Since a and b are arbitrary, we see that (2.1) holds on the real line. This completes the proof of the theorem. $\square$

Corollary 2.2. *The neutrix composition $\delta^{(s)}[(\sinh^{-1}|x|)^r]$ exists and*

$$\delta^{(s)} \left[\left(\sinh^{-1}|x| \right)^r \right] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^k \binom{k}{i} \frac{[(-1)^k + 1] c_{r,s,k,i}}{2^{k+1} k!} \delta^{(k)}(x), \tag{2.19}$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$

In particular, the composition $\delta(\sinh^{-1}|x|)$ exists and

$$\delta(\sinh^{-1}|x|) = \frac{1}{2}\delta(x). \quad (2.20)$$

Proof. To prove (2.19), we note that

$$\begin{aligned} \int_{-1}^1 \delta_n^{(s)} \left[\left(\sinh^{-1}|x| \right)^r \right] x^k dx &= n^{s+1} \int_{-1}^1 \rho^{(s)} \left[\left(n \sinh^{-1}|x| \right)^r \right] x^k dx \\ &= n^{s+1} \left[1 + (-1)^k \right] \int_0^1 \rho^{(s)} \left[n \left(\sinh^{-1}x \right)^r \right] x^k dx, \end{aligned} \quad (2.21)$$

and (2.19) now follows as above.

Equation (2.20) follows on noting that in the particular case $s = 0$, the usual limit holds in (2.10). This completes the proof of the corollary. $\square$

Theorem 2.3. The neutrix composition $\delta^{(2s-1)}[\sinh^{-1}(\operatorname{sgn} x \cdot x^2)]$ exists and

$$\delta^{(2s-1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] = \sum_{k=0}^{2s-1} \sum_{i=0}^{k+1} \binom{k}{i} \frac{(-1)^k b_{s,k,i}}{2^{k+1}(2k+1)!} \delta^{(k)}(x), \quad (2.22)$$

for $s = 1, 2, \dots$, where

$$b_{s,k,i} = (k - 2i + 1)^{2s-1} + (k - 2i - 1)^{2s-1}. \quad (2.23)$$

Proof. To prove (2.22), we now have to evaluate

$$\int_{-1}^1 \delta_n^{(2s-1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] x^k dx. \quad (2.24)$$

We have

$$\begin{aligned} \int_{-1}^1 \delta_n^{(2s-1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] x^k dx &= n^{2s} \int_{-1}^1 \rho^{(2s-1)} \left[n \sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] x^k dx \\ &= \begin{cases} 2n^{2s} \int_0^1 \rho^{(2s-1)} \left[n \left(\sinh^{-1}x^2 \right) \right] x^k dx, & k \text{ odd,} \\ 0, & k \text{ even.} \end{cases} \end{aligned} \quad (2.25)$$

Making the substitution $t = n(\sinh^{-1} x^2)$, we have for large enough n

$$\begin{aligned}
 & \int_{-1}^1 \delta_n^{(2s-1)} \left[\sinh^{-1} (\operatorname{sgn} x \cdot x^2) \right] x^k dx \\
 &= 2n^{2s} \int_0^1 \rho^{(2s-1)} \left[n(\sinh^{-1} x^2) \right] x^{2k+1} dx \\
 &= \frac{n^{2s-1}}{2^{k+1}} \sum_{i=0}^k \binom{k}{i} (-1)^i \int_0^1 \left\{ \exp \left[\frac{(k-2i+1)t}{n} \right] + \exp \left[\frac{(k-2i-1)t}{n} \right] \right\} \rho^{(2s-1)}(t) dt,
 \end{aligned} \tag{2.26}$$

where

$$\begin{aligned}
 & n^{2s-1} \int_0^1 \left\{ \exp \left[\frac{(k-2i+1)t}{n} \right] + \exp \left[\frac{(k-2i-1)t}{n} \right] \right\} \rho^{(s)}(t) dt \\
 &= \sum_{p=0}^{\infty} \int_0^1 \frac{[(k-2i+1)^p + (k-2i-1)^p] t^p}{p! n^{p-2s+1}} \rho^{(2s-1)}(t) dt.
 \end{aligned} \tag{2.27}$$

