Research Article

Optimal Inequalities between Harmonic, Geometric, Logarithmic, and Arithmetic-Geometric Means

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We find the least values p , q , and s in $(0, 1/2)$ such that the inequalities $H(pa + (1-p)b, pb + (1-p)a) > AG(a, b)$, $G(qa + (1-q)b, qb + (1-q)a) > AG(a, b)$, and $L(sa + (1-s)b, sb + (1-s)a) > AG(a, b)$ hold for all $a, b > 0$ with $a \neq b$, respectively. Here $AG(a, b)$, $H(a, b)$, $G(a, b)$, and $L(a, b)$ denote the arithmetic-geometric, harmonic, geometric, and logarithmic means of two positive numbers a and b , respectively.

1. Introduction

The classical arithmetic-geometric mean $AG(a, b)$ of two positive real numbers a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{aligned} a_0 &= a, & b_0 &= b, \\ a_{n+1} &= \frac{a_n + b_n}{2}, & b_{n+1} &= \sqrt{a_n b_n}. \end{aligned} \tag{1.1}$$

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (a - b)/(\log a - \log b)$, $I(a, b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $A(a, b) = (a + b)/2$, and $M_p(a, b) = [(a^p + b^p)/2]^{1/p}$ ($p \neq 0$) and $M_0(a, b) = \sqrt{ab}$ be the harmonic, geometric, logarithmic, identric, arithmetic, and p -th

power means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that

$$\begin{aligned} \min\{a, b\} < H(a, b) = M_{-1}(a, b) < G(a, b) = M_0(a, b) < L(a, b) \\ < I(a, b) < A(a, b) = M_{-1}(a, b) < \max\{a, b\} \end{aligned} \quad (1.2)$$

for all $a, b > 0$ with $a \neq b$.

Recently, the inequalities for means have been the subject of intensive research. In particular, many remarkable inequalities for arithmetic-geometric mean can be found in the literature [1–9].

Carlson and Vuorinen [2], and Bracken [9] proved that

$$L(a, b) < AG(a, b) \quad (1.3)$$

for all $a, b > 0$ with $a \neq b$.

In [3], Vamanamurthy and Vuorinen established the following inequalities:

$$\begin{aligned} AG(a, b) < I(a, b) < A(a, b), \\ AG(a, b) < M_{1/2}(a, b), \\ AG(a, b) < \frac{\pi}{2} L(a, b), \\ M_1(a, b) < \frac{AG(a^2, b^2)}{AG(a, b)} < M_2(a, b) \end{aligned} \quad (1.4)$$

for all $a, b > 0$ with $a \neq b$.

We recall the Gauss identity [6, 7]

$$AG(1, r')\mathcal{K}(r) = \frac{\pi}{2} \quad (1.5)$$

for $r \in [0, 1)$ and $r' = \sqrt{1 - r^2}$. As usual, $\mathcal{K}$ and $\mathcal{E}$ denote the complete elliptic integrals [8] given by

$$\begin{aligned} \mathcal{K}(r) &= \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n}, \quad \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{E}(r) &= \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1/2, n)(1/2, n)}{(n!)^2} r^{2n}, \quad \mathcal{E}'(r) = \mathcal{E}(r'), \end{aligned} \quad (1.6)$$

where $(a, 0) = 1$ for $a \neq 0$, and $(a, n) = \prod_{k=0}^{n-1} (a + k)$.

For fixed $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$, let

$$f_1(x) = H(xa + (1-x)b, xb + (1-x)a), \quad (1.7)$$

$$f_2(x) = G(xa + (1-x)b, xb + (1-x)a), \quad (1.8)$$

$$f_3(x) = L(xa + (1-x)b, xb + (1-x)a). \quad (1.9)$$

Then it is not difficult to verify that $f_1(x)$, $f_2(x)$, and $f_3(x)$ are continuous and strictly increasing in $[0, 1/2]$, respectively. Note that $f_1(0) = H(a, b) < AG(a, b) < f_1(1/2) = A(a, b)$, $f_2(0) = G(a, b) < AG(a, b) < f_2(1/2) = A(a, b)$ and $f_3(0) = L(a, b) < AG(a, b) < f_3(1/2) = A(a, b)$.

Therefore, it is natural to ask what are the least values p , q , and s in $(0, 1/2)$ such that the inequalities $H(pa + (1-p)b, pb + (1-p)a) > AG(a, b)$, $G(qa + (1-q)b, qb + (1-q)a) > AG(a, b)$, and $L(sa + (1-s)b, sb + (1-s)a) > AG(a, b)$ hold for all $a, b > 0$ with $a \neq b$, respectively. The main purpose of this paper is to answer these questions. Our main results are Theorems 1.1–1.3.

Theorem 1.1. *If $p \in (0, 1/2)$, then inequality*

$$H(pa + (1-p)b, pb + (1-p)a) > AG(a, b) \quad (1.10)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 1/4$.

Theorem 1.2. *If $q \in (0, 1/2)$, then inequality*

$$G(qa + (1-q)b, qb + (1-q)a) > AG(a, b) \quad (1.11)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $q \geq 1/2 - \sqrt{2}/4$.

Theorem 1.3. *If $s \in (0, 1/2)$, then inequality*

$$L(sa + (1-s)b, sb + (1-s)a) > AG(a, b) \quad (1.12)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $s \geq 1/2 - \sqrt{3}/4$.

2. Lemmas

In order to establish our main results we need several formulas and lemmas, which we present in this section.

For $0 < r < 1$, the following derivative formulas were presented in [6, Appendix E, pp. 474-475]:

$$\begin{aligned} \frac{d\mathcal{K}}{dr} &= \frac{\xi - r'^2 \mathcal{K}}{rr'^2}, & \frac{d\xi}{dr} &= \frac{\xi - \mathcal{K}}{r}, \\ \frac{d(\xi - r'^2 \mathcal{K})}{dr} &= r\mathcal{K}, & \frac{d(\mathcal{K} - \xi)}{dr} &= \frac{r\xi}{r'^2}, \end{aligned} \quad (2.1)$$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r). \quad (2.2)$$

The following Lemma 2.1 can be found in [6, Theorem 3.21(7) and Exercise 3.43(4)].

Lemma 2.1. (1) $(1 + r'^2)\xi(r) - 2r'^2 \mathcal{K}(r)$ is strictly increasing from $(0, 1)$ onto $(0, 1)$;
 (2) $\xi(r)/r'^{1/2}$ is strictly increasing from $(0, 1)$ onto $(\pi/2, +\infty)$.

Lemma 2.2. *Inequality*

$$\frac{2}{\pi} \mathcal{K}(r) \sqrt{1 - \frac{1}{2}r^2} > 1 \quad (2.3)$$

holds for all $r \in (0, 1)$.

Proof. Let

$$f(r) = \log \left[\frac{2}{\pi} \mathcal{K}(r) \sqrt{1 - \frac{1}{2}r^2} \right]. \quad (2.4)$$

Then simple computations lead to

$$f(0) = 0, \quad (2.5)$$

$$f'(r) = \frac{\xi(r) - r'^2 \mathcal{K}(r)}{rr'^2 \mathcal{K}(r)} - \frac{r}{2 - r^2} = \frac{(1 + r'^2)\xi(r) - 2r'^2 \mathcal{K}(r)}{rr'^2(2 - r^2)\mathcal{K}(r)}. \quad (2.6)$$

It follows from Lemma 2.1 (1) and (2.6) that $f'(r) > 0$ for $r \in (0, 1)$, which implies that $f(r)$ is strictly increasing in $(0, 1)$.

Therefore, inequality (2.3) follows from (2.4) and (2.5) together with the monotonicity of $f(r)$. $\square$

Lemma 2.3. *Inequality*

$$\frac{2\sqrt{3}}{\pi} r \mathcal{K}(r) > \log \left(\frac{2 + \sqrt{3}r}{2 - \sqrt{3}r} \right) \quad (2.7)$$

holds for all $r \in (0, 1)$.

Proof. Let

$$g(r) = \frac{2\sqrt{3}}{\pi} r \mathcal{K}(r) - \log \left(\frac{2 + \sqrt{3}r}{2 - \sqrt{3}r} \right). \quad (2.8)$$

Then simple computations lead to

$$\begin{aligned} g(0) &= 0, \\ g'(r) &= \frac{2\sqrt{3}}{\pi} \left(\mathcal{K}(r) + r \frac{\xi(r) - r'^2 \mathcal{K}(r)}{r r'^2} \right) - \frac{4\sqrt{3}}{4 - 3r^2} = \frac{2\sqrt{3}}{\pi(1 + 3r'^2)} \left(\frac{1 + 3r'^2}{r'^{3/2}} \frac{\xi(r)}{r'^{1/2}} - 2\pi \right). \end{aligned} \quad (2.9)$$

Clearly the function $r \rightarrow (1 + 3r'^2)/r'^{3/2}$ is strictly decreasing from $(0, 1)$ onto $(4, +\infty)$. Then (2.10) and Lemma 2.1 (2) lead to the conclusion that $g'(r) > 0$ for $r \in (0, 1)$. Thus, $g(r)$ is strictly increasing in $(0, 1)$.