It follows that

$$\begin{aligned}
 & N - \lim_{n \rightarrow \infty} n^{2s-1} \int_0^1 \left\{ \exp \left[\frac{(k-2i+1)t}{n} \right] + \exp \left[\frac{(k-2i-1)t}{n} \right] \right\} \rho^{(s)}(t) dt \\
 &= N - \lim_{n \rightarrow \infty} \sum_{p=0}^{\infty} \int_0^1 \frac{[(k-2i+1)^p + (k-2i-1)^p] t^p}{p! n^{p-2s+1}} \rho^{(2s-1)}(t) dt \\
 &= \frac{-(k-2i+1)^{2s-1} + (k-2i-1)^{2s-1}}{2} \\
 &= \frac{b_{s,k,i}}{2},
 \end{aligned} \tag{2.28}$$

and so by using the neutrix limit, we have

$$N - \lim_{n \rightarrow \infty} \int_{-1}^1 \delta_n^{(2s-1)} \left[\sinh^{-1} (\operatorname{sgn} x \cdot x^2) \right] x^{2k+1} dx = \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{i+1} b_{s,k,i}}{2^{k+1}}, \tag{2.29}$$

for $k = 0, 1, 2, \dots$

When $k = 2s$, we have

$$\begin{aligned}
 \int_{-1}^1 \left| \delta_n^{(2s-1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] x^{4s+1} \right| dx &= n^{2s} \int_{-1}^1 \rho^{(2s-1)} \left[n \left(\sinh^{-1} x^2 \right) \right] x^{4s+1} dx \\
 &\leq \frac{n^{2s-1}}{2^{s-1}} \exp(s+1) \int_{-1}^1 \left| \left[1 - \exp\left(-\frac{2t}{n}\right) \right]^{2s} \rho^{(2s-1)}(t) \right| dt \\
 &= \frac{n^{2s-1}}{2^{s-1}} \exp(s+1) \int_{-1}^1 \left| \left[\frac{2t}{n} + O(n^{-2}) \right]^{2s} \rho^{(2s-1)}(t) \right| dt \\
 &\leq 2^{2s+1} n^{-1} \exp(s+1) \int_{-1}^1 \left| 1 + O(n^{-2/r}) \right| \rho^{(2s-1)}(t) dt \\
 &= O(n^{-1}).
 \end{aligned} \tag{2.30}$$

Thus, if φ is an arbitrary continuous function, then

$$\lim_{n \rightarrow \infty} \int_{-1}^1 \delta_n^{(2s-1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] x^{4s+1} \varphi(x) dx = 0. \tag{2.31}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1, 1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{4s} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{4s+1}}{(4s+1)!} \varphi^{(4s+1)}(\xi x), \tag{2.32}$$

where $0 < \xi < 1$, and so

$$\begin{aligned}
 N - \lim_{n \rightarrow \infty} \left\langle \delta_n^{(2s-1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right], \varphi(x) \right\rangle \\
 &= N - \lim_{n \rightarrow \infty} \sum_{k=0}^{2s-1} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \int_{-1}^1 \delta_n^{(2s-1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] x^{2k+1} dx \\
 &\quad + \lim_{n \rightarrow \infty} \frac{1}{(4s+1)!} \int_{-1}^1 \delta_n^{(4s+1)} \left[\sinh^{-1}(\operatorname{sgn} x \cdot x^2) \right] x^{4s+1} \varphi^{(4s+1)}(\xi x) dx \\
 &= \sum_{k=0}^{2s-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{i+1} b_{s,k,i} \varphi^{(k)}(0)}{2^{k+1} (2k+1)!} + 0 \\
 &= \sum_{k=0}^{2s-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{i+k+1} b_{s,k,i}}{2^{k+1} (2k+1)!} \left\langle \delta^{(k)}(x), \varphi(x) \right\rangle,
 \end{aligned} \tag{2.33}$$

on using (2.25) to (2.31), proving (2.22) on the interval $(-1, 1)$. However, it is clear that $\delta_n^{(2s-1)}[\sinh^{-1}(\operatorname{sgn} x \cdot x^2)] = 0$ for $|x| > 0$ and so (2.22) holds on the real line, completing the proof of the theorem. $\square$

Corollary 2.4. *The composition $\delta'[\sinh^{-1}\operatorname{sgn} x \cdot x^2]$ exists and*

$$\delta'[\sinh^{-1}(\operatorname{sgn} x \cdot x^2)] = \frac{\delta'(x)}{4 \cdot 3!} - 2\delta(x). \quad (2.34)$$

Proof. To prove (2.34) note that in the particular case $s = 1$, the usual limits hold and then (2.34) is a particular case of (2.22). This completes the proof of the corollary. $\square$

For further related results on the neutrix operation of distributions, see [12–22] and [2, 3, 23].

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