Therefore, inequality (2.7) follows from (2.8) and (2.9) together with the monotonicity of $g(r)$. $\square$

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. Let $\lambda = 1/4$, then from the monotonicity of the function $f_1(x) = H(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.10) we only need to prove that

$$\text{AG}(a, b) < H(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) \quad (3.1)$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.7) we clearly see that both $\text{AG}(a, b)$ and $H(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0, 1)$ and $r = (1 - t)/(1 + t)$, then from (1.5) we have

$$H(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) - \text{AG}(a, b) = \frac{(t + 3)(3t + 1)}{8(1 + t)} - \frac{\pi}{2\mathcal{K}(t')}. \quad (3.2)$$

Let

$$F(t) = \frac{(t + 3)(3t + 1)}{8(1 + t)} - \frac{\pi}{2\mathcal{K}(t')}. \quad (3.3)$$

Then making use of (2.2) we get

$$F(t) = \frac{(2 + r)(2 - r)}{4(1 + r)} - \frac{\pi}{2(1 + r)\mathcal{K}(r)} = \frac{\pi}{8(1 + r)\mathcal{K}(r)} F_1(r), \quad (3.4)$$

where $F_1(r) = (2/\pi)(4 - r^2)\mathcal{K}(r) - 4$. Note that

$$\begin{aligned} F_1(r) &= \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n} (4 - r^2) - 4 \\ &= 4r^2 \sum_{n=0}^{\infty} \frac{(1/2, n+1)^2}{[(n+1)!]^2} r^{2n} - r^2 \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{[(n+1)!]^2} (3n^2 + 2n) r^{2(n+1)} > 0. \end{aligned} \quad (3.5)$$

Therefore, inequality (3.1) follows from (3.2)–(3.5). $\square$

Next, we prove that the parameter $p = \lambda = 1/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.10) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < p < 1/2$ and small $x > 0$,

$$\text{AG}(1, 1-x) = \frac{\pi}{2\mathcal{K}(\sqrt{2x-x^2})} = 1 - \frac{1}{2}x - \frac{1}{16}x^2 + o(x^3), \quad (3.6)$$

$$H(p(1-x) + 1-p, (1-p)(1-x) + p) = 1 - \frac{1}{2}x + \left(-p^2 + p - \frac{1}{4}\right)x^2 + o(x^3). \quad (3.7)$$

It follows from (3.6) and (3.7) that inequality $\text{AG}(1, 1-x) \leq H(p(1-x) + 1-p, (1-p)(1-x) + p)$ holds for small x only $p \geq 1/4$.

Remark 3.1. For $0 < p < 1/2$ and $x > 0$, one has

$$\lim_{x \rightarrow 0} \frac{H(px + 1-p, (1-p)x + p)}{\text{AG}(1, x)} = \lim_{x \rightarrow 0} \frac{4[px + 1-p][(1-p)x + p]}{(1+x)\pi} \mathcal{K}(x') = +\infty. \quad (3.8)$$

Equation (3.8) implies that there does not exist $p \in (0, 1/2)$ such that $\text{AG}(1, x) > H(px + 1-p, (1-p)x + p)$ for all $x \in (0, 1)$.

Proof of Theorem 1.2. Let $\mu = 1/2 - \sqrt{2}/4$, then from the monotonicity of the function $f_2(x) = G(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.11) we only need to prove that

$$\text{AG}(a, b) < G(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \quad (3.9)$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.8) we clearly see that both $AG(a, b)$ and $G(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0, 1)$ and $r = (1 - t)/(1 + t)$, then from (1.5) we have

$$G(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) - AG(a, b) = \sqrt{[\mu + (1 - \mu)t][\mu t + (1 - \mu)]} - \frac{\pi}{2\mathcal{K}(t')}. \quad (3.10)$$

Let

$$G(t) = \sqrt{[\mu + (1 - \mu)t][\mu t + (1 - \mu)]} - \frac{\pi}{2\mathcal{K}(t')}. \quad (3.11)$$

Then making use of (2.2) we have

$$G(t) = \frac{\pi}{2(1+r)\mathcal{K}(r)} \left[\frac{2}{\pi} \mathcal{K}(r) \sqrt{1 - \frac{1}{2}r^2} - 1 \right]. \quad (3.12)$$

Therefore, inequality (3.9) follows from (3.10)–(3.12) together with Lemma 2.2. $\square$

Next, we prove that the parameter $q = \mu = 1/2 - \sqrt{2}/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.11) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < q < 1/2$ and small $x > 0$,

$$G(q(1-x) + 1 - q, (1-q)(1-x) + q) = 1 - \frac{1}{2}x + \frac{1}{8}(-4q^2 + 4q - 1)x^2 + o(x^3). \quad (3.13)$$

It follows from (3.6) and (3.13) that inequality $AG(1, 1-x) \leq G(q(1-x) + 1 - q, (1-q)(1-x) + q)$ holds for small x only $q \geq 1/2 - \sqrt{2}/4$.

Remark 3.2. For $0 < q < 1/2$ and $x > 0$, one has

$$\lim_{x \rightarrow 0} \frac{G(qx + 1 - q, (1-q)x + q)}{AG(1, x)} = \lim_{x \rightarrow 0} \frac{2}{\pi} \sqrt{[qx + 1 - q][(1-q)x + q]} \mathcal{K}(x') = +\infty. \quad (3.14)$$

Equation (3.14) implies that there does not exist $q \in (0, 1/2)$ such that $AG(1, x) > G(qx + 1 - q, (1-q)x + q)$ for all $x \in (0, 1)$.

Proof of Theorem 1.3. Let $\beta = 1/2 - \sqrt{3}/4$, then from the monotonicity of $f_3(x) = L(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.12) we only need to prove that

$$AG(a, b) < L(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) \quad (3.15)$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.9) we clearly see that both $AG(a, b)$ and $L(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0, 1)$ and $r = (1 - t)/(1 + t)$, then from (1.5) one has

$$\begin{aligned} & L(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) - AG(a, b) \\ &= \frac{\sqrt{3}(1 - t)}{2 \log \left[\left((2 - \sqrt{3})t + 2 + \sqrt{3} \right) / \left((2 + \sqrt{3})t + 2 - \sqrt{3} \right) \right]} - \frac{\pi}{2\mathcal{K}(t)}. \end{aligned} \quad (3.16)$$

Let

$$J(t) = \frac{\sqrt{3}(1 - t)}{2 \log \left[\left((2 - \sqrt{3})t + 2 + \sqrt{3} \right) / \left((2 + \sqrt{3})t + 2 - \sqrt{3} \right) \right]} - \frac{\pi}{2\mathcal{K}(t)}. \quad (3.17)$$

Then from (2.2) we get

$$J(t) = \frac{\pi}{2(1 + r)\mathcal{K}(r) \log \left(\left(2 + \sqrt{3}r \right) / \left(2 - \sqrt{3}r \right) \right)} g(r), \quad (3.18)$$

where $g(r)$ is defined as in Lemma 2.3.

Therefore, inequality (3.15) follows from (3.16)–(3.18) together with Lemma 2.3. $\square$

Next, we prove that the parameter $s = \beta = 1/2 - \sqrt{3}/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.12) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < s < 1/2$ and small $x > 0$,

$$L(s(1 - x) + 1 - s, (1 - s)(1 - x) + s) = 1 - \frac{1}{2}x + \frac{1}{12}(-4s^2 + 4s - 1)x^2 + o(x^3). \quad (3.19)$$

It follows from (3.6) and (3.19) that inequality $AG(1, 1 - x) \leq L(s(1 - x) + 1 - s, (1 - s)(1 - x) + s)$ holds for small x only $s \geq 1/2 - \sqrt{3}/4$.

Remark 3.3. For $0 < s < 1/2$ and $x > 0$, one has

$$\lim_{x \rightarrow 0} \frac{L(sx + 1 - s, (1 - s)x + s)}{AG(1, x)} = \lim_{x \rightarrow 0} \frac{2}{\pi} \mathcal{K}(x') \frac{(1 - 2s)(1 - x)}{\log[(sx + 1 - s)/((1 - s)x + s)]} = +\infty. \quad (3.20)$$

Equation (3.20) implies that there exist no values $s \in (0, 1/2)$ such that $AG(1, x) > L(sx + 1 - s, (1 - s)x + s)$ for all $x \in (0, 1)$.

